DISCOVERING MEANINGS IN ELEMENTARY SCHOOL MATHEMATICS

FOSTER E. GROSSNICKLE

LEO J. BRUECKNER

JOHN RECKZEH

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PREFACE

During the past decade changes in the elementary school mathematics curriculum have occurred at such a rapid pace that they have frequently been termed revolutionary. A number of factors have been responsible for these changes, but the most important has been the experimental work in the teaching of mathematics conducted by various centers, notably the School Mathematics Study Group (SMSG). The research of these groups has resulted in what is popularly known as modern mathematics.

In a period of rapid change there is always a tendency to select the new and discard the old, regardless of their relative value. For example, drill for retention of learning, which was an integral part of the traditional program, was given a very minor place, if any, in the first programs of modern mathematics. This omission has since been shown to be a serious error, and in today's programs provision is made for retention of learning. In other, similar ways the modern mathematics curriculum has had time to stabilize. Topics of fringe or doubtful value have been eliminated or put in their proper place while those of major importance are now included in the curriculum of a large number of elementary schools.

Modern mathematics is an extension of the program that emphasizes meaning and understanding in learning. This movement began over two decades ago, as evidenced by the title of the first edition of this text, *How To Make Arithmetic Meaningful*, which was published in 1947.

The present volume is the fifth edition in the series now entitled, *Discov-*

ering Meanings in Elementary School Mathematics. The third edition appeared in 1959, the fourth in 1963, and the fifth five years later. These three editions were published during the period of greatest change in the elementary school mathematics program. The third edition thus dealt largely with what may be termed the traditional arithmetic program, while the fourth edition reflected the period when the mathematics program was in a transitional state and lacked stability. With this edition the modern program has approached a state of equilibrium, so that the present volume presents a program that has been tested in a number of elementary schools throughout the country.

All of the tried and tested features that characterized earlier editions have been retained in the present text. Learning by discovery is again stressed, and as in former editions, step-by-step procedures are given for introducing new topics and for enriching pupils' understanding through participation in a variety of activities. The features new to this edition may be summarized as follows:

1. A mathematics program for the kindergarten is included (Chap. 7).

2. Four new chapters have been added (4, 12, 18, 23). The first three of these deal with material pertaining to modern mathematics. Chapter 4 deals with sets and Chapter 12 discusses special numbers and the set of integers. Chapter 18 is a comprehensive treatment of nonmetric geometry. Chapter 23 deals with the slow learner. With today's emphasis on the education of the school dropout, this material is particularly important, since the program for the dropout and the slow learner have many elements in common.

3. More emphasis is placed on *structure.* Structure, perhaps the greatest contribution of mathematics, implies that there are certain properties of number that govern the basic operations with number. The present edition is completely structured with respect to the operations with whole numbers, fractional numbers, and decimals.

4. Emphasis is given to the application of the principles of learning. Six principles are listed that govern the learning of number (Chap. 3). Through a sample presentation of a new topic the authors demonstrate how these principles apply.

5. Supplementary instructional aids are available for use with the present edition. Included is a teacher's manual with tests for each chapter as well as a comprehensive test for the entire text. Also, a study guide will be available for the student who needs help in modern mathematics.

With this edition of *Discovering Meanings in Elementary School Mathematics* Dr. John Reckzeh joins the authorship. Dr. Reckzeh, who is chairman of the mathematics department at Jersey City State College, has taught mathematics in programs sponsored by the National Science Foundation since the inception of that program.

Just as the manuscript for this edition was submitted to the publishers, Dr. Leo J. Brueckner passed away. In this and earlier editions Dr. Brueckner wrote with insight and understanding on the subject of teaching mathematics to children. It is fitting that the present text, the last of Dr. Brueckner's many publications, should be dedicated to his memory.

F.E.G.

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DISCOVERING MEANINGS IN ELEMENTARY SCHOOL MATHEMATICS

TRENDS IN ELEMENTARY SCHOOL MATHEMATICS

"CHANGE IS THE LAW OF LIFE"

People living in the world today face many problems of adjustment to it—personal, social, and professional being just a few. They search out principles with which to order their lives; sometimes each area of existence requires a different principle. People have only to go to literature and philosophy in order to be aided in their search. In 1963, President Kennedy addressed West Berliners assembled at the Wall. In offering those divided people a law of life, he quoted Goethe: "For time and the world do not stand still. Change is the law of life. And those who look only to the past are certain to miss the future." Some would prefer to heed Santayana's warning: "He who neglects the past is condemned to repeat it."

Ideally today's mathematics teacher should emulate the Roman god Janus, who was supposed to have had a face looking to the future and one looking to the past. It is true of mathematics as of any endeavor that the changes that are essential for progress should integrate the best features of the old and the new.

The elementary school mathematics program has changed rapidly within the past decade. Some of the changes reflect the findings of the various experimental projects that have been underway in the last several years. One such program, the School Mathematics Study

Group (SMSG), which has received federal support, has been particularly significant in bringing about changes in the elementary mathematics program. Indeed, as knowledge of the subject increases and as new discoveries are made concerning the process of learning, further changes may be anticipated as the result of experimental programs such as the SMSG.

Not only is the content of mathematics changing today but the applications of the discipline are expanding. Even at the elementary and secondary levels widespread use of computers, calculators, and other automatic devices has made an impact on the mathematics program. The shortage of persons who can program computers has created a demand for students who have a broader background in mathematics than was necessary a decade ago. Although the supply of persons skilled in programing is less than the demand at the present time, this special skill may become largely obsolete within the next decade as computers that can program themselves come into general use. Thus the development of specific skills that are used in a vocation should not be the guideline for teaching elementary mathematics. Rather, a student needs the kind of education that will give him flexibility and versatility in meeting new situations as he encounters them.

It is quite possible that by the time the pupil in today's elementary school is employed he will be called upon to utilize skills that are not known today. Thus a mathematics program that stresses the acquisition of mathematical skills through rote learning and not through understanding is at present unsatisfactory. A pupil needs to understand the principles that govern the structure of our numeration system, must discover number relationships and patterns of number behavior in order to be able to adjust to new quantitative situations. Additionally the pupil needs to acquire computational proficiency to help enrich his understanding of number and to use this knowledge effectively in a changing scientific society.

In the past the average employed individual changed jobs at least three times during his lifetime. Some leading economists and industrialists predict that in the future a person may need to change his *vocation* three times during his working years. To meet these changing conditions the individual must learn to discover patterns that he can apply to increase his knowledge in a given field. Emphasis in teaching mathematics should therefore be placed on developing an understanding of work with number. Ideally the elementary school mathematics program should stress mastery of the basic principles underlying number, supplemented by a reasonable proficiency in the application of these principles. This text is designed to implement that philosophy.

ORGANIZATION OF THE TEXT

Part I of this text contains three chapters which deal with the objectives of elementary school mathematics, the factors that affect the curriculum in this field, and a set of general principles that underlie instruction. The authors believe that the curriculum should include essential elements of the traditional arithmetic program as well as introductory work in algebra and geometry. Emphasis should be placed on helping the student discover a pattern that he can apply in further learnings. The text includes discussions of the experimental programs of several leading research centers and their impact on the elementary mathematics curriculum.

Chapter 3, which considers principles of teaching mathematics, stresses the importance of making the work mathematically meaningful to the students. Through numerous examples the text demonstrates how children can be led to discover the facts, ideas, relationships, and principles that underlie the basic number operations. Mathematical principles form the basis of teaching procedures that are discussed in many of the chapters of this book.

Part II is made up of 16 chapters dealing with the teaching of the basic subject matter of elementary school mathematics, beginning with a treatment of sets and number sentences, new to this edition.

Chapter 4 introduces many of the concepts of modern mathematics for the student or teacher who lacks a background in the subject or who needs a review. The next two chapters deal with systems of numeration and systems of numbers and their characteristics. Chapter 5 describes the distinguishing features of our decimal system of numeration and compares it with other systems. The chapter also illustrates how the principle of regrouping numbers in base ten applies to regrouping numbers that are expressed in a different base. Chapter 6 emphasizes structure in systems of numbers. The chapter defines the set of numbers that is of concern to the elementary school teacher as well as the essential principles that govern the basic operations in that number system.

Chapters 7 through 11 are devoted to work with whole numbers. Chapter 7 is concerned with the introduction of number concepts in the kindergarten and the first steps in beginning number work in grades 1 and 2, while the remaining chapters in this group treat the four basic operations with whole numbers. Emphasis is given to the relationship between an operation and its inverse, as between addition and subtraction.

The material of Chapter 12 on the set of integers and the prime and composite numbers is also new to this edition. The integers include positive and negative numbers. Many schools today introduce negative numbers below the level of the junior high school. Some of the characteristics of prime and composite numbers are discussed and methods are given for determining the prime factors of a number.

Chapters 13 through 16 deal with rational numbers expressed as fractional numerals, decimals, and per cents. A rational number is the quotient of two whole numbers, as $\frac{a}{b}$, provided b does not equal 0. The text emphasizes the properties of rational numbers and the relationships between the set of whole numbers and the set of rational numbers.

Mathematical thinking and problem solving are the subjects of Chapter 17. The use of an equation to solve a verbal problem is explained, as well as the procedure for translating a verbal statement into a number sentence. The chapter also considers the different levels of maturity that apply to the solution of equations.

Chapters 18 and 19 introduce geometry. The first of these chapters, new to this edition, presents nonmetric geometry which is concerned with sets of points as they appear in figures of one, two, or three dimensions. Metric geometry, the concern of Chapter 19, deals with the measures of sets of points. The set of points forming a line segment *AB* would be treated as nonmetric geometry, while the measure of the set of points would be treated as metric geometry. The measure of this line segment is a number that indicates the number

of standard units in the measure, for example, a centimeter, an inch, a foot, or some other standard linear unit.

Part III, which includes Chapters 20 through 24, deals with equipping the classroom, procedures to appraise pupil progress, and providing for individual differences in learning. The classroom must be adequately equipped to carry out a program that stresses meaningful learning. Chapter 20 describes the materials needed to equip the classroom as a learning laboratory. Chapter 21 contains a comprehensive discussion of methods of evaluating pupil achievement in relation to the desired outcomes of instruction in mathematics. In Chapter 22, which treats the guidance of learning, emphasis is placed on developing an instructional program that takes into consideration the strengths as well as the weaknesses of the individual learner. Stress is given to directing the learning activities so that a minimum of remedial measures need to be applied. Reliable diagnostic and corrective measures are also discussed.

Chapters 23 and 24 deal with making provisions for the less able and the more able students. The financial support given by the federal government to the education of deprived children in recent years indicates the need for developing techniques for teaching the slow learner. Chapter 23 considers procedures and materials for instructing pupils in this group, while Chapter 24 presents ways of enriching the program for the more able learners. For the latter students these methods include differentiated curriculum content, expansion and extension of mathematical topics, grouping of pupils according to achievement, use of the school library for independent study of topics of interest, and other means of fostering power in mathematics.

Discovering Meanings in Elementary School Mathematics, fifth edition, is a blueprint for teaching mathematics in the elementary school. Use of this text should enable the teacher to understand and implement a program of modern mathematics. The teacher with a limited background in modern mathematics will find a variety of instructional aids to enrich the teaching program. In combination with the best features of a conventional program, such aids can be used to develop a method of instruction that emphasizes the discovery of number patterns, the structure of the decimal system of numeration, and the properties of number.

THE CHANGING MATHEMATICS PROGRAM OF THE ELEMENTARY SCHOOL

The uses to which mathematics will be put in the future baffle the imagination. The world today demands more mathematical knowledge of more people than ever was true of the world of yesterday, and tomorrow's world will doubtlessly make even greater demands.[1] An understanding of the role of mathematics in the rapidly changing industrial society of the space age is now a prerequisite for intelligent citizenship.[2] No one can foretell what requirements will be made of mathematics in any occupation of the future. It is important, therefore, that mathematics be presented in such a way that students will be able to apply the ideas and skills they learn today to the new mathematical problems of the future. The mathematicians of tomorrow are enrolled in the schools of today.

In this chapter the following topics are discussed. scope of the elementary school mathematics program: the changing mathematics program. ideas under-

[1] National Council of Teachers of Mathematics. *The Revolution in School Mathematics* (Washington, D.C.: The Council, 1961). A report of regional orientation conferences in mathematics.

[2] See Morris Kline, *Mathematics in Western Culture* (New York: Oxford University Press, 1955).

lying the modern program, organization of the mathematics program; gradation of curriculum content; experimental research projects; algebra and geometry in the elementary school program.

SCOPE OF THE ELEMENTARY SCHOOL MATHEMATICS PROGRAM

The purpose of instruction in mathematics in the elementary school is to develop an intelligent understanding of basic mathematical ideas and an appreciation of the role of mathematics in daily life in a time of rapid social change and scientific progress.[3] In the modern mathematics curriculum of grades K–6, emphasis is placed on the structure of our numeration system and why the system functions as it does in the various mathematical procedures. Stress is given to making basic number operations mathematically meaningful. Definite steps are also being taken to include the simpler elements of algebra and geometry in the mathematics program of the elementary school. It is becoming clear that we need an improved mathematics curriculum, one that teaches students of all levels of ability not only the basic computational skills but also the basic concepts as well as the structure of mathematics.

A current trend in experimental programs in mathematics is to place increased emphasis on *structure*. Structure implies that number has certain properties that govern the basic operations (see Chap. 6). In a well-constructed elementary school program, both structure and applied mathematics should receive a carefully integrated treatment.

[3]*Goals for School Mathematics*, Report of the Cambridge Conference on School Mathematics (Boston: Educational Services, Inc., 1963). (See pp. 19–20 of this text for a discussion of this report.)

Today emphasis is being placed on helping the more able student to handle abstract mathematical concepts and ideas with facility and insight. In fact, the mathematics program of the elementary school must be so planned that a sound basis for the study of mathematics beyond that level is established. At the same time, special attention is being given to the development of a mathematics program for the slow learner at all grade levels.[4]

The teacher must also take into consideration the broader objectives of education in which all areas of the curriculum play a part. These include outcomes related to the development of desirable aspects of the learner's personality, including his interests, attitudes, emotional adjustment, social traits, and even his physical well-being. There is thus the need for a rich, vitalized, well-integrated program adapted to the interests, aptitude, and stage of maturity of the various children.

THE CHANGING MATHEMATICS PROGRAM

The rapid extension of the use of computers and automatic devices for recording information raises problems of profound significance for curriculum makers in the field of mathematics. The following questions are adapted from an important report of the secondary school curriculum committee of the National Council of Teachers of Mathematics. The reader should keep these questions in mind as he reads this chapter.

1. What is the place of mathematics in a changing society?

2. What are some of the more significant new interpretations and uses of mathematics?

3. What are some of the more recently developed areas of mathematical subject matter from which pertinent adaptations for the elementary level of instruction can be made?

4. What are the criteria to be used in planning a program in mathematics that will guarantee opportunity for maximum benefit both to the slow learner and to the mathematically gifted?

5. What are the criteria to be used in planning a program that will provide equal assurance for appropriate functional competence to both terminal students and college-capable students?

6. How can we secure better counseling of pupils so that they may be informed about avenues of potential interest and challenge in mathematics as well as warned against possible frustrations and disappointments in the unwise attempt to reach unattainable goals of mathematical achievement?

7. Which of the conventional topics of mathematics, if any, should be radically changed or eliminated?

8. Which of the newer developments in subject matter have become of more than mere specialized significance?

9. What of the new can be used to enrich the traditional, and what are approved procedures for the accomplishment of such enrichment appropriate to elementary school pupils?

10. What of the old can be used to orient and clarify the new in such a manner that elementary school pupils may profit most from experience with the new?[5]

Questions such as these emphasize the urgent need to reappraise the place of mathematics in our changing society and to reexamine the mathematics curriculum in our schools and the instructional procedures used to help pupils derive the greatest benefit from this curriculum.

[5]*Mathematics Teacher*, May 1959, 52:389–417.

BASIC IDEAS UNDERLYING THE MODERN PROGRAM

The traditional elementary school mathematics program had as its goal the mastery of more or less isolated, unrelated facts and operations through repetitive practice. Meanings and understandings were not given adequate consideration, and relationships among operations and topics were not developed.

In the modern elementary school, a definite effort is made to emphasize the structure of mathematics. The modern program stresses the relationships that exist among the various number operations and attempts to develop power in quantitative thinking and in utilizing mathematical procedures in all areas of daily life. Emphasis is placed on problem solving. The Twenty-fourth Yearbook of the National Council of Teachers of Mathematics, *Growth of Mathematical Ideas, K–12* (1959), is devoted to a discussion of vital mathematical ideas that can serve as the basis for the mathematics program at all levels of the school The following discussion of ten key ideas adapted from that yearbook identifies important strands that can serve as the framework for the mathematics program.

Sets, numbers, and numeration

The very young child has primitive number ideas. He senses the size of a set of objects, such as a set of blocks, without counting them. He later learns to tell how many there are in similar

small groups; he then learns to count the objects, then to tell how to compare the sizes of small groups. Finally, he learns to read and write the symbols for the numerals. Gradually over the years the pupil learns many different kinds of numerals and sets of numbers that man has invented to deal effectively with the quantitative aspects of his environment. (Chapters 5 and 6 describe different systems of numeration and of numbers.)

Properties of numbers underlying mathematical procedures

The traditional arithmetic program stressed the mastery of number operations; it gave little consideration to creating a mathematical understanding of the operations. In recent years, however, mathematicians have made an effort to show how the mathematical meanings of number operations can be presented more effectively. The result has been emphasis on properties of numbers that underlie the basic operations. (These properties are described in detail in Chapter 6.)

Structural relationships

The relationships that exist between number operations are an important means of bringing out the structure of mathematics. In turn, a knowledge of the structure of mathematics is a valuable aid in learning and understanding the basic mathematical concepts. Addition and multiplication are the two basic operations in terms of which the opposite or inverse operations can be defined. The relationship in the set of numbers $\{3, 5, 8\}$ suggests the four basic addition and subtraction facts, $3 + 5 = 8$, $5 + 3 = 8$, $8 - 3 = 5$, and $8 - 5 = 3$. The teaching of such number facts in related groups emphasizes the underlying structure of the system of numeration and makes it possible for the teacher to help the pupils make generalizations about the relationships among the facts of the various operations.

Mathematical symbols and sentences

The child must learn a large technical vocabulary related to number and number operations in the elementary school mathematics program. He must learn the meaning of many symbols used in mathematical notation, for example, $+$, $-$, $\times$, $\div$, $=$, $\neq$, $>$, $<$, 10^2, and 10_4. In addition, he must become familiar with the symbols and abbreviations that are used to represent quantitative ideas, such as 2 ft. and 30°.

The pupil must also learn to use and make *number sentences.* A number sentence is a mathematical statement, such as $3 + 5 = 8$, $5 - \square = 1$, $2 \times 6 = n$, and $5 > 3$. (Chapter 4 describes number sentences and their role in elementary mathematics.)

Measurement

Measurement involves the use of standard units that have been arbitrarily chosen in order to deal effectively and precisely with quantitative aspects of the environment. The learner should have experiences that will help him to understand the nature and variety of standard units and the relationships among them. Through their use the student should become familiar with the many measuring devices that are used in the affairs of daily life, both in and out of school. He should also learn that new units and methods of measurement are continually being devised by scientists as new needs arise, for example, micron, millisecond, and nanosecond.

Nonmetric and metric geometry

The expansion of geometry as a part of the elementary school mathematics

curriculum to include both nonmetric and metric aspects of the subject grows out of the realization that it is important for an individual living in the space age to be familiar with geometric ideas.

Nonmetric geometry deals with the properties of sets of points (Chap. 18) while metric geometry deals with measures of figures. The measures include those for length, area, volume, and angles (Chap. 19). These topics of metric geometry have long been a part of the traditional elementary school mathematics program.

Estimation

One of the most common applications of mathematics in everyday life is in the making of estimates. Estimates are judgments as to the likely answers of some computation or problem. Many of the applications of measurement are estimates of amounts. Estimates serve as a basis for judging the correctness of a solution. Much of the mental arithmetic used in science, business, industry, and the home involves estimation.

Approximation

We are often inclined to think of a measure as being exact. Thus a child may use a ruler to find that the length of a line segment is apparently $8\frac{1}{2}$ inches. The pupil should have experiences to enable him to discover that measurements are approximate. Thus the length of a line segment that seems to be $8\frac{1}{2}$ inches may be a little more or a little less than $8\frac{1}{2}$ inches. The use of round numbers (p. 147) and of scientific notation (p. 63) are two of the most common uses of approximation.

Statistics

The reading and interpretation of tables, graphs, charts, and diagrams have become increasingly important in the elementary mathematics program. Such materials are often found in books dealing with science and the social studies, and their analysis provides excellent experience in applying mathematical thinking.

Proof

Formal mathematical proof is not usually appropriate at the elementary level. A foundation for understanding the nature of proof can be established in the elementary school (see Chapter 12).

ORGANIZATION OF THE MATHEMATICS PROGRAM

Principles for selecting mathematical experiences

To develop an appreciation of the role of mathematics and to make it function in their thinking, children should have experiences that will help them to discover the mathematical relationships that exist in the affairs of the world.

The selection of the necessary experiences is an important undertaking that requires the participation of mathematics specialists, curriculum consultants, school administrators, and teachers who understand children and plan their learning activities. In the last analysis it is the classroom teacher who must select from the many situations that arise those that hold specific possibilities in which items of mathematical importance can be emphasized. A good modern textbook provides a developmental background of content that should be of great assistance in planning the elementary mathematics program.

The following principles may be of help to teachers in selecting and planning mathematical experiences:

1. Because every child has his own rate and his own ways of learning and growing, mathematical experiences in

number, measurement, and form should be considered on every grade level in terms appropriate to the individual needs, abilities, and interest of each child. Implied in this is opportunity for individualized growth and for a curriculum which provides for continuity in understanding mathematics, for recurrence of opportunities in order to sharpen or broaden concepts, and for practice necessary to produce proficiency in using mathematics.

2. Because certain characteristics seem to be exhibited by children of similar chronological ages, a sequence of major experiences seems pertinent to each grade level. Allowance must be made within each group, however, for individual needs, interests, and abilities if all children are to be challenged to grow.

3. Because mathematics pervades all areas of living, opportunities for development of mathematical understanding pervade all areas of the curriculum.

4. Mathematics gets its meaning from the environment, and understanding mathematics develops greater understanding of the environment. Therefore, children should be given many opportunities to explore mathematical situations, relationships, and possibilities in the environment. An understanding of mathematics occurs only in the mind of the individual; concepts are developed and broadened from many meaningful experiences. Since the quality of thought is revealed in speech and action, the teacher must give each child many opportunities to express and clarify in a variety of ways his understanding of mathematics.[6]

Organizing learning experiences

Curriculum-makers are faced with the problem of organizing the content of the mathematics program. Mathematics itself has an internal logic and coherence that determine the sequential organization of the subject matter to be taught. This is especially true of number and number operations. Topics dealing with measurement and its applications and with the form and position of objects, that is informal geometry, allow for a much greater flexibility of organization and content. In its broad outlines the mathematical facet of the curriculum can be developed through a logical analysis of its content. The grade placement of this content can be determined in the light of the capacity of the children and the demands of society.

GRADATION OF CURRICULUM CONTENT

Approaches

There are several different ways in which to approach grade placement. One method is to assign areas of subject matter in terms of the interests and developmental levels of the children. In order to do this teachers must know the interests, backgrounds, and abilities of their students and must select subject matter accordingly. Another approach is to assign instructional content to particular grade levels and then to adjust the child to this arrangement. This method emphasizes such factors as prerequisite knowledge and general experience of the children. A third approach is to adjust instructional goals by adapting contents and activities to the child, even to the extent of modifying the objectives of instruction in terms of the needs and abilities of individual children.

It is more likely that these approaches to grade placement will be used in conjunction with one another rather than separately. Unless this is done, the selection and location of curriculum content and experience will stress the developmental needs and interests of children to the neglect of adequate

[6]*Looking Ahead in Mathematics* (Sacramento, Calif.: State Department of Education, 1961), p. 17.

content, or else subject matter will be stressed without regard to the child and his development.

The logical nature of mathematics, as well as the increasing complexity of the elements that make up the subject, are decisive factors to be considered in determining the sequence of topics. One reason for the difficulty of long division is the complexity of the component skills involved. Thus it is inevitable that the teaching of divison by two-place divisors must be delayed until the underlying skills have been mastered.

Procedures

Surveys of current practices involve the examination of textbooks and courses of study to determine the average or the most frequent grade in which a given topic or some phase of it is taught. In varying degrees the validity of current practices is determined by the extent to which the gradation is based on the judgment of competent, informed individuals and groups in the light of known facts about the relative difficulty of the various elements of mathematics as determined by scientific studies and by experimentation. The analysis of current practices has yielded disappointing results, since the wide variations in the location of materials from school to school make it difficult to establish standards of placement. The problem is further complicated by what is known about individual differences in the capabilities of children, a fact the teacher in the classroom must take into consideration in adjusting the curriculum to meet their needs. Not only should the curriculum be flexible but provisions should be made for adapting it to varying environmental conditions and to the needs of children in local situations. Ideally the teacher should study the needs of the children and try to arrange learning experiences in view of their capabilities and interests.

Relative difficulty of number facts and processes

The difficulty of the subject matter of mathematics can be best determined by the success with which children master it. A method that has been widely used is to administer tests containing selected items to children at various grade levels and of different levels of ability and to determine the percentage of correct responses to the various items. The theory is that judgments based on objective data as to difficulty offer a more dependable basis for gradation than unaided personal judgments. An illustration of the types of information this procedure yields is given in Table 2.1. The data show the relative difficulty of a rather wide sampling of examples in the four number operations with whole numbers, fractions, and decimals. The level of difficulty for each type of example is measured by the per cent of correct responses on each test item for a random sampling of grade 7.1 (October) children from schools in all parts of the country. The pupils were all of approximately normal age and intelligence. The basic data were supplied by the California Test Bureau, Monterey, California.

Curriculum-makers and teachers of grades 6–8 should examine carefully the data included in Table 2.1 and consider their implications for the grade placement of subject matter and the levels of achievement to be expected of children at the end of the elementary school. It seems altogether probable that a number of the examples in sets (E–I) were too difficult for the students with IQ's below 100. It is possible, of course, that the level of achievement

would have been much higher if instructional methods and materials of the kind described in this book had been used to make learning more meaningful.

Growth of mathematical ability

Teachers must recognize the fact that there is a gradual growth in all aspects of mathematical ability from grade to grade. The rate and moment of growth in ability has not been determined objectively for algebra and geometry as has been done for computational skills. Even if a pupil's performance is not judged to be satisfactory at a given grade level, with added maturity and well-planned review he will usually perform at a higher level in following years. This fact suggests the necessity of setting up

TABLE 2.1

Per Cents of Correct Responses in Selected Items for Grade 7.1 Children of Normal Age and 100 IQ

		A—90–100%		
$12 + 34$	$67 - 23$	500×4	$6\overline{)42}$	400×60
		B—80–89%		
$4\frac{1}{2} + \frac{1}{2}$	$375 - 108$	607×8	$3\overline{)186}$	$\frac{3}{8} + \frac{3}{8}$
		C—70–79%		
$\$26.23 + 3.86 + .57 + 7.18$	$6804 - 3529$	$\frac{4}{5} - \frac{2}{5}$	684×23	$4\overline{)436}$
		D—60–69%		
$\$4.00 + \$.50 =$ $\$14 + \$3.30 =$	$7\frac{1}{3} - 3$	$\frac{2}{3} \times \frac{3}{8} =$	$31\overline{)6355}$	$12\frac{1}{3} + 4\frac{1}{4}$
		E—50–59%		
$3\frac{5}{6} + 2\frac{1}{4}$	$\frac{5}{6} - \frac{2}{3}$	5032×708	$4\overline{)6.72}$	$\frac{2}{3} \div 3 =$
		F—40–49%		
$\$300 - \$12.74 =$	$\frac{1}{4} \times \frac{1}{4} =$	$8 \times 3\frac{1}{4} =$	$\frac{3}{5} \div \frac{1}{5} =$	$2\frac{5}{6} + 4\frac{4}{5}$
		G—30–39%		
$24\frac{3}{4} + 11\frac{1}{2} + 10\frac{2}{3}$	$5\frac{1}{3} \times 3\frac{1}{2} =$	$42.3 \times .053$	$78\overline{)6293}$	$8 - 4\frac{1}{3}$
		H—20–29%		
$40.6 - 4\frac{1}{2} =$	$45\frac{5}{6} \times 13$	$120 \div 1\frac{1}{3} =$	$.04\overline{).672}$	
		I—Below 20%		
$63.07 - 3.0527 =$	$6\frac{1}{4} \div 2\frac{2}{3} =$	$.03\overline{)6}$	$.16\frac{2}{3} + 19.3 =$	4 yd. 2 ft. 9 in. $\times 5$

standards of achievement that are adjusted to the level of ability of the various pupils; all children should not, however, be expected to progress at the same rate.

Investigations are needed to answer such questions as the following: Is there an optimum grade level at which examples of each topic should be introduced in the curriculum? In view of the evident growth of achievement from year to year, what standards of performance should be set up for children of different levels of ability at successive grade levels? What methods of teaching will assure a higher level of performance? If number operations are made mathematically meaningful and are more frequently used in practical situations, will the situation be improved? What can be done to improve instructional materials?

A caution about gradation procedures

The results of experimental studies of the gradation of curriculum content should be used with a full recognition of their implications and limitations. The following statement by Tyler and Brownell presents the issue very clearly:

> Of course, knowing that we *can* teach this or that topic in a particular grade does not necessarily signify that we *should* do so. Yet, the wisdom of earlier instruction is not questioned often enough. Some "experimenters," having satisfied themselves that something can be taught earlier than commonly offered, then argue for instruction in this part of their discipline with no regard for the remainder of the curriculum. For example, as far as mathematics is concerned, it may be better to spend time throughout the first five or six grades in attempting to produce a high level of intelligent competence in computation instead of trying to teach more advanced topics of different number systems. Questions about "readiness" are, of course, interesting, both theoretically and practically, but the limited answers which are commonly accepted do not automatically determine and define curricular content and sequence. Ability to learn a certain skill or principle or generalization is not necessarily the sole criterion to be accepted for inclusion in the school curriculum.[7]

EXPERIMENTAL RESEARCH PROJECTS

Since 1955, a number of experimental programs have been undertaken with the purpose of updating the elementary school mathematics program and bringing it into line with the emerging needs of mathematics in modern life. We shall review here the main contributions of several of these investigations. A consideration of the outcomes of these studies should be of great value to curriculum-makers in evaluating the mathematics program at all grade levels.

These projects and the educational philosophies on which they are based are exerting a far-reaching influence on mathematics programs at all levels of the elementary school. Some have investigated the contents of a broad program for all grade levels; others have developed and experimented with units that can be taught at various levels; still others have been limited largely to specific areas, for example, sets, geometry, and algebra. An examination of these units should be of value to all elementary school teachers, since they suggest the kinds of changes that can and should be made in the mathematics curriculum at all levels.

[7] R. Tyler and W. A. Brownell, *Individualizing Instruction*, Sixty-first Yearbook of the National Society for the Study of Education (Chicago: University of Chicago Press, 1962), p. 322.

School mathematics study group (SMSG)

The content of the SMSG program includes:

1. Extensions in depth and breadth of the content now covered in elementary mathematics in grades K–6, with greater emphasis placed on the laws and properties of mathematics underlying number operations.

2. The introduction of simple concepts throughout grades K–6 on an intuitive level by methods that are consistent with later approaches to the topics. New topics include intuitive geometry and simple algebraic ideas. Discovery is the basis of learning.

3. An emphasis on precision of mathematical language as an aid to logical thinking and on symbols as a means of abbreviating language.[8]

The outlines of the contents for each grade level will assist schools in the selection and gradation of the contents of the program for each grade level.[9]

The following chapter headings outline briefly the contents of the courses at each grade level.

Kindergarten A commentary for the teacher, with mathematical background and instructional suggestions, contains the following chapters:

1. Sets
2. Recognizing Geometric Figures
3. Comparison of Sets
4. Subset of a Set
5. Joining and Removing Sets
6. Comparison of Sizes and Shapes
7. Ordering Sets
8. Using Geometric Figures for Directions and Games
9. Using Numbers with Sets

Book 1

1. Sets and Numbers
2. Numerals and the Number Line
3. Sets of Ten
4. Introduction to Addition and Subtraction
5. Recognizing Geometric Figures
6. Place Value and Numeration
7. Addition and Subtraction
8. Arrays and Multiplication
9. Partitions and Rational Numbers
10. Linear Measurement

Book 2

1. Sets and Numbers: Review
2. Addition and Subtraction: Review
3. Sets of Points
4. Addition and Subtraction: Further Facts and Techniques
5. Linear Measurement
6. Computing Sums and Differences
7. Congruence of Angles and Triangles
8. Arrays and Multiplication
9. Division and Rational Numbers

Book 3

1. Sets of Points
2. Addition and Subtraction: Review and Extension
3. Describing Points as Numbers
4. Arrays and Multiplication
5. Addition and Subtraction: Shorter Forms of Computation
6. Length and Area

[8] Adapted from: Edwina Deans, *Elementary School Mathematics: New Directions* (Washington, D.C.: Government Printing Office, 1963), pp. 20–21. (Dr. E. G. Begle of Stanford University is the director of this program.) Robert Eads, ed., *New Curricula* (New York: Harper & Row, Publishers, Inc., 1964), Chaps. 2, 3, and 4. William Wooten, *SMSG: The Making of a Curriculum* (New Haven, Conn.: Yale University Press, 1965).

[9] Copies of the SMSG textbooks and teacher commentaries for grades K–6 may be obtained from School Mathematics Study Group, Yale University Press, 92 Yale Station, New Haven, Conn. 06520.

7. Multiplication, Quotients, and Division
8. Rational Numbers
9. Division

Grade 4

1. Concepts of Sets
2. Numeration
3. Properties and Techniques of Addition and Subtraction, I
4. Properties of Multiplication and Division
5. Sets of Points
6. Properties and Techniques of Addition and Subtraction, II
7. Techniques of Multiplication and Division
8. Recognition of Common Geometric Figures
9. Linear Measurement
10. Concept of Rational Numbers

Grade 5

1. Extending Systems of Numeration
2. Factors and Primes
3. Extending Multiplication and Division, Part I
4. Congruence of Common Geometric Figures
5. Extending Multiplication and Division, Part II
6. Addition and Subtraction of Rational Numbers
7. Measurement of Angles
8. Area
9. Ratio
10. Summary

Grade 6

1. Exponents
2. Multiplication of Rational Numbers
3. Side-Angle Relationships of Triangles
4. Introducing the Integers
5. Coordinates
6. Division of Rational Numbers
7. Volume
8. Organizing and Describing Data
9. Summary
10. Sets and Circles

Greater Cleveland mathematics program (GCMP)

The GCMP was launched in 1959 with the purpose of improving the mathematics program at all levels. The program sought to develop a curriculum that could present to all children in a logical, systematic way the basic mathematical concepts before the introduction of computational schemes or algorisms.

To accomplish its purpose, the program makes extensive use of the discovery approach to learning. Problem situations are presented to the pupils as if they had not already been explored by the great minds of the past and present. Thus students are led to the established symbolism to be mastered. The exploration of the logical structure of mathematics stimulates the student's imagination and leads him to appreciate mathematics as a dynamic and meaningful study as he experiences the thrill of discovering or recreating some mathematics for himself. The elementary program was begun in the primary grades and has progressed upward until it now (1968) is nearing completion. Geometry is begun in grade 4.

The content of the GCMP draws heavily upon the principles of mathematics to help children learn the underlying structure of the materials presented. Some of the content is placed several years below the usual placement of the topics. In addition to the topics in the traditional mathematics program, GCMP contains units or exercises on:

1. Number sequences
2. Factors and multiples
3. Prime and composite numbers

4. Other numeration systems (bases other than 10)
5. An introduction to powers, roots, and negative numbers
6. Physical geometry (nonmetric)
7. Linear, area, and volume measurement (metric geometry)
8. The concept and language of sets carried through the topics at all grade levels
9. Simple informal proofs
10. Mathematical sentences and conditional statements presented to describe the corresponding ideas with numbers.

Schools in all parts of the country cooperated in trying out the materials prepared by the central committee. On the basis of reports submitted by individual teachers, the materials were revised one or more times before being made available for general distribution.[10]

Stanford projects

Sets and numbers Patrick Suppes, author of *Sets and Numbers*, Books I and II, believes that all mathematics can be developed from notions of sets and operations on sets.[11] In these two volumes he states that his primary objective was to develop the elementary concepts and laws of arithmetic in a manner that is both pedagogically simple and mathematically correct. He explains that he finds the term "sets" useful in helping children to develop basic concepts, and he holds that operations on sets are more concrete and comprehensible to young children than abstract operations on numbers. Emphasis is placed on understanding through the development of a technical vocabulary and by a carefully constructed set notation for recording ideas.

Geometry for the primary grades Books I and II by Hawley and Suppes are designed for grades 2 and 3.[12] The authors believe that an understanding of geometry will help children to understand and analyze the physical world. They include systematic work in simple intuitive geometry to stimulate and challenge the more able learners. Emphasis is placed on correctness in work with the pencil and on precision in the use of the compass and straight edge in order that accurate ideas will result from constructions. The technical vocabulary necessary for reading and communicating geometric ideas is presented as the need arises in construction work.

Hawley and Suppes make clear their belief that geometry is not a substitute for arithmetic, which also must be taught as an essential branch of mathematics.

The Madison project

The content of the Madison Project for grades 2–9 is intended for enrichment and draws heavily upon intuitive algebraic and geometric ideas.[13] The first lessons develop the concepts of equations, the open sentence, the truth set, and inequality. Set symbolism and language provide an orderly arrangement for recording the results of thinking. Negative numbers as well as operations with them are introduced through

[10] GCMP textbooks and teacher commentaries are distributed by Science Research Associates, Chicago, Illinois.

[11] Professor Suppes, Veterans' Hall, Stanford University, Stanford, California, has extended his primary programs to grade 6. His series, *Sets and Numbers*, was published by Random House (1965–1966).

[12] Professor Newton S. Hawley, Department of Mathematics, Stanford University, Stanford, California, is the director of the project.

[13] See Robert B. Davis, *Discovery in Mathematics—Madison Project* (Reading, Mass.: Addison-Wesley Publishing Company, 1964).

a variety of games and experiences. It is evident that this material is not to replace the conventional program in elementary school mathematics.

Several other centers are at work on the development of new types of curriculum content in elementary school mathematics, including the Minnesota Elementary Curriculum Project and the University of Illinois Arithmetic Project.

The following articles contain critical discussions of the new mathematics programs:

David P. Ausubel, "Some Psychological and Educational Limitations of Learning by Discovery," *The Arithmetic Teacher*, May 1966, 11:290–302.

John R. Clark, "Perspective in Programs of Elementary Mathematics," *The Arithmetic Teacher*, March 1965, 12:604–611.

C. G. Corle, "The New Mathematics," *The Arithmetic Teacher*, April 1964, 11:241–247.

H. F. Fehr, "Sense and Nonsense in a Modern Mathematics Program," *The Arithmetic Teacher*, February 1966, 13:83–92.

———, "Modern Mathematics and Good Pedagogy," *The Arithmetic Teacher*, March 1963, 10:402–411.

J. L. Marks, "The Uneven Progress of the Revolution in Elementary School Mathematics," *The Arithmetic Teacher*, December 1963, 10:474–478.

D. Rappaport, "Mathematics—Logical, Psychological, Pedagogical," *The Arithmetic Teacher*, February 1962, 9:67–70.

An appraisal of experimental programs

A few years ago the National Council of Teachers of Mathematics appointed a committee to evaluate different experimental programs.[14] This committee evaluated the programs in light of the following eight criteria:

1. *Social applications*. How much emphasis does the program give to social applications of mathematics?
2. *Placement*. At what grade level is a given topic presented?
3. *Structure*. What emphasis does a program give to structure to enable a pupil to understand mathematics?
4. *Vocabulary*. Is the vocabulary adapted to grade level and is the vocabulary part of accepted mathematical language?
5. *Methods*. What methods are used in presenting the material?
6. *Concepts or skills*. What is the relationship between the development of concepts and skills? When should each be developed?
7. *Proof*. At what level should proofs be introduced and how rigorous should they be?
8. *Evaluation*. Are there available measures that the teacher can apply to determine the effectiveness of a given program?

According to the report, most of the programs give a limited amount of consideration to the social applications of number and lack adequate means of evaluation. On the other hand, these programs seem to meet the other criteria described.

A significant point of view on goals for school mathematics

One of the most important examinations of goals in mathematics that has appeared in recent years is the bulletin *Goals for School Mathematics, The Report of the Cambridge Conference on*

[14] *An Analysis of New Mathematics Programs* (Washington, D.C.: The Council, 1963), p. 68.

School Mathematics.[15] This report was prepared by a representative group of professional mathematicians and public school educators and reflects the view of this group on what the future role of mathematics at all levels of the elementary school is likely to be. It contains a detailed analysis of goals for elementary school mathematics, grade K–2 and grades 3–6, that embodies many of the ideas that have been presented in the preceding pages as well as more advanced ideas.[16]

Stone has summarized the point of view of the Cambridge Report as follows:

> The grand goal proposed by the Committee Report is to compress the mathematical program so that what is now taught in twelve years of school plus three of college can be completed by the end of high school; that is, in twelve years. How do the authors propose to achieve this goal? The means proposed are essentially those which have been put forward by everyone who has seen the need for this kind of compression: the introduction of a great deal more mathematics into the elementary school program; better use of the opportunities for moving ahead in grades 7 and 8; a more or less drastic reevaluation of topics to be included in the curriculum; a more tightly and skillfully organized presentation of the essential elements of school mathematics; and finally, more stimulating and efficacious pedagogical methods aimed at developing important insights into the structure of mathematics as well as basic manipulative skills.[17]

In discussing the Cambridge Report, Allendoerfer pointed out that the report deals primarily with mathematics for students in the upper third of the class and he emphasized the necessity for a new formulation of the curriculum for students in the lower third.[18] However, he made no specific recommendations as to what the content should be. Much experimentation is needed along this line.

ALGEBRA AND GEOMETRY IN THE ELEMENTARY SCHOOL PROGRAM

Algebra and arithmetic

Until recent years the mathematics curriculum in grades 1–6 in American schools has devoted very little time to algebraic concepts, while European schools have introduced some of the basic ideas in this area as early as grades 4 and 5, especially in classes for the more able learners.

The modern approach is to teach mathematics as a whole rather than to present the subject in separate branches as arithmetic, algebra, and geometry. It has become increasingly difficult to determine when the pupil is dealing with algebra as opposed to arithmetic. He learns that number sentences or equations of the type $2 + 3 = 5$ and $2 + 3 = 3 + 2$ represent the subject matter of arithmetic. On the other hand, when he learns that the equation $a + b = b + a$ is true for all numbers, he is learning both arithmetic and algebra. In many

[15]Published for Educational Services, Inc., by Houghton Mifflin Company, Boston, 1963. See also a recent critical discussion by Irving Adler, "The Cambridge Conference Report: Blue Print or Fantasy," *The Arithmetic Teacher*, March 1966, 13:179–186. (Adler sees values and limitations in the report. He points out that many of the changes the report proposes have already been tried successfully.)

[16]See especially Section 5 of the report.

[17]Marshall Stone, "Review of the Goals of Mathematics: The Report of the Cambridge Conference on School Mathematics," *The Mathematics Teacher*, April 1965, 58:353–360.

[18]Carl B. Allendoerfer, "The Second Revolution in Mathematics," *The Mathematics Teacher*, December 1965, 58:690–695.

instances it is impossible to separate the subject matter of algebra from that of arithmetic; indeed, no effort should be made to separate the two. A distinctive feature in the beginning chapters of the ninth-year algebra prepared by the SMSG was the limited number of equations of the type $n - 5 = 8$. To the traditional teacher the introductory work appeared to involve more arithmetic than algebra.

From a modern point of view, both arithmetic and algebra involve the study of numbers systems and their properties. Algebra is more concerned with examining these properties through the use of expressions and sentences involving *variables*, such as $x^2 - y^2$ and $3x + 2 = 7x - 1$.

A number sentence of the type $\square + 1 = 5$ or $n + 2 = 6$ is called an *open-number sentence.* Open-number sentences are an integral part of mathematics in the elementary school. These sentences are not introduced merely as preparation for future work in algebra but rather as more effective ways of learning important ideas at every level of instruction. Open sentences are given throughout this text and provide a number of benefits:

1. Open sentences provide a fitting balance between vertical and horizontal patterns in arithmetic. Number statements in traditional arithmetic were almost entirely in vertical form, as shown at the right. This practice created an unnecessary and artificial barrier in the transition from arithmetic to algebra, since number statements in traditional algebra were nearly always expressed horizontally. The use of number sentences throughout the elementary program should help break down the barrier between arithmetic and algebra.

$$\begin{array}{r} 2 \\ +3 \\ \hline 5 \end{array}$$

2. Early introduction to the solution of open sentences provides useful additional variety in activities necessary to fix the basic facts.

3. The use of open sentences can help pupils learn important patterns on a nonverbal level more effectively than the old rule method (see p. 53).

4. Open sentences provide a major breakthrough in the treatment of verbal problems (see Chapter 17).

The work with formulas is an excellent illustration of the use of algebra in problem solving. Some of the simpler formulas with which children in the intermediate grades should become familiar are the following:

1. Cost = number of items × price of one, or $c = np$.
2. Distance = rate × time, or $d = rt$.
3. Perimeter of a square = 4 × length of one side, or $P = 4s$.
4. Perimeter of a rectangle = 2 × length + 2 × width, or $P = 2l + 2w$.
5. Perimeter of an equilateral triangle = 3 × length of one side, or $P = 3s$.
6. Area of a rectangle = length × width, or $A = lw$.

In the elementary school the study of signed or directed numbers should be largely limited to the number line and the thermometer. As will be shown, the number line can be used to demonstrate the meaning of positive and negative integers. The use of negative numbers in equations and in problem solving should be delayed until the upper grades.

Incorporating algebra and geometry into the elementary program

There are several ways of incorporating algebra and geometry into the elementary school mathematics program. One plan is to introduce elements of this subject matter into the work in

arithmetic whenever it seems feasible to do so. This procedure would result in the isolated, piecemeal treatment of the content of these two aspects of mathematics.[19] Another method is to introduce special units of algebra and geometry into the curriculum sequentially arranged and alternating with arithmetic units. This seems to be the SMSG plan. However, the units can be introduced at such times as the teacher may see fit. This plan is excellent for more capable learners. Another way is the general European plan, a parallel track plan. In some schools that follow this plan one class period a week is devoted to algebra and one to geometry. In Greece, geometry is taught as such for three hours a week in grades 4, 5, and 6.[20] Systematic arithmetic is taught during the remaining periods of the week. This plan practically amounts to having two or three separate courses in mathematics above the third grade.

American schools have been experimenting with these and various other plans with more or less satisfactory results. The goal should be a well-integrated program that provides a comprehensive, rounded, systematic organization of all aspects of mathematics from the kindergarten through the secondary school and beyond. A detailed analysis of the topics in geometry included in experimental courses of study is presented in Chapter 18.

The following references are helpful on the subject of the role of algebra in the elementary school:

W. L. Bradfield, "Algebraic Concepts for Elementary School," *The Arithmetic Teacher*, March 1965, 12:183–186.

Edwina C. Deans, "Algebraic Approaches to Developmental Work with the Operations," *The Arithmetic Teacher*, April 1964, 11:266–267.

Elizabeth King, "Greater Flexibility in Abstract Thinking through Frame Arithmetic," *The Arithmetic Teacher*, April 1963, 10:183–187.

Rachael A. La Roe, "Algebraic Concepts in the Elementary School," *The Arithmetic Teacher*, March 1965, 12:181–183.

Lola B. May, "Three Problems of Using Equations in Elementary Arithmetic Programs," *The Arithmetic Teacher*, March 1964, 10:166–168.

David Page, *Number Lines, Functions, and Fundamental Topics*. New York: Crowell-Collier and Macmillan, Inc., 1964.

[19] I. H. Brune, "Some K–6 Geometry," *The Arithmetic Teacher*, October 1967, 14:441–447; G. Edith Robinson, "The Role of Geometry in Elementary School Mathematics," *The Arithmetic Teacher*, January 1966, 13:3–10.

[20] G. H. Miller, "Geometry in the Elementary Grades: A Comparative Study of Greek Mathematics Education," *The Arithmetic Teacher*, February 1964, 11:85–88.

EXERCISES

1. Why is the scope of the elementary mathematics program being extended today?
2. Why should mathematics be taught in connection with its applications?
3. How are the mathematics programs of your local elementary schools being modified to meet present trends?
4. Do you regard the principles of subject-matter selection on pages 9–11 as valid?
5. How should the grade placement of

the content of the elementary mathematics program be determined?

6. What is the significance for instruction of the data in Table 2.1 concerning the difficulty of selected types of examples?
7. What are the contributions to curriculum-making of the various experimental programs discussed in this chapter?
8. At what grade level should algebraic concepts be introduced? How should this be done?
9. What topics in geometry do you think should be included in the work at the various grade levels?
10. The reference cited on page 19 deals with the evaluation of experimental programs. Have a member of the class report on the evaluation of one of these programs, such as SMSG.

SELECTED READINGS

An Analysis of New Mathematics Programs. Washington, D.C.: The National Council of Teachers of Mathematics, 1963. A 68-page report.

The Changing American School, Sixty-fifth Yearbook of the National Society for the Study of Education. Part II. Chicago: University of Chicago Press, 1966. Chapters 2 and 11.

Child Development and the Curriculum, Thirty-eighth Yearbook of the National Society for the Study of Education. Bloomington, Ill.: Public School Publishing Company, 1939. Chapters 15 and 16.

Goals for School Mathematics: The Report of the Cambridge Conference on School Mathematics. Boston: Houghton Mifflin Company, 1963.

Hogben, L. *Mathematics in the Making.* New York: Doubleday & Company, Inc., 1960.

Instruction in Arithmetic, Twenty-fifth Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1960. Chapters 2 and 12.

Kline, Morris *Mathematics—A Cultural Approach.* Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1962.

———, *Mathematics in Western Culture.* New York: Oxford University Press, 1953.

Lee, J. Murray and Doris M., *The Child and His Curriculum.* New York: Appleton-Century-Crofts, 1960. Chapter 11.

Saylor, G. G., and W. Alexander, *Curriculum Planning in Modern Schools.* New York: Holt, Rinehart and Winston, Inc., 1965. Chapter 9.

PRINCIPLES OF TEACHING AND LEARNING MATHEMATICS

The elementary school mathematics program encompasses three interdependent aspects of the subject—arithmetic, algebra, and geometry. The teacher faces the problem of presenting these areas in such a way that the interrelationships among them become increasingly evident to students as they progress through school. A number of basic ideas and concepts are common to all these areas, for example, sets, place value, and the properties underlying operational procedures. The relationships systematically developed and utilized among these properties will give meaning and unity to the whole program. In the traditional drill program the tendency was to teach masses of isolated number facts and computational steps as discrete, unrelated elements. With this approach students acquired a body of learning that lacked the basic coherence that the modern mathematics program seeks to build into the subject.

This chapter will consider the following topics: three aspects of any learning situation; modern principles of teaching mathematics; application of modern principles; unit teaching in mathematics.

THREE ASPECTS OF ANY LEARNING SITUATION

There are three closely knit, interrelated aspects of any learning experience that should be considered by the teacher:

1. What is to be learned
2. The search for structure (pattern)
3. The application of elements of structure in new situations.

What is to be learned

In mathematics there are many kinds of information, meanings, mathematical procedures, understandings, appreciations, and interests that students should acquire. Some of them are acquired in the home before the child enters school, for example, counting. The simpler elements of some ideas that function in all number operations are introduced in the primary grades, such as the regrouping procedure used in addition of whole numbers. This procedure is encountered again and again in a spiral fashion in operations with rational numbers (fractions, $\frac{1}{2}$ or .5) and measures. The idea of regrouping is thus one of many means of developing increased mathematical power and insight. If the teacher ignores such basic relationships, learning is not organized as it should be to insure understanding.

The search for structure

The mere acquisition of knowledge is not sufficient for effective learning in mathematics. The pupil must be helped to see how the new material presented is structured, or how it fits into the subject as a whole. There are many patterns in mathematics. Taken together, they form the structure of the subject. When the student discovers a pattern, it gives organization and structure to the new learnings. He can then apply the pattern he has discovered to further his learning.

To illustrate the discovery of patterns, let us consider the sets A and B.

A	B
$4+5=9$	$1+0=0+1=1$
$5+4=9$	$2+0=0+2=2$
$9-5=4$	$3+0=0+3=3$
$9-4=5$	$4+0=0+4=4$

The teacher lists the four related facts given under set A and asks the children to discuss what they notice about them. The class will quickly discover that the four facts are related to the set of three numbers $\{4, 5, 9\}$. To extend this notion, the teacher then writes the set $\{3, 2, 5\}$ on the chalkboard and asks, "What number facts does this set of numbers suggest?" Thus the children quickly learn that a set of related numbers suggests a group of four related facts, an important pattern of thinking.

In a similar way, using set B, the pupils can discover the addition fact that when we add a number to 0 or 0 to a number the sum is the same as the number. This generalization applies to all such zero number facts in addition, and it leads directly to the discovery of the identity element for addition. There are hundreds of such patterns in mathematics. A pattern at a higher level is the concept underlying regrouping in a place in addition. The pattern when once understood can be applied to addition of any kind of number. The idea of regrouping thus serves as a strand that reconstructs at higher levels the process of addition.

The important role of structure in learning has been well expressed by Bruner as follows:

> The curriculum of a subject should be determined by the most fundamental understanding that can be achieved of the

underlying principles that give structure to the subject. Teaching specific topics or skills without making clear their context in the broad fundamental structure of a field of knowledge is uneconomical in several deep senses. In the first place, such teaching makes it difficult for the student to generalize from what he has learned to what he will encounter later. In the second place, learning that has fallen short of the grasp of general principles has little reward in terms of intellectual excitement. The best way to create interest in a subject is to render it worth knowing, which means to make the knowledge gained usable in one's thinking beyond the situation in which the learning was secured. Third, knowledge has been acquired without sufficient structure to tie it together. Knowledge is likely to be forgotten as an unconnected set of facts has a pitiably short half-life in memory. Organizing facts in terms of principles and ideas from which they may be inferred is the only known way of reducing the quick rate of loss of human memory. Designing curricula in a way that reflects the basic structure of a field of knowledge requires the most fundamental understanding in that field. It is a task that cannot be carried out without the active participation of the ablest scholars and scientists.[1]

The application of elements of structure in new situations

The third aspect of a learning situation that must be considered is the application of the pattern to new situations. The addition generalization growing out of set *A* can readily be applied to other such sets of related numbers. In a modified form, the addition generalization applies to a set of four multiplication and division facts based on such a related set of numbers as {3, 5, 15}. The identity elements for multiplication can also be developed in a way similar to the procedures used with set *B*. Problem solving consists in searching for elements in a new situation that may be similar or identical to elements with which the pupil is already familiar.

It should be pointed out that the three aspects of a learning situation discussed above are not separate phases of a lesson. Rather, they merge and are interrelated, although it is not possible to tell just when one particular aspect merges with another. One learner may suddenly and intuitively discover an idea while the other students do not. Slower learners can be led to see a generalization by a skillful teacher, although they may not be able to express the idea in words, since verbalizing an idea represents a high level of thinking. The teacher may be unaware of these three aspects of a learning situation and may assume that children make discoveries of procedures instinctively and also apply new learning to new situations. This point of view is faulty and leads to incomplete and meaningless learning. When an understanding of structure is carefully developed, retention of what is learned usually is assured.

MODERN PRINCIPLES OF TEACHING MATHEMATICS

Contrasting theories of learning

There was a time when the mind was regarded as a reservoir into which knowledge could be poured. This view has generally been discarded. Another theory saw the mind as a muscle that could be developed and strengthened by exercising it through the study of difficult, uninteresting facts and skills. This theory is also out of line with current thinking. A third theory, the so-called associationist point of view, took the position that the best way to assure

[1]Jerome S. Bruner, *The Process of Education* (Cambridge, Mass.: Harvard University Press, 1961), pp. 31–32.

the learning of computational procedures was through long periods of intensive, repetitive drill. Little if any consideration was given to whether or not the drill was meaningful to the learner or whether he understood what he was practicing. Unfortunately the drill theory still determines practices in many classrooms.

The authors believe that the child should understand what he is learning, that what he is learning should be mathematically meaningful to him, and that the process of learning is even more important than the end product. The structure of mathematics should be emphasized at all levels of the school, and no child should be assigned repetitive practice to develop efficiency in computational skills until the teacher is sure that he understands the operations to be practiced.

APPLICATION OF MODERN PRINCIPLES

The procedures used by the teacher in the following description of a new topic in addition exemplify the principles of learning that the authors endorse. These principles will be presented later in this chapter.

An illustrative lesson

The teacher's goal was to have the class learn how to deal effectively with two addends in which the sum in ones' place must be regrouped, as in the example at the right. The three aspects of any learning situation described before can be illustrated by demonstrating how the teacher led the class to discover the pattern of the procedure for solving an example of this type. The numbers named[1] represent the number of tickets sold by the boys and the girls of the class for a school play. The problem was to find the number of tickets sold by both groups. The teacher capitalized on this situation to introduce addition involving regrouping in the sum. First the teacher provided a quick review of three addition examples in which no regrouping in ones' place was necessary. The lesson then proceeded as follows:

$$\begin{array}{r} 34 \\ +\ 48 \\ \hline \end{array}$$

Teacher

1. What is the sum in the ones' place in the example?

$$\begin{array}{r} 34 \\ +\ 48 \\ \hline \end{array}$$

2. What was the greatest sum in ones' place in the three examples we added before?

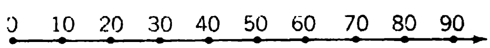
3. Now we must learn to deal with a sum when the sum in ones' place is 10 or more. We can say that the ones' place in the example is *overloaded*. Someone suggest a way to find the sum of 34 and 48.

Pupils

1. We can use our markers to represent the numbers and then find the sum.

2. We can think: $48 + 10 = 58$; $58 + 10 = 68$; $68 + 10 = 78$; $78 + 4 = 82$.

3. We can find the sum by using a number ray.[2]



4. We can add the tens, then add the ones, and then add the two sums: $34 + 48 = (30 + 40) + (4 + 8) = 70 + 12 = 82$.

Teacher

1. Let us use our squares and strips to find the sum.

[2] The usual name is "number line," but "ray" is more precise and is a term the pupil needs to know. A number ray has one endpoint, but a number line has no endpoints.

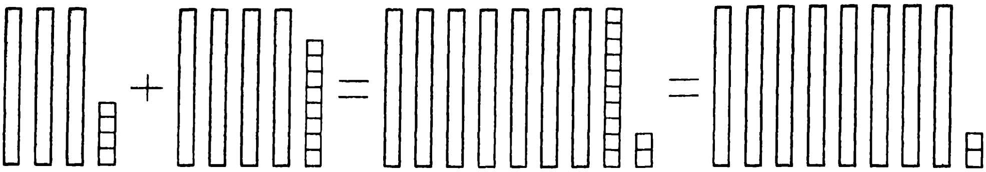


Figure 3.1

Each pupil found the sum by using his materials, as shown in Figure 3.1.

2. There are how many ones in ones' place?

3. How do we regroup the cards in ones' place? Tell why?

4. Let us use our place-value chart to find the sum (see Fig. 3.2). The teacher had a pupil place the cards in the respective pockets as the class described the steps to follow.

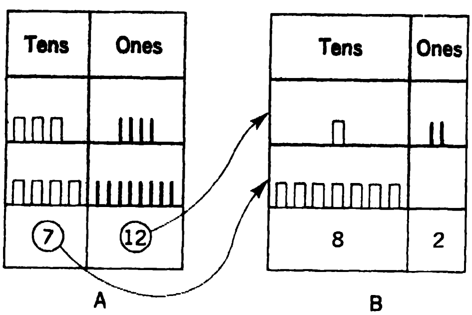


Figure 3.2

Which place in the sum is overloaded?

How do we regroup the markers in ones' place in the sum in (A)?

5. Let us write the numerals in expanded form and then add.

$$\begin{array}{rl} 34 = & 30 + 4 \\ +48 = & 40 + 8 \\ \hline & 70 + 12 = 70 + (10 + 2) \\ & \qquad = (70 + 10) + 2 = 82 \end{array}$$

The teacher wrote the numerals on the chalkboard, but the class gave the sequence of steps. The class first stated that 34 = 30 + 4 and then gave each succeeding step as shown. The class identified the use of the associative property in the grouping 70 + (10 + 2).

6. The class added the ones and then the tens. The teacher wrote each sum as shown. The class then added the two sums, as no regrouping was involved. The procedure in this solution stressed the place value of the numerals.

$$\begin{array}{r} 34 \\ +48 \\ \hline 12 \\ 70 \\ \hline 82 \end{array}$$

7. In the situation at the right written on the chalkboard the teacher had the class tell what to think. "First, the sum in ones' place is 12 ones, which is the same as 1 ten and 2 ones." The teacher wrote the 2 ones in ones' place and then wrote the 1 ten in tens' place as shown and added the tens. A few lessons later when dealing with regrouping the sum, the teacher directed the class to find the sum without writing the 1 ten.

$$\begin{array}{r} {}^{1}34 \\ +48 \\ \hline 82 \end{array}$$

The pupil who understands the work just outlined should learn to do the following problem much more readily.

$$\begin{array}{r} 3\frac{1}{2} \\ +2\frac{3}{4} \\ \hline \end{array} \qquad \begin{array}{rl} 3 + \frac{2}{4} & \\ 2 + \frac{3}{4} & \\ \hline 5 + \frac{5}{4} & = 5 + (1 + \frac{1}{4}) \\ & = (5 + 1) + \frac{1}{4} \\ & = 6\frac{1}{4} \end{array}$$

8. Now the teacher wished to stress the new element in the learning situation. This new learning pertained to the regrouping of a number in a different overloaded place in the sum.

When is the sum in ones' place overloaded?

When this place is overloaded, what do we do with the number named?

Let us add in the example I shall write on the chalkboard.

What is the sum in ones' place? Is this place overloaded?

What is the sum in the tens' place? Is this place overloaded? How can you tell? What must we do with this number of tens?

$$\begin{array}{r} {\scriptstyle 1} \\ 384 \\ +142 \\ \hline 526 \end{array}$$

What is the sum in the hundreds' place? Is this place overloaded?

9. Finally, the teacher had the class turn to the page in the textbook that presented the work dealing with regrouping in ones' place in the sum. The pupils now had the necessary background to read and understand the formal textbook presentation. The pretextbook activities were directed at giving experiences that assured an understanding of the printed page. The class then copied a set of four examples involving regrouping in work from the textbook and found the sums. The teacher quickly looked over the written work to locate and correct possible errors.

The above lesson illustrates the three aspects of a new learning situation. First, the pupils acquired the new knowledge in the situation. In this case it pertained to a difficulty that arises when a place is overloaded in the sum. Second, with the help of the teacher the pupils discovered a way to deal with an overloaded place in the sum. The pattern to follow is to regroup in ones' place in the two-place numeral. In this illustration, the regrouped numeral 12 is named as 1 ten and 2 ones. We write the ones in ones' place and add the 1 ten to the number of tens in tens' place. Finally, the teacher had the class apply this pattern in a new situation, as in regrouping in tens' place when the sum is greater than 10. Some teachers may wish to postpone the last activity until a later lesson because of time limitations. Regardless when it is presented, the same plan as described should be followed when regrouping is in tens' place. As soon as a pupil discovers the pattern for regrouping when a place in the sum is overloaded, he understands and actually knows how to add any two whole numbers regardless of the number of places in the numeral. The teacher was interested in having the pupil understand the pattern for regrouping rather than in having him learn a specific technique for "carrying one" to the next column. The latter technique typifies rote learning in dealing with two addends. It adds little knowledge of the structure of mathematics to what is being learned.

Principles of learning implicit in the lesson

Certain principles of learning underlie the teaching situation described in the illustrative lesson. We shall identify and discuss six of these principles, as follows:

1. *Learning should be purposeful and goal centered.* The need for learning the new step in addition arose in a realistic situation. The class wanted to know how many tickets in all the two groups sold. The problem situation created the social setting for the new learning. The problem per se was not the important element in the lesson. The class could find the answer to the problem in a variety of ways, but not by the use of the conventional algorism of addition already known. The teacher skillfully led the class to see that the ones' place in the sum was overloaded. Therefore, the new learning consisted in finding a way

to use the algorism for addition when a place in the sum is overloaded. The objective was clear to *both teacher and pupil.* The specific activities could be performed to deal with the new learning involved in the given problem.

2. *The discovery of facts, meanings, and procedures leads to insight and understanding so that the learner can generalize about a particular situation or set.* Every new learning presents a situation in which self-discovery is possible. The teacher could have told the class how to deal with a place that is overloaded and then given practice in applying that technique or procedure. But when the pupil *discovers for himself,* as the class did, he solves the problem encountered because he understands what he is doing. He draws upon his background to interpret the new situation. Discovery should not be confused with creativeness. In the given lesson the pupil discovers how to regroup, but most certainly he does not invent or create the technique of regrouping. The pupil should discover how the numeration system operates.

After the pupil discovers how to regroup in a place in a numeral, he should be able to make a statement about the procedure to follow. We call such a statement a *generalization.* A generalization is a statement that is true for every element of a set.

In case of a place in a sum, that place is overloaded when the number named in that place cannot be expressed by one of the ten digits in our system of numeration. The numeral in a place that is overloaded must be regrouped. In the given lesson, the 12 ones were regrouped as 1 ten and 2 ones.

Kerch described two different teaching procedures in the use of discovery in learning.[3] One involves having the teacher give hints concerning the new learning. The teacher supplies instructions that lead to answer giving. In the other plan the teacher does not give hints concerning the new principle or generalization involved, but instead suggests alternative plans of procedure. The pupil then selects the plan that seems best suited to the situation. The first plan is effective for teaching a specific principle or generalization, while the second is useful for teaching a technique for identifying principles or generalizations. The second procedure is preferable because the pupil learns to discover a pattern for identifying principles.

3. *The use of a wide variety of learning experiences and instructional materials adapted to the learner's level of development extends and enriches meanings and background.* Learning takes place most readily and effectively when what is being learned is experienced in vivid, realistic, and meaningful situations. The learning experiences used in the lesson on addition varied in degree of concreteness. The class used strips and squares of oaktag to represent tens and ones, markers in a place-value chart to represent numbers, numerals expressed in different notations, and finally the written explanation given in the textbook. Each new kind of material used represented a different level of abstraction. The use of such a wide variety of learning experiences enabled the teacher to adapt instruction to differences in the ways and rates at which the class learned. Emphasis was directed to one particular new element in the learning situation, namely, how to deal with a place that is overloaded. The

[3]Bert Y. Kerch, "Learning by Discovery. Instructional Strategies," *The Arithmetic Teacher,* October 1965, 12:414–417. See also Ernest R. Ranucci, "Discovery in Mathematics," *The Arithmetic Teacher,* January 1965, 12:14–18.

different activities that the class performed all emphasized and clarified the new learning.

4. *Learning is a growth process from an immature level of dealing with numbers to an adult level of operation with them.* A pupil frequently can learn important patterns on a nonverbal basis and apply them effectively before being able to state or understand them on a verbal basis.

Piaget has identified five stages of learning. They vary from individual to individual, and the stages gradually merge with increasing age. They may be described briefly as follows:

a. The stage when "sensory motor intelligence" appears, from birth to two years. The child performs motor actions and experiences through which he learns. Language appears.

b. The period of "preconceptual" thoughts, from two to four years, during which stage the child reasons from the particular to the particular but does not generalize.

c. From four to seven years there follows a period of "intuitive" thought, when thought is not freed from perception. Intuitive thought may come to the child suddenly. He cannot explain how he arrived at the thought and is not able to verbalize it. He cannot derive any general principles.

d. From seven to eleven years the period is one of "concrete operations," during which the reasoning processes are logical but are not altogether dissociated from the concrete data. The capacity for complete conceptual generality has not yet been obtained.

e. In the final period, from eleven to fifteen years and beyond, the stage of "the propositional or formal operations" is reached. The child attains complete conceptual generality and achieves the capacity for hypothetic-deductive reasoning. As Piaget says, "Instead of just coordinating facts about the actual world, hypothetical-deductive reasoning draws out the implications of possible statements and thus gives rise to a unique synthesis of the possible and the necessary."[4] In a word, the child can readily generalize and sense relationships.

Piaget points out that the transition from one stage of learning to the next can be accelerated or retarded by the kind of teaching we do.

Learning as a growth process is closely related to learning that results from a variety of experiences. There are different levels of abstraction in learning a process. In the lesson on addition, the teacher arranged a series of exploratory experiences that provided a meaningful background for the new learning. The gradual sequence of activities through the use of exploratory and visual aids made it certain that even the slowest learners would understand the work. It is quite probable that the more able learners would have understood the new step without the use of some of the concrete materials in the sequence of activities, but even in their case, richness of meanings resulted from the variety of experiences.

A distinction should be made between the process of learning and the end product. The process of learning a basic operation, such as addition, involves a developmental sequence of behavior in which more mature methods of reacting replace less mature methods, thus gradually leading to increased understandings and proficiency in dealing

[4]Jean Piaget, *Logic and Psychology* (New York: Basic Books, 1957), p. 19. See also Piaget's "How Children Form Mathematical Concepts," in R. C. Anderson and D. Ausubel, *Readings in the Psychology of Cognition* (New York. Holt, Rinehart, and Winston, Inc., 1965), p. 409.

with that operation. The steps leading to complete mastery of the addition operation involved a wide variety of experiences. The mastery of the operation may be regarded as the end product in the learning situation. Through these experiences the pupil gradually learns the basic facts in addition, the meaning of the operation, the basic properties that govern addition, as well as the ability to add with skill and competence. Understanding of the properties of addition and skill in computation should increase from grade to grade.

In a drill program that stresses skill in computation as the end product, teaching procedures are quite different from those described in the model lesson. In a drill program the initial teaching of a topic presents the mature mental processes that are used at the adult level of operation. Usually the teacher employs the telling technique in presenting a new step, hence rote learning is emphasized and the pupil often does not understand the work. Practice of the operations results in mechanical skill involving the given operation or procedure.

5. *The new learning should be structured so that it serves as a pattern useful for further learnings.* Structure implies that the material is so arranged that it will fit into a recognizable pattern. When a pupil understands the structure of a subject, he learns how the different elements of that subject are related. A basic rule or property may govern or characterize the new learning. Structure does not consist in learning a given skill or fact. Instead, it involves the recognition of a pattern or principle that can be applied in similar situations. In the model lesson the principle governing regrouping of an overloaded place in ones' place applies to regrouping of an overloaded place in any place in the sum. The pupil understands the meaning of regrouping in a specific place so that he can transfer that technique to all situations of an overloaded place in addition. According to Bruner, "teaching and learning of structure rather than simply the mastery of facts and techniques is at the center of the classic problem of transfer."[5]

Grasping the structure of a subject enables the pupil to discover how the new learning can become an integral part of his background in that subject. Discovery, structure, and generalization are interrelated in a meaningful learning situation. A chain reaction takes place in the learner's thinking when he applies these catalysts to derive meaning in a new situation.

The teacher should bear in mind that the essential part of the learning situation pertaining to structure consists in the pupil's discovery of that structure. Piaget made a very insightful remark with regard to teaching structure:

> The question comes up whether to teach the structure, or to present the child with situations where he is active and creates the structure himself. . . . The goal in education is not to increase the amount of knowledge, but to create the possibilities for a child to invent and discover. When we teach too fast, we keep the child from inventing and discovering himself. . . . Teaching means creating situations where structure can be discovered; it does not mean transmitting structure which may be assimilated at nothing other than a verbal level."[6]

The pupil should discover a pattern of procedure before he generalizes about it. For example, he should be able to find a pattern for renaming fractional numerals before he can make a mean-

[5]Bruner, p. 12.

[6]*Piaget Rediscovered* (Ithaca, N.Y.: Cornell University Press, 1964), p. 3.

ingful statement about the procedure. The renaming procedure for $\frac{3}{4}$ is as follows:

$$\frac{3}{4} \times \frac{2}{2} = \frac{3 \times 2}{4 \times 2} = \frac{6}{8}$$

$$\frac{3}{4} \times \frac{3}{3} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$$

The pupil can readily discover that both terms of the fractional numeral $\frac{3}{4}$ are multipled by the same number without changing the fractional number. He can discover this pattern more readily than he can verbalize the procedure. In a program that emphasizes drill and not understanding, the pupil is given a verbal statement pertaining to the procedure to follow in order to rename a fraction in higher terms. He then tries to interpret the rule, which often results in mere manipulation of numerals.

After the pupil discovers the pattern, he shows growth in dealing with the process by recognizing that the renaming of fractional numbers illustrates an application of the identity element of multiplication. The identity element of 1 is a basic property of multiplication (see p. 81). Therefore there is growth in maturity of dealing with a given topic as a pupil's knowledge progresses from the discovery of a pattern of how numbers behave to the recognition of the basic property involved.

6. *Necessary practice to develop control and proficiency of skills should not be assigned until the learner understands what he is to practice.* Practice for the fixation of learning and the acquisition of skill is an integral part of a learning program. The important element is the time and amount of practice. In the model lesson the teacher did not assign practice exercises to fix the procedure for regrouping until the class understood the procedures involved. When the learner understands the regrouping of an overloaded place in a sum and how to proceed, we say that he understands how to perform the algorism in addition. However, he must practice the step systematically in order to develop reasonable skill and proficiency in that operation. With practice and usage, the pupil progresses from the slow and cumbersome methods of responses that are characteristic of immaturity to smooth, rapid procedures and thought patterns approaching those used by adults.

Traditionally the word "drill" has been used to identify the repetitive procedure used in a program that looks upon learning as a mechanical process. The term "practice," however, implies a different conception of the nature of learning and of the function of repetitive procedures. Organized practice exercises are essential if in addition to understanding it is necessary to develop structure and a mature level of dealing with a given operation. Learning is essentially complete when the student understands the basic principles of what he has learned. This is the time to provide for systematic practice to develop an acceptable level of performance and proficiency.

Adler summarized 12 of the guiding principles underlying the Cambridge Report as follows:

> 1. Beginning with the earliest grades there should be a parallel and integrated development of algebra and geometry.
> 2. Teach for understanding, not merely for manipulative skill.
> 3. The first approach to each topic should be intuitive. Use many approaches to illuminate the topics from many sides. Provide experience in the manipulation of physical objects as the basis for abstract learning.
> 4. Pay serious attention to the development of suitable problem material. In

particular, provide the children with problems that give them opportunities to explore and make discoveries that are within their reach.

5. Replace drill for drill's sake by the use of past learnings in new, meaningful situations.

6. Use the spiral approach, in which the same subject arises at different times with increasing degrees of complexity and rigor.

7. Make fuller use of the historical background of a topic to develop an appreciation of how it arose and why it is studied.

8. Many significant mathematical topics can be approached through exciting games, tricks or puzzles. Exploit the recreational aspects of mathematics, especially in the lower grades.

9. Use supplementary pamphlets for individual work by the student who is ready to pursue a topic more broadly and deeply.

10. Show how mathematics is applied in the physical sciences or to other studies of the real world. But keep in mind that many important applications are internal, that is, they are applications to mathematics itself. . . .

11. Aim to develop a growing awareness of the nature of logical reasoning. . . .

12. In the development of postulational thinking avoid excessive delicacy and austerity. If proofs are too long and seem to be only laborious ways of arriving at what seems obvious to the student the deductive method is not likely to look either attractive or powerful.[7]

Levels of maturity in learning

In most elementary school mathematics classrooms all children in the class work on the same topic or concept at the same time. Teachers, however, have discovered how to make provisions for different levels of learning maturity among their students.

The teacher must recognize the fact that children do not all learn in the same way and that they almost always learn on several levels of maturity. For example, some of the more mature learners in a class will understand the steps in the algorism for the subtraction examples shown in wholly symbolic, abstract form in (a). Others would need diagrams or pictures to help them to visualize the transformation of $3\frac{1}{4}$ to $2\frac{5}{4}$.

a
$$\begin{array}{r} 3\frac{1}{4} = 2\frac{5}{4} \\ \underline{-1\frac{3}{4}} = \underline{-1\frac{3}{4}} \\ 1\frac{2}{4} = 1\frac{1}{2} \end{array}$$

b
$$\begin{array}{rl} 3\frac{1}{4} = 3 + \frac{1}{4} = & 2 + 1 + \frac{1}{4} \\ \underline{-1\frac{3}{4}} = \underline{-1 + \frac{3}{4}} = & \underline{-1 + \frac{3}{4}} \end{array}$$

$$\begin{array}{rl} = 2 + \frac{4}{4} + \frac{1}{4} = & 2 + \frac{5}{4} \\ \underline{-1 + \phantom{\frac{4}{4} +} \frac{3}{4}} = & \underline{-1 + \frac{3}{4}} \\ & 1 + \frac{2}{4} = 1\frac{1}{2} \end{array}$$

Some students would need the detailed steps shown in (b) to see the transformation. Still others whose learning is at a low level of maturity would find it necessary to work with exploratory materials such as fractional parts in a fraction kit before they can perceive the transformation shown in the example.

Obviously the first of these three procedures represents the greatest maturity in thinking and indicates ability to operate at a high level of abstract reasoning. At a somewhat lower level the use of visual and pictorial aids is clearly a less mature procedure than the first. The lowest level of learning is characterized by a need for concrete experiences with manipulative and exploratory materials.

Ordinarily in mixed classes all levels of learning maturity will be found. For this reason the teacher in presenting a new step in a process should initially include experiences at each level of learning, as was done by the teacher of the illustrative lesson in addition. This procedure will enrich the learning for all

[7] Irving Adler, "The Cambridge Report: Blue Print or Fantasy," *The Arithmetic Teacher*, March 1966, 13:179–186.

children. After the initial presentation the teacher should divide the children into groups according to evident differences in levels of learning and thinking and then proceed according to the needs of the pupils.[8] At least two groups should be formed, one made up of those who can proceed with the abstract work and are ready for systematic practice, and the other consisting of those who still need help with visual aids. Some of the slower children may even need further experiences with manipulative materials.

The teacher faces the problem of adjusting methods and materials to differences in the learning levels so that each pupil is challenged to proceed at his highest level of thinking.

Although a variety of instructional materials is of value for meeting different abilities of the class, the teacher should not neglect the nonverbal approach. Page 31 pointed out that a pupil is able to recognize a pattern of procedure more readily than he can verbalize it. The teacher must be aware, however, that the pupil's ability to recognize and reproduce patterns does not represent a mature level of understanding of a topic.

UNIT TEACHING IN MATHEMATICS

Kinds of units

There are two kinds of units dealing with mathematics in the elementary school. The first type involves areas of mathematics, such as the addition or subtraction of fractional numbers. Most modern textbooks handle this subject by breaking the topic into subunits, such as the addition of two fractional numbers named by fractions having like denominators, as in the example $\frac{1}{5}+\frac{3}{5}$. Another subunit in this field may include fractional numbers named by fractions in which one denominator is a common denominator, as in the example $\frac{1}{2}+\frac{3}{8}$. Finally, there may be another subunit in which the denominators of the fractions have no common factor, as in the example $\frac{2}{3}+\frac{3}{4}$. The pupil should discover a pattern of procedure that will enable him to add or subtract any two fractional numbers regardless of the type of denominator. He reaches this level of maturity by first learning to add simpler type of fractional numbers and progressing successively to the more complex types as expressed by their denominators.

The second type of unit deals with a topic that is not restricted to mathematics. The informational phase of a topic in mathematics, such as the story of measures, provides a good illustration of this kind of unit. Many teachers think of a unit as a learning experience that is not limited to one subject area. We shall designate a unit of this kind as an enriched unit of experience.

Enriched units of mathematics

Modern educational theory stresses that instruction should be organized so that pupils not only learn subject matter and skills of significance but also acquire patterns of thought and effective habits of thinking. The core of the mathematics program is the unit of subject matter into which the logical sequential development of number, number operations, and geometry is organized. The structure of mathematics must be systematically developed in this program. In textbooks this development is paralleled by subject matter dealing with the applications of these number processes,

[8] Harold H. Leich, "Intra-Class Grouping for Arithmetic Instruction—Critique and Criteria," *The Arithmetic Teacher*, December 1961, 8:404–407.

including problems involving measures, groups of word problems involving the use of mathematics in other curriculum areas, and number situations based on the affairs of daily life. In most of this work emphasis is placed on the mastery of subject matter that may be more or less vital and interesting to children. In more recent books special attention is given to the development of effective techniques of thinking, including problem solving, the discovery of patterns of thinking, the perception of relations between the elements given in a problem, the development of generalizations, and methods of securing, organizing, and evaluating information about some topic or problem. (See Chapter 17 for more detailed information).

Enriched units of experience of the greatest value grow out of the cooperative organization by the teacher and pupils of a plan of attack on some problem or topic that is of significance to the children and that, in the judgment of the teacher, is likely to lead to valuable kinds of group and individual learning. An enriched unit of work consists of a series of learning experiences that is focused on the achievement of a common goal which the children have accepted as their own. It cuts across subject lines and thus makes subject matter more meaningful. Such a learning experience makes the relationships among the various curriculum areas more apparent. A wisely chosen unit of work offers rich and vital experiences in which the children use subject matter and skills in a functional way.

With units dealing with the applications of mathematics, such as telling time, reading a thermometer, counting money, weighing things, figuring taxes, and the like, the whole class can work together. However, the teacher can adjust the various activities of the class to the differences in the ability levels of the children. The more able pupils can be given special assignments on some aspect of the topic that require research and the preparation of reports to be given to the class as a whole. In this way they are also given training in leadership and they experience the satisfaction that grows out of worthwhile, unselfish contributions to their associates, including those at the lower levels of ability.

According to Ragan, the essential features of such group experiences are as follows:

> 1. Learning takes place through many types of experiences rather than through a single activity such as reading and reciting.
> 2. The activities are unified around a central theme, problem, or purpose.
> 3. The unit provides opportunities for the socialization of pupils by means of cooperative group planning.
> 4. The role of the teacher is that of a leader rather than that of a taskmaster.[9]

The success of a unit of work depends in large measure upon the ability of the teacher to (1) create an interest in the unit, (2) help the pupils to see the significance of the unit, (3) relate the unit to past experiences of the pupils, (4) utilize the resources of the local community in orienting the children to the problem, and (5) provide a classroom environment that stimulates interest in the unit. Modern courses of study contain many illustrations of enriched units found to be of value in the classroom. The teacher should utilize these units to enrich learning experiences.

[9] W. B. Ragan, *Modern Elementary Curriculum*, rev. ed. (New York: Holt, Rinehart and Winston, Inc., 1960), pp. 150–151.

EXERCISES

1. Why should the arithmetic classroom be regarded as a learning laboratory?
2. Under what circumstances is the use of concrete materials justified?
3. Why is the process of learning arithmetic so important?
4. Illustrate by concrete examples th· six principles of learning discussed in this chapter. You may use some item contained in the description of the lesson (p. 27) or some other example.
5. How would you apply these same principles to the teaching of the new step in subtraction of fractions in the example $3 - 1\frac{1}{2}$?
6. Why is an intensive drill program of the traditional type defective?
7. Discuss the implications for teaching of the principles underlying the organization of practice (p. 33).
8. What is meant by levels of maturity in learning?
9. How can a teacher decide when to introduce enriched units of mathematics into the work of the classroom?
10. Some teachers maintain that enriched units of experience are not effective for teaching mathematics because of the structure of the subject. Evaluate this statement.

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TEACHING ELEMENTARY SCHOOL MATHEMATICS

SETS AND SENTENCES

The set concept is one of the most basic and universal ideas in mathematics. While advanced set theory is difficult and complex, basic set ideas are simple and may be understood on the most elementary level. It is now generally recognized that these ideas are useful in simplifying and clarifying many fundamental concepts in arithmetic, algebra, and geometry. While there is controversy about how much set language and notation should be used at various levels, it is generally agreed that many set ideas are valuable in the teaching of elementary mathematics. Elementary school teachers should therefore become familiar with the simplest set ideas and with the ways in which these ideas may be effectively used.

"Sets of books," "sets of dishes," and many similar phrases occur naturally in everyday speech. These examples illustrate that the conversational use of the word "set" differs little from the use of the term in elementary mathematics. A *set* is a collection of things. Things that belong to sets are called *members* or *elements* of those sets. Members of the same set need not have any property in common other than belonging to that set. A set may contain the planet Venus, the concept of abstractness, and an elephant, although such a set is not likely to have much value or importance. In elementary mathematics the most common sets are sets of numbers and sets of points. Arithmetic and algebra are concerned with sets of numbers and

their properties, while geometry is concerned with sets of points and their properties.

The definition of key ideas is of fundamental importance in science and mathematics. Defining every term, however, results in circular definitions. To avoid circular definitions, then, a basic set of undefined terms is usually chosen. "Set" and "element" are usually included in the basic set of undefined terms for mathematics. "Collection," "group," and similar words are frequently employed to describe the set concept. While such words are helpful for descriptive purposes, they should not be confused with precise terminology.

This chapter discusses fundamental set concepts and the manner in which these concepts can be used in the teaching of elementary school mathematics. The following topics are included: basic set ideas, introducing sets in the elementary school; and open sentences.

BASIC SET IDEAS

One of the first activities associated with learning about sets is describing and naming them. Sets may be described verbally. A set commonly described in this manner is the set of whole numbers less than 10 (frequently called single-digit numbers). When this method of describing sets, commonly called the *roster method,* is used, the names of the members are usually listed between braces. Three sets described by the roster method are as follows:

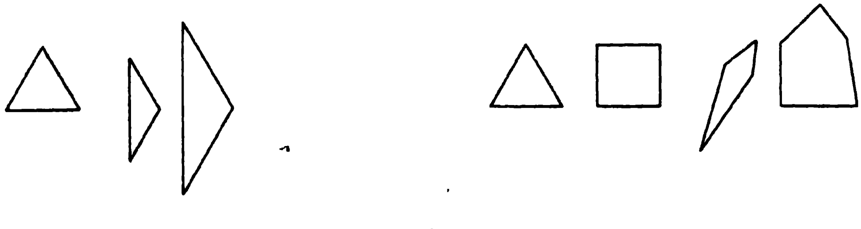
A {1, 2, 3, 4} C {blue, red, green}
B {2, 5}

Sets are often named by capital letters. Set A has four members or elements, 1, 2, 3, and 4; set B has two members; and set C has three members, the colors blue, red, and green.

Set A may be described verbally as the set of the first four natural numbers. Set C may be described verbally as the set of three primary colors.

A third method of describing sets is called the *rule* or *set-builder* notation. The set $\{x\ x = 2n,\ n \text{ an integer}\}$ is interpreted as the set of all x such that x is twice an integer (another way of describing the set of even numbers). This method of describing sets is not usually employed in the elementary school.

Textbooks for elementary schools frequently indicate sets of objects by enclosing the objects within circles or other closed curves, as in Figure 4.1.



Set of 3 Triangles Set of 4 Polygons

Figure 4.1

Fundamental vocabulary of sets

The following words are among the most basic in the vocabulary of sets: set; element; member; cardinal number; infinite set; empty set; null set, well defined, universal set; finite set.

Since "set" and "element" are undefined, they must be introduced by description and illustration.

Every set has a *cardinal number.* The number 2 is the cardinal number of a set containing a pair of members. A set with a single member has a cardinal number of 1. The cardinal number of a set indicates how many members are in the set. The cardinal number of {a, b, c, d, e} is 5. This fact is sometimes written as follows: $N\{a, b, c, d, e\} = 5$. The letter N placed before set braces indicates the cardinal number of that set. The notation $N(A)$ or $n(A)$ is frequently used to represent the cardinal number of set A.

An *empty set* is also called a *null set.* Either word is descriptive and suggests a set without any members or elements. The cardinal number of the empty set is 0. Zero is a number that indicates how many members are contained in the empty or null set.

It may seem strange to talk about a set with no members but this concept is important in mathematics. The empty set may be illustrated in many ways. The set of whole numbers less than 0 is an empty set (using the most common definition of whole numbers). The set of artificial satellites launched before 1957 is an empty set. The empty or null set is useful in helping to understand the nature of the number 0, a number that has caused much confusion throughout the ages, even among mathematicians.

Two symbols in common use for the empty set are { } and ϕ. The symbol { } is more common at the elementary level since it suggests a set with no members. The symbol ϕ is used almost universally in advanced work. It is a common error to confuse $\{\phi\}$ with the symbol for the empty set. The symbol $\{\phi\}$ indicates a set with the empty set as its only element. The cardinal number of $\{\phi\}$ is 1. $N\{\phi\} = 1$; $N\phi = 0$; $N\{\ \} = 0$.

The *universal set* is the set of all members that can enter into a given discussion. The universal set in arithmetic may be the set of whole numbers or the set of *rational numbers* (see page 84). The universal set in geometry may be the set of all the points on a line or all the points in a plane. It is important to recognize that the universal set for one discussion may be different for another discussion. It is characteristic of good mathematics to be very specific in designating the universal set for each discussion. The universal set is frequently referred to as the universe.

A *finite set* has a specific number of elements. The cardinal number of a finite set is a member of the set of whole numbers. A finite set may be very large. The set of all grains of sand in the world is a finite set even though it is very large. The set $\{x, y, z\}$ is a finite set with three members (cardinal number of 3).

An *infinite set* is not finite. An infinite set has an unlimited number of elements. The set of whole numbers is an infinite set. The roster method for indicating the infinite set of whole numbers is $\{1, 2, 3, 4, \cdots\}$. The leaders indicate that the set continues indefinitely and therefore is infinite. This method of indicating infinite sets is effective when there is a clear pattern to the set (as in the example or for such sets as even numbers and odd numbers).

Every mathematical set is formed from some universe. It is required that sets be formed from a universe in which there exists no doubt as to whether a member belongs to a set. The "set" of tall men is not a set in this respect, since

it is not clear whether some men are or are not tall. For this reason it is desirable to refer to the collection of tall men. Members that clearly do or do not belong to a set are called *well defined.*

Relations between sets

In the previous section, the basic vocabulary of sets was introduced. This section considers the words that describe the relationship between sets. These words include: equal sets; subsets; disjoint sets; overlapping sets; equivalent sets.

Equal sets have exactly the same members. In Example 1, set C is equal to set D because both sets contain the same members, 2, 4, and 6. Members of a set may be listed in any order, but sets of numbers are usually listed in order of size, as in the example, with the exception of set D. The statement that two sets are equal, as $C = D$, indicates that there is one set with two different symbols, C and D, just as the statement $1 + 1 = 2$ indicates that there is one number with two different numerals, $1 + 1$ and 2.

A {0, 1, 2, 3, 4, 5, 6, 7}
B. {1, 3, 5}
C: {2, 4, 6}
D {6, 4, 2}
E. { }
F: {1, 2, 3}

The set of even numbers may be defined as the set of multiples of two. This definition indicates that the set of even numbers is equal to the set of multiples of two. In a similar manner, every good definition must contain two equal sets, the set designated by the word being defined and the set designated by the definition. If these two sets are not equal, the definition is faulty.

In the example above, set B is a *subset* of set A because every member of set B is also a member of set A. Subsets are sometimes described as sets within sets or as parts of sets.[1] Set F is not a subset of set B, since the member 2 in set F is not contained in set B. Every set in Example 1 is a subset of set A. The empty set is defined to be a subset of every set. This definition cannot lead to a contradiction, since it is not possible to show that the empty set has an element not contained in another set.

In Figure 4.2, the set of triangles is a subset of the set of polygons.

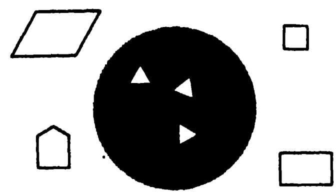


Figure 4.2

Disjoint sets have no members in common. In the previous example, sets B and C are disjoint. The sets of even and odd numbers are disjoint. In Figure 4.3, the set of circles and the set of

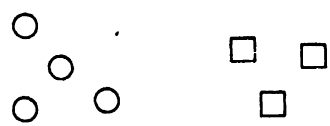


Figure 4.3

[1]Set A is a subset of itself by the definition given above, since every member of set A is contained in set A. By the same argument, every set is a subset of itself. Set A is defined to be an *improper subset* of itself. By this definition, if set A is an improper subset of set B, then $A = B$. Some authors also refer to the empty set as an improper subset, but this practice is not universal. The symbolic statement for "A is a subset of B" is $A \subset B$.

squares are disjoint. The set of red checkers and the set of black checkers are disjoint.

Two sets are *overlapping* if they are *not* disjoint and neither is a subset of the other. Figure 4.4 illustrates why set B and set F, from Example 1, are overlapping. They are not disjoint because they have the members 1 and 3 in common. Set B is not a subset of set F because 5 is a member of set B and not of set F. Set F is not a subset of set B because set F has 2 as a member and set B does not.

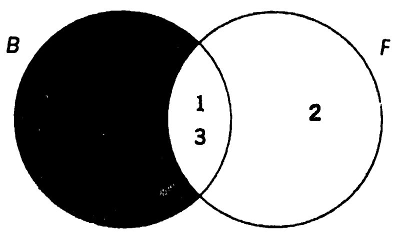


Figure 4.4

The set of isosceles triangles and the set of right triangles are overlapping. A right triangle with three unequal sides is not isosceles, so the set of right triangles is not a subset of the set of isosceles triangles. Some isosceles triangles are not right triangles, so the set of isosceles triangles is not a subset of the set of right triangles. Since a right isosceles triangle belongs to both sets, they are not disjoint and the two sets are overlapping. In the same way, the set of red objects and the set of all books is overlapping. A red car is not a member of the set of books and a green book is not a member of the set of red objects, so neither set is a subset of the other. A red book belongs to both sets, so the two sets are not disjoint and are therefore overlapping.

Equivalent sets are sets in which each member of one set can be paired with exactly one member of the other set. When two sets are paired in this manner they are said to be in one-to-one correspondence. Figure 4.5 illustrates several ways in which sets B and C in Example 1 may be put into one-to-one correspondence with each other.

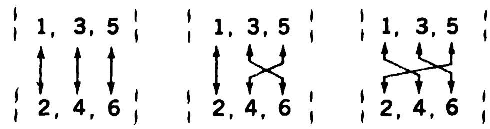


Figure 4.5

The manner in which the one-to-one pairing is achieved is unimportant. It is important, however, to know whether such pairing can be done. It should be clear that any set can be placed into one-to-one correspondence with itself. Therefore equal sets are equivalent but equivalent sets are not necessarily equal.

The most important fact about equivalent sets is that they have the same cardinal number. It follows that if the cardinal numbers of two sets are not equal, the sets are not equivalent. On the elementary level equivalent sets are frequently called *matching sets*. The activity of pairing members of different sets is also described as the matching of sets. In Figure 4.6 the set of triangles is equivalent (or matches) to the set of circles. The set of circles does not match the set of squares because one square is without a partner or mate.

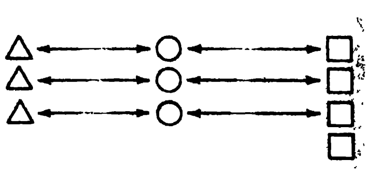


Figure 4.6

EXERCISES

1. It is a universally accepted convention that a member of a set is not listed more than once. Therefore the set of letters required to spell the word "need" is $\{n, d, e\}$. Which of the following sets is sufficient to spell the word "noon"?
   A: $\{a, b, n\}$ C: $\{n, o\}$
   B: $\{n\}$ D: $\{r, n, t\}$
2. A: $\{2, 3, 4\}$ C: $\{5, 6\}$ E: $\{1, 2\}$
   B: $\{1, 2, 3\}$ D: $\{2, 3, 4\}$
   Use these sets to choose the following: (a) A pair of equal sets; (b) a set and its proper subset; (c) a pair of disjoint sets; (d) a pair of overlapping sets; (e) a pair of unequal equivalent sets.
3. How is the set of even numbers related to the set of whole numbers?
4. How is the set of triangles related to the set of rectangles?
5. How is the set of squares related to the set of rectangles?
6. Which of the following is an infinite set?
   A: $\{1, 2, 3, 4, 5, 6, 7, 8\}$
   B: $\{1, 2, 3, 4, 5, \cdots\}$
7. Which of the following statements are correct?
   a. $\{2, 3, 4\} \subset \{2, 3\}$
   b. $\{2, 3\} \subset \{2, 3, 4\}$
   c. $\{1\} \subset \{1, 2, 3, 4, 5, 6\}$
   d. $\emptyset \subset \{1, 2, 3\}$

Set diagrams

Relations between sets are often illustrated by set diagrams, called Venn or Euler diagrams.[2] These diagrams are illustrated in Figure 4.7. The rectangles are not required but are desirable to stress the necessity of a universal set (universe).

Figure 4.8 illustrates that the set of squares is a subset of the set of rectangles. The diagram shows that every square is a rectangle but not every rectangle is a square.

Figure 4.9 indicates that the set of right triangles and the set of scalene triangles (no sides equal) are overlapping. A right triangle with no sides equal (scalene right triangle) belongs to both sets (they are not disjoint). No set and subset relation exists because some right triangles are not scalene (they might be isosceles) and some scalene triangles are not right triangles.

Figure 4.10 illustrates that the set of rectangles and the set of triangles are disjoint. A triangle is not a rectangle and a rectangle is not a triangle. Both are polygons.

It is clear from Figure 4.11 that the set of odd numbers is equal to the set of numbers obtained by adding one to each even number. Therefore a good definition for the set of odd numbers is that it is the set obtained by adding one to each even number.

Figure 4.12 shows how four sets are related. It shows that A and B as well as D and B are pairs of overlapping sets. It also shows that D is a proper subset of A and that C is a proper subset of B, as well as that C is disjoint with respect to both A and D.

Test your understanding of set relations by drawing Venn diagrams to show how the following sets are related:

a. $\{1, 2, 3, 4\}$ and $\{5, 6, 7\}$
b. $\{2, 3, 4, 5\}$ and $\{4, 5, 6\}$
c. $\{1, 2, 3\}$ and $\{0, 1, 2, 3, 4\}$

[2] The terms "Venn diagram" and "Euler diagram" are usually used interchangeably in the literature. Venn lived more than 100 years after Euler and used diagrams that were somewhat more specialized and more sophisticated than those of Euler (pronounced "oiler").

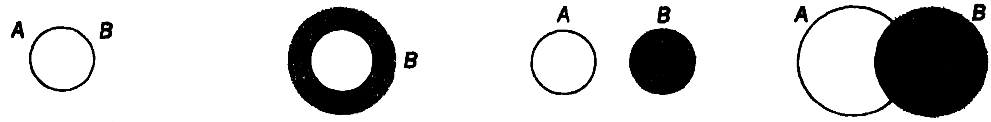


Figure 4.7

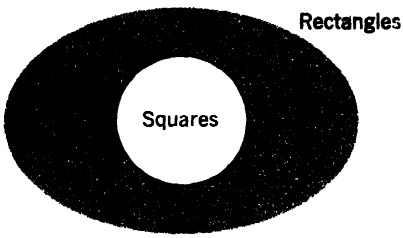


Figure 4.8

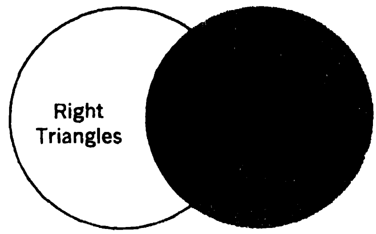


Figure 4.9

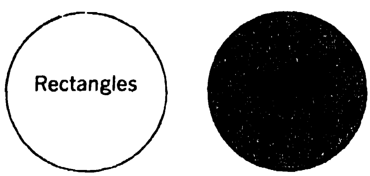


Figure 4.10

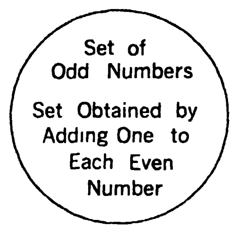


Figure 4.11

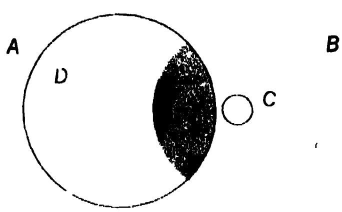


Figure 4.12

Operations on sets

Operations may be applied to pairs of numbers to obtain a number. Similarly, operations may be applied to pairs of sets to obtain a set. In this section the following set operations are discussed: union; intersection, set difference, set product, partition.

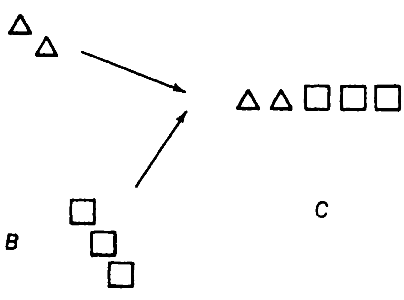
When the operation of addition is applied to two numbers, the result is a single number. Addition is a *binary operation* because it is applied to exactly two numbers at a time. When the binary set operation of *union* is applied

to two sets, the result is a single set containing all the members of the first set or the second set or of both sets.[3]

The symbol $\cup$ indicates the set operation of union, as illustrated below:

$$\{1,2,3\} \cup \{3,4,5\} = \{1,2,3,4,5\}$$
$$\{2,3\} \cup \{4,5\} = \{2,3,4,5\}$$
$$\{a,b,c,d\} \cup \{b,c\} = \{a,b,c,d\}$$

Applying the set operation of union to two sets is frequently described as combining the sets. When two disjoint sets are combined into a single set, the number (cardinal number) of the final set is the sum of the number of the first set and the number of the second set. Early work with addition is illustrated by combining pairs of disjoint sets of objects. Figure 4.13 illustrates the union of two disjoint sets and indicates that the cardinal number of the union is the sum of the numbers of the original sets.



$$A \cup B = C$$
$$n(A) + n(B) = n(C), \text{or}$$
$$2 + 3 = 5$$

Figure 4.13

The *intersection of two sets* is a set that contains all members belonging to both the first *and* the second set. It may be helpful to note that the key word in the definition of union is "or," while the key word in the definition of intersection is "and." The symbol $\cap$ indicates the binary set operation of intersection, as illustrated below:

$$\{1,2,3\} \cap \{3,4,5\} = \{3\}$$
$$\{a,b,c,d,e\} \cap \{b,c,d,e,f,g\} = \{b,c,d,e\}$$
$$\{1,2,3\} \cap \{4,5,6\} = \{\} \text{ or } \phi$$

The intersection of the set of whole numbers less than 10 and the set of whole numbers greater than 5 is the set of whole numbers between 5 and 10, or $\{6, 7, 8, 9\}$. The intersection of the set of teenagers and the set of car owners is the set of teenage car owners. The intersection of the set of even numbers and the set of odd numbers is the empty set. Figure 4.14 illustrates that the intersection of the set of right triangles and the set of isosceles triangles is the set of right isosceles triangles.

The *set difference* of set A with respect to set B is the set of all members of set A that are not members of set B. The symbol used to indicate the operation of set difference is $-$,[4] as shown below:

$$\{1,2,3,4\} - \{4,5,6\} = \{1,2,3\}$$
$$\{4,5,6\} - \{1,2,3,4\} = \{5,6\}$$
$$\{1,2,3\} - \{4,5\} = \{1,2,3\}$$
$$\{a,b\} - \{a,b,c\} = \{\ \}$$

It is desirable to read $A - B$ as "A not B," even though "A minus B" is correct. The former is more descriptive.

One scheme for determining the set $A - B$ is to list all the members of set A and then cross out those that belong to set B. The set of remaining members is the set $A - B$. If A represents the set of whole numbers and B the set of odd numbers, then $A - B$ represents the set of even numbers.

[3]The phrase "or of both sets" is not necessary, since the mathematical use of "or" includes the possibility of both. However, it is probably desirable to use it in elementary work.

[4]Some texts use the symbol $\sim$ to indicate the operation of set difference, but the symbol for subtraction is more common.

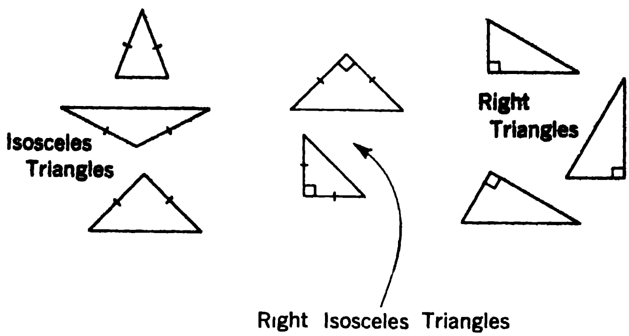


Figure 4.14

The symbol (2, 3) represents an *ordered pair* of the numbers 2 and 3. The ordered pair (2, 3) is a different ordered pair than (3, 2). The ordered pair (a, b) is the same as (equal to) the ordered pair (c, d) if and only if $a = c$ and $b = d$. In the ordered pair (2, 3), the number 2 is called the *first element* or *component* and 3 is called the *second element* or *component*.

The *cross* or *set product*[5] of two sets is the set of all ordered pairs obtained by choosing the first element of the ordered pair from the first set and the second element from the second set. The set product of sets A and B is represented by the symbol $A \times B$.

If set $A = \{1, 2\}$, set $B = \{x, y\}$, and set $C = \{1, 2, 3\}$, the following set products may be written:

$$\begin{aligned}
A \times B &= \{1,2\} \times \{x,y\} \\
&= \{(1,x),(1,y),(2,x),(2,y)\} \\
B \times A &= \{x,y\} \times \{1,2\} \\
&= \{(x,1),(x,2),(y,1),(y,2)\} \\
A \times C &= \{1,2\} \times \{1,2,3\} \\
&= \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}
\end{aligned}$$

The product $A \times B$ is different from the product $B \times A$ because the ordered pair $(1, x)$ is different from the ordered pair $(x, 1)$. $A \times B$ and $B \times A$ contain the same pairs but have different ordered pairs.

Some people find it helpful to write the set product in a table, as in Table 4.1.

TABLE 4-1

The Set Product

	$A \times B$			$B \times A$	
	x	y		1	2
1	$(1, x)$	$(1, y)$	x	$(x, 1)$	$(x, 2)$
2	$(2, x)$	$(2, y)$	y	$(y, 1)$	$(y, 2)$

The set product may be used to define multiplication of whole numbers without reference to addition. If $A \times C = D$, the product of the cardinal number of A and the cardinal number of C is the cardinal number of D. In the example above,

$$\begin{aligned}
n(A) \times n(C) &= 2 \times 3 = 6 \\
&= n(A \times C), \text{ or } n(D)
\end{aligned}$$

To *partition* a set is to separate it into disjoint subsets such that the union of these subsets is the original set. Partitioning a set may be described as breaking up the set without losing any of its members. The set of whole numbers may be partitioned into the set of even numbers and the set of odd numbers. A set of checkers may be partioned into the set of red checkers and the set of

[5]The set product is frequently called the Cartesian product in honor of the French mathematician René Descartes. The set product is a basic concept in analytic geometry invented by Descartes.

black checkers. Figure 4.15 illustrates the partitioning of a set of geometric figures into a set of circles, a set of squares, and a set of triangles.

Combining (union of) sets is associated with addition. Separating (partition of) sets is associated with subtraction. The simplest partition of a set breaks it into two disjoint subsets. When a set of 2 apples is removed from a set of 3 apples, the original set of 3 apples is partitioned into a set of 1 apple and a set of 2 apples. Set difference is also sometimes used as a basis for introducing subtraction.

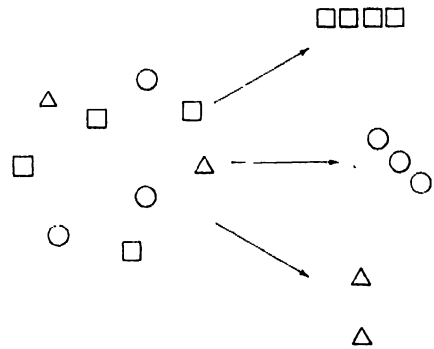


Figure 4.15

EXERCISES

Use the following sets to perform the operations indicated:

Universe: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

A: {1, 2, 3, 4}	D: {3, 4}
B: {3, 4, 5}	E: {0}
C: {5, 6, 7}	F: { } or ϕ

1. Perform the following operations:

a. $A \cup B$	f. $B \times C$
b. $A \cup C$	g. $D \times E$
c. $D \cup C$	h. $A - B$
d. $A \cup F$	i. $C - D$
e. $E \cup F$	j. $A \times F$

2. Partition set A into two sets, one of which is {1, 2}.
3. Partition the universe into two sets, one of which contains all the numbers greater than 3.
4. Describe verbally the union of the set of things that are green and the set of hats.
5. Describe verbally the intersection of the two sets described in problem 5.
6. Perform the following operations:

a. $(A \cup B) \cap C$	c. $(A \cup B) \cap (A \cup C)$
b. $(A \cup C) - B$	d. $(B \cap C) \cup (D \cup E)$

INTRODUCING SETS IN THE ELEMENTARY SCHOOL

The following activities are natural and important in the earliest stages of the elementary school:

1. Recognizing the cardinal number of a set

2. Pairing elements of sets (matching sets) to determine whether the sets are equivalent.

Identifying the cardinal number of a set has always been a major activity in the early elementary grades. When a pupil says that there are two animals in the picture, he is identifying the cardinal number of a set of animals. It is not necessary to use the word "set" in all such activities but the word should be used frequently and naturally to help pupils think in terms of collections of things.

The matching of sets has not been a common activity on the elementary level but it is useful in teaching facts about numbers and relations among them. Sets that match (are equivalent) have the same cardinal number. For classroom purposes it is acceptable to say that such sets have the same number. If an attempt to match one set with

another shows that the first set has one or more members without a partner or mate in the second, the two sets do not match and have different numbers or cardinality. Some activities involving matching of sets are as follows:

1. Choose a set of objects in the classroom and have the pupils identify the number of this set (such as 3). Help the class find as many sets as possible that match the original set. Help the pupils discover that each set which matches the original set also has a cardinal number 3. Use sets of books, sets of chairs, sets of pencils, and so on. The greater the variety in the types of elements used, the less likely pupils are to associate the concept of number with a particular type of object.

2. Choose a set as in the previous activity and have the class identify the number (as 4). Choose another set and identify its number (as 6). Have the class attempt to match the two sets and recognize that the second set has members that do not have mates in the first set; record that fact as 6 is greater than 4, or $6 > 4$; 4 is less than 6, or $4 < 6$.

When it is recognized that there are exactly two members in the second set without a mate in the first set, this fact can be recorded: $4 + 2 = 6$.

Both the inequality $4 < 6$ and the equation $4 + 2 = 6$ are called *number sentences* or *sentences*.

3. Talk about the set of cookies in an empty box. Talk about the set of living people more than 200 years old. Ask for the number of the set of students usually in the classroom at three o'clock in the morning. Help the class to recognize the empty set and that the number of the empty set is 0. Ask the class for suggestions for descriptions of the empty set. Repeat this type of work until the pupils readily recognize that 0 is the number of the empty set.

4. Have the pupils attempt to match a set with two members with the empty set as an extension of the activity described above. These sentences should result: $0 < 2$; $2 > 0$; $0 + 2 = 2$.

Appropriate repetition of this activity in early years will provide readiness for recognition of 0 as the *identity element* for addition (see p. 8).

5. Choose a set and identify its number (as 3). Choose a second set and identify its number (as 4). Combine these two sets and identify the number of the single set that results (7). The result of combining these two *disjoint* sets into a single set can be described by the sentence $3 + 4 = 7$. Of course, this activity must be repeated many times with a large variety of sets and should not be uncommon in traditional arithmetic, because the distinction between the set and its number is not always clear to pupils. It is important to stress the distinction between combining sets and adding the numbers associated with sets (see p. 48).

The combining of disjoint sets is described by adding the cardinal numbers associated with each set. Chairs are not added to chairs, but a set of chairs is combined with another set of chairs and this action is described by adding the numbers associated with the sets of chairs.[6]

6. Choose a set and identify its number, as 6. Remove a subset of two mem-

[6] It has frequently been said in traditional mathematics that only like numbers can be added. This statement is not sensible in light of the current discussion. Only numbers can be added, not chairs or people. A set of 4 elephants combined with a set of 3 pins is a set of 7 things. The 3 and 4 are added as easily in this case as they are if the sets contain only elephants or only pins. The difficulty in the case of a set of elephants combined with a set of pins is interpreting the meaning of the 7 obtained by adding the 3 and 4. It is neither 7 elephants nor 7 pins, but may be called 7 things

bers. This activity may be described in two ways:

a. A set of 6 members is partitioned into a set of 2 members and a set of 4 members. The sentence (equation) associated with this action is $6 = 2 + 4$.

b. A subset of 2 members is removed from a set of 6 members. A subset of 4 members remains. The sentence used to describe this action is $6 - 2 = 4$.

These and similar activities should be repeated often to lay a sound foundation for understanding the close relation between the sentences $6 = 2 + 4$ and $6 - 2 = 4$ (see p. 121).

Similar set activities can be used to help the pupils understand the nature of multiplication and division (see p. 158).

Recognizing subsets

A subset may be described as a set within a set. The set of even numbers is a subset of the set of whole numbers. The set of circles is a subset of the set of geometric figures. The set of West Coast states is a subset of the 50 states. The set of railroad engineers is a subset of the set of transportation workers. The set of policemen is a subset of the set of law officers.

The above examples help to illustrate that sets and subsets exist in a wide variety of areas. Recognizing sets and subsets is a worthwhile activity in subjects other than mathematics.

The following activities may be introduced in the early elementary grades:

1. Have the class identify the subset of red books in a set of books.
2. Have pupils identify the subset of the first five letters in the alphabet.
3. Ask the class to identify the subset of round objects in a group of objects. Be certain that there is no confusion in what is meant by round objects.
4. Ask a pupil to give a subset of the set of the first five whole numbers (there are many such subsets).
5. Help the class to recognize that the set of numbers less than 5 is a subset of the set of whole numbers.
6. Ask the class to write the set of fractional numerals with a denominator of 5 and a numerator that is less than the denominator.

Beginning examples should utilize familiar concrete objects, for example, books, blocks, and similar materials. Examples involving numbers and other ideas should be introduced gradually. Good definitions frequently place the term being defined in a familiar set as a subset. Additional information is then given to distinguish between the term being defined and other members of the parent set. Some examples of this procedure are:

A square is a rectangle with equal sides.

A triangle is a polygon with three sides.

A prime number is a whole number greater than 1 that is exactly divisible only by itself and 1.

OPEN SENTENCES

Sentences, for example, $5 < 7$ and $1 + 3 = 4$, are either true or false. All the number sentences in column (a) below are true; those in column (b) are false.

a.	b.
$1 + 1 = 2$	$1 + 4 = 6$
$6 - 5 = 1$	$4 - 6 = 2$
$2 \times 3 = 6$	$3 \times 5 = 10$
$11 > 5$	$6 > 8$
$4 < 20$	$13 < 5$

Some sentences, for example, $n + 3 = 12$ or $\square - 4 = 7$, are neither true nor false and are called *open sentences.* An *open sentence* contains a symbol that represents an unspecified number. This

symbol may be a letter (n as shown) or a frame ($\square$ as shown). When this letter or frame is replaced by the name of a number (numeral), the sentence then becomes true or false. The symbol, such as the letter or frame illustrated above, that holds a place for a numeral in an open sentence is called a *place holder* or *variable*. It is not usually desirable to refer to a variable by name on the elementary level. It is sufficient to say that the letter or frame is replaced by whatever numeral is involved. Frames occur in a variety of shapes, as illustrated in the following:

$+3=7 \qquad -4=7 \qquad 3\cdot\ =6 \qquad \div 2=8$

Frames are used almost exclusively in early elementary work. As pupils mature frames are gradually replaced by letters.

There are two sets associated with every open sentence. The first set is the set of all numbers that may come under consideration (the universal set). In beginning elementary work the universal set is usually the set of whole numbers. By the middle elementary grades the universal set becomes the set of numbers represented by fractional numerals (but no negative numbers). It will rarely be necessary to refer to the universal set on the elementary level. The second set associated with an open sentence is the *solution set*. The *solution set* of an open sentence is the set of numbers that makes the sentence true. Again it is not necessary to mention specifically the solution set as such on an elementary level. Finding the solution set of an equation (or inequality) is usually referred to as solving the equation (or inequality). The teacher should understand the relation between the universal set (universe) and the solution set of an open sentence and should use this information as a guide for discussing the work with pupils in less technical language. (For a discussion of various levels of solving open sentences, see p. 134.)

Open sentences are introduced in the grade 1 in modern mathematics programs. Once pupils can work effectively with simple open sentences, some of the work with sets outlined earlier can be modified as follows:

1. Choose a set and identify its number (such as 4). Choose another set and designate its number by the frame $\square$. Have the class attempt to match the second set with the first. When it is determined that the second set has 5 members without partners, the pupil should be able to write an open sentence that describes the situation $4 + 5 = \square$.

2. Combine a set of 3 members with a set of 7 members and help the pupils recognize that this activity can be described by the sentence $3 + 7 = \triangle$.

3. Partition a set of 6 members into a set of 4 members and a set of 2 members. Write the open sentence $6 = \triangle + \square$ and ask the pupils to write numerals in the frames that describe the result of the partition.

4. Remove a set of 4 members from a set of 10 members and help the pupils recognize that this activity can be described by the sentence $10 - 4 = \square$. It is useful to recognize that this activity can also be described as partitioning a set of 10 members into two sets, one of which has 4 members. This latter interpretation leads to the equation $10 = 4 + \square$, which illustrates the relation between addition and subtraction.

5. Write the equation $\triangle + \square = 8$ and ask the pupils which pairs of numbers make the sentence true. Help them recognize that both the pairs (3, 5) and (4, 4) make the sentence true. At this early stage it is probably not necessary to use the term "ordered pair," although

the teacher may exercise judgment in terms of the experience and ability of the class.

6. Write the equation $\square + \square = 8$ and help the pupils to recognize that the pair (4, 4) makes the sentence true but the pair (3, 5) does not apply, since both variables must represent the same number because they have the same shape. It is important that pupils recognize the different shaped frames (as in item 5) may represent the same or different numbers, but frames of the same shape must represent the same number when they occur in the same sentence.

7. Write on the chalkboard the open sentence $\triangle + 2 = \nabla$. Ask the class to find two sets whose numbers will make the above sentence true. At least several sets with different numbers should be found. Help the pupils to recognize that any pair of sets will work if the matching attempt produces two members in one set without partners or mates in the other.

8. On a day when there are three empty chairs in the classroom assign the frame $\square$ to represent the number of chairs (not counting the teacher's chair). Assign the frame ∇ to represent the number of pupils in the classroom. Write an open sentence to show how the number of chairs is related to the number of pupils ($\nabla + 3 = \square$).

9. Write the sentence $\triangle + 3 = 8$ on the chalkboard. Ask the class to find a set whose number will make the sentence true. Ask for a set situation that can be described by the true sentence. (Combining a set of 5 members with a set of 3 members gives a set of 8 members, or attempting to match a set of 5 members with a set of 8 members leaves 3 members of the latter set without partners.)

10. Write the sentence $\square + \nabla = 11$ on the chalkboard. Have the class find pairs of sets that will make the above open sentence true. Discuss the set situations that can be described by the resulting true sentences (combining or matching sets).

Activities similar to those above are most valuable when pupils are learning to identify numbers of sets and basic number facts on a manipulative level. Such activities help to provide some of the variety desirable in the learning process.

EXERCISES

1. Find pairs of sets that match (are equivalent) among the following:
   A: $\{x, z\}$ D: $\{x, y, z\}$
   B: $\{1, 2, 3\}$ E: $\{0, 1\}$
   C: $\{w, x, y, z\}$
2. Use the sets listed in problem 1 to perform the following operations:
   a. $A \cup B$ d. $A \cap B$
   b. $B \cup E$ e. $B \times E$
   c. $A \cap D$ f. $C \cap D$
3. Give the (cardinal) number of each set listed in problem 1.
4. Draw set diagrams to show how the following pairs of sets are related:
   a. A and B c. B and E
   b. C and D d. A and C
5. Identify the number sentences among the following:
   a. $2 + 3$ d. $1 + 1 = 3$
   b. $1 < 2$ e. $7 - 4 = 3$
   c. $10 - 4$ f. 2×3
6. Which of the sentences in problem 5 are true sentences?
7. Which of the following open sentences

have solution sets that are not empty if the universal set is the set of whole numbers?

a. $\square + \triangle = 1$ c. $3n = 12$
b. $n - 3 = 5$ d. $5 \times \square = 7$

8. Describe the role of the variable in the sentence $n + 3 = 7$.
9. Which of the following has more than one number in its solution set if the universe is the set of whole numbers?

a. $n + 11 = 17$ c. $n < 3$
b. $n - 8 = 3$ d. $3n = 15$

10. What universal set is necessary in order that the solution set of each of the following open sentences not be empty?

a. $2n = 3$ c. $4n = 6$
b. $5 \times \square = 3$ d. $10n = 13$

11. Illustrate how combining two disjoint sets may be used to provide understanding of addition situations.

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CHARACTERISTICS OF NUMERATION SYSTEMS

One of the goals of space exploration is to make a landing on Mars. Indeed, the directors of our space agency anticipate that this objective will be attained before the end of this century. If the planet is inhabited, it is highly probable that the Martians have some means of dealing with quantities. A knowledge of numeration systems, then, would aid space explorers in deciphering the methods of dealing with numbers on Mars if the latter are encountered there.

This chapter deals with the characteristics of numeration systems. The topics treated include the following: systems of numeration; the decimal system, different number bases, changing from base *b* to base ten and vice versa; addition and subtraction in base *b*.

Martians may follow the two procedures for dealing with quantities used by the inhabitants of our planet. Our ancestors made a record of the number of their possessions by 1) *tallying* or *matching* or by 2) a *system of numeration.*

A matching between two sets represents the lowest level of dealing with numbers. Precivilized man kept a record of his flock of sheep by using pebbles. The set of pebbles matched the set of sheep in the flock: For each

pebble there had to be one and only one sheep. Similarly, for each sheep there had to be one and only one pebble. In that way there was a one-to-one correspondence between the two sets.

The elements of the set of matching objects were not restricted to pebbles; tally marks, sticks, drawings, or other objects were also used.

If the Martians have not progressed beyond the level of our early tribesmen in dealing with quantities, today's space explorers could readily discover the pattern used. On the other hand, if they do use a system of numeration, it is almost certain that it is different from the one we use. Space travelers could then try to decipher that system in light of their knowledge of systems of numeration. We may speculate on the nature of this system from the different systems developed on our own planet. The history of various civilizations records a variety of numeration systems.

SYSTEMS OF NUMERATION

It is necessary to have symbols to name numbers in a system of numeration. One of the most commonly used symbols is a stroke or a tally mark to represent the number 1 and three strokes to represent the number 3. In the same way, five strokes, as IIIII, would represent the number 5. To facilitate recognition of the number of strokes in a group, we often unite or combine them, as ~~IIII~~, to designate a given grouping, such as five. It would be unwieldy and impractical to represent large numbers by repeating the same symbol; therefore other symbols are used to name different numbers.

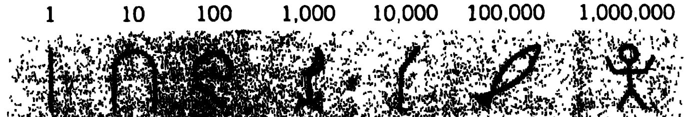
Basically two distinct procedures are employed for representing numbers. First, a new symbol is used to represent a group that is greater than that represented by the preceding group. Second, a place in a numeral has a given value, known as *place value*. The value a symbol represents depends upon the position of that symbol in a numeral.

The choice of symbols used to represent a new group and the number in that group is arbitrary, but a study of most ancient numeration systems reveals that the number of fingers on one or both hands determined the number in a group. In our numeration system there are 10 ones in 1 ten and 10 tens in 1 hundred. In the same way, 10 of a group make one of the next greater group. This relationship between successive groups or places in a numeral is the *base* of that system of numeration. The base of our numeration system is therefore 10.

Egyptian system of numeration

The ancient Egyptian system of numeration illustrates the use of a new symbol to represent the next greater group. Table 5.1 gives the different

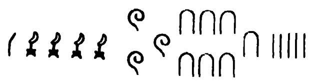
TABLE 5.1
Egyptian Symbols and Their Values



1	10	100	1,000	10,000	100,000	1,000,000
Stroke	Arch	Coiled Rope	Lotus Flower	Pointed Finger	Tadpole	Astonished Man

symbols and their corresponding values in our system of numeration.

Table 5.1 shows that 10 stokes would have the same value as 1 arch. In the same way, each succeeding new symbol represents a number ten times as great as the preceding symbol. The number represented by a numeral is the sum of the numbers represented by its various symbols. We may illustrate the procedure by considering the numeral for 14,375:



The number represented by the numeral using Egyptian symbols is the sum of 10,000, 4000, 300, 70, and 5, or 14,375.

The symbols in the numeral shown are arranged in sequence of value. The symbol of greatest value is on the left and the symbol of least value is on the right. The sequence of the symbols could have been in the reverse order or in a random order. Therefore, the position or place that a symbol occupied in a numeral did not affect the value of that symbol. The Egyptian system of numeration used neither order nor place value.

The Egyptian numeral for 14,375 contains 20 symbols while in our numeration system the number is represented by 5 symbols. Since an Egyptian symbol may be repeated in a given place in a numeral, there could be 9 symbols in each place compared with 1 symbol used in each place in our system.

Roman system of numeration

The Roman system of numeration also illustrates the principle of introducing a new symbol for each succeeding greater group. The different symbols used in the Roman notation are as follows:

I	V	X	L	C	D	M
1	5	10	50	100	500	1000

A bar placed above a numeral multiplies the value of the number represented by 1000. Thus $\overline{\text{C}}$ represents 100,000 instead of 100.

The plan of counting by five, as in tallying votes, offers an explanation for the usage of the symbols I, V, and X in Roman notation. Four vertical strokes and one cross stroke suggest V to represent five. The representation for the numbers 5–10 would be as follows:

Tally	卌	卌 I	卌 II	卌 III	卌 IIII	卌 卌
Numeral	V	VI	VII	VIII	VIIII	X

A cross diagonal suggests X to represent 10. According to Irwin,[1] the Romans used Etruscan symbols to represent larger numbers, such as 50, 100, and 1000. The letters to represent these numbers were not included in the Latin alphabet. We are not certain why L was selected to represent 50. In the case of C to represent 100 and M to represent 1000, the first letter of the words *centum* (hundred) and *mille* (thousand) seemed appropriate. An old Roman symbol for 1000 was CIↃ. By splitting this symbol in half and using the half to the right, we have a symbol that is similar to D, the present symbol to represent 500, or half of 1000.

Value represented by Roman numerals

The Roman system of notation is a combination of a system of tens and a system of fives. The value of a number

[1] Keith G. Irwin, *The Romance of Writing* (New York: The Viking Press, Inc., 1956), pp. 136–137.

represented by Roman numerals usually is the sum of the numbers represented by each numeral, as in the following:

MMM CCCC XX V II
= 3000 + 400 + 20 + 5 + 2
= 3427

Multiplication is also used in finding the value of a numeral when that numeral is capped by a vertical bar, as $\bar{X}$, to represent 10,000. Subtraction is used under certain conditions to find the value of a numeral. In case a symbol of lesser value precedes a symbol of greater value, the number represented is the difference between the two values provided that difference is divisible by either 4 or 9. Thus, the number represented by the numeral CIX is equal to 100 + (10 − 1), or 109. Similarly, XLIV is equal to (50 − 10) + (5 − 1), or 44. Addition is largely used to find the value of a number represented by Roman numerals. The Roman system has the property of order but not of place value as we use the latter. The Roman system is therefore superior to the Egyptian system, which contained neither of these traits.

There is no relationship between the symbol used to represent a smaller group and the next greater group in both the Roman and Egyptian systems of notation. There is no discernible relationship between X and L, L and C, C and D, 𓎆 and 𓍢, 𓍢 and 𓆼, or 𓆼 and 𓂭. If a person forgets the number represented that precedes a new symbol, there is no way to find that number from the given numerals.

Same symbol—different values

The second way to construct a system of numeration is to use the same set of symbols but assign different values to a symbol. The value of a symbol would also depend upon the place of that symbol in a numeral. We can illustrate the principle of assigning a different value to a symbol by the use of pebbles in numeration. Instead of the symbols that we use to represent numbers, primitive man used objects, such as pebbles. A chieftain in Madagascar determined the number of men in his command by having each soldier pass through an arch. As a soldier passed the chieftain would drop a pebble. When a pile contained 10 pebbles, an aide would gather it up and put one pebble in a different place to form a new pile. For each 10 pebbles in the first pile the enumerator placed one pebble in the second pile. He repeated the process until the second pile contained 10 pebbles, and then he formed a third pile at a different place. In a similar manner, the enumeration continued until all of the soldiers were matched with pebbles. Then one pebble twice removed from the original pile would have the same value as 100 pebbles in that pile.

According to the second plan, it was not necessary to have a variety of pebbles of different sizes. All pebbles were approximately the same size, but their value depended upon the position they occupied. However, this feature was the chief defect of the plan: There was no way to identify the value of a position represented by a pebble unless the person using the method remembered two things about the pebble. First, he had to remember from which pile a pebble was taken, and second, the value of a pebble in that pile. There was no identifying characteristic of a pebble to indicate its value. Practically all ancient peoples solved this problem by using the rods on an abacus to hold places in a numeral.

The abacus

The invention of the *abacus* made it possible to assign a fixed value to a group of pebbles. The *counting board* was the forerunner of the abacus. The counting board usually consisted of an area on sand marked with grooves to correspond to the places in a numeral. The principle of the counting board can be represented by making grooves in a board and assigning values to these grooves. Pebbles or markers may be inserted in a groove to represent the digit in a given place in a numeral. The counting board in Figure 5.1 shows the numeral 403. The 4 pebbles in the hundreds' place represent 400. The 3 pebbles in ones' place represent 3 ones. There are no pebbles in tens' place because there are no tens in tens' place in the numeral 403. A groove would be *overloaded* if it contained more than 9 pebbles. Thus, if there were 13 pebbles in the ones' groove, these pebbles would be rearranged to represent 1 ten and 3 ones. The abacus operates on the same principle as the counting board.

Most ancient peoples used a counting board or an abacus for computation. This invention made it possible to perform computations that could not be done otherwise. Multiplication with Roman numerals was very difficult, but the operation was made practical by using a counting board. The product could be recorded in Roman numerals, but the computation was performed on a counting board by repeated addition.

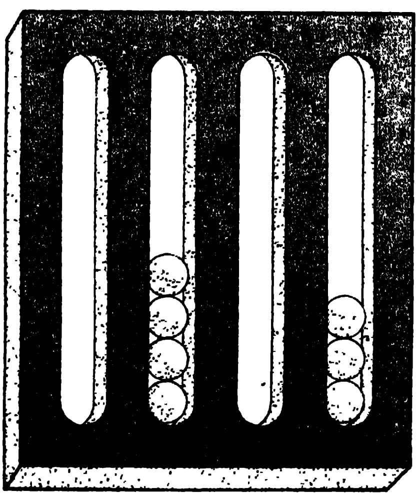


Figure 5.1

THE DECIMAL SYSTEM

Our system of notation for numbers is called the Hindu-Arabic system in honor of the people from whom we received it. The system is a decimal system and has the characteristic of place value. The Hindus invented the system and the Arabs were instrumental in introducing it into Western civilization. Although Hindu-Arabic numerals were introduced into Western Europe before the beginning of the eleventh century, it was not until the beginning of the thirteenth century that a European author wrote a comprehensive discussion of them.

The Egyptian and Roman systems of notation had one or more of the following deficiencies:

1. A new symbol was used to represent successively greater groups.
2. A numeral contained duplicates of the same value.
3. There was no fixed order to the symbols in a numeral in the Egyptian system.
4. Supplementary aids were used to hold a place in a numeral.

The Hindu-Arabic system of notation, however, overcame all of these shortcomings. This system uses only 10 digits, no duplicates of symbols of the same value occur in a numeral, a digit's value depends upon the place the digit occupies in a numeral, and no supplementary aids are necessary to have com-

plete place value to represent whole numbers. We may enumerate the characteristics of the decimal system of numeration for whole numbers as follows:

1. The base of the system is 10.
2. The digits used are in the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}$.
3. Each digit in a numeral performs two functions: it holds a place and it shows the frequency of that place.
4. Each digit in a two-or-more-place numeral that names a whole number has three values: (1) its *cardinal value*, sometimes called its *face value;* (2) its *positional value;* and (3) its *total value.* The positional value is the value of the place the digit holds in a numeral. This value is a power of the base. The power is the same as the number of places the digit is to the left of ones' place. The total value is the product of the cardinal and the positional values. In the numeral 247, the values of each digit are as follows: 2 has a cardinal value of 2, a positional value of 100 (10^2), and a total value of 200 (2×100); 4 has a cardinal value of 4, a positional value of 10, and a total value of 40; 7 has a cardinal value of 7, a positional value of 1, and a total value of 7. The cardinal and the total values of a digit are the same for every digit in ones' place. These two values are also the same for the digit 0 because the product of 0 and a number is 0.
5. The number named by a numeral is the sum of the total values of the digits in the different places.
6. Moving a digit to the left in a whole number multiplies its place value by a power of 10. If 3 is moved two places to the left, the value of the has been multiplied by 10^2. We read the numeral 10^2 as, "ten to the second power" or "ten square."
7. Moving a digit to the right divides its place value by a power of 10. Since dividing by 10 is the same as multiplying by $\frac{1}{10}$, moving a digit to the right in a numeral multiplies the value of that digit by a power of $\frac{1}{10}$. The power is the same as the number of places the digit is moved.
8. Every two-or-more-place numeral can be expressed in *expanded notation* to show the total value of each digit. Thus we may express 349 in expanded form as in (a) and (b).

a $349 = 300 + 40 + 9$
b $349 = 3 \times (10)^2 + 4 \times (10)^1 + 9 \times (10)^0$

The expanded numeral in (b) is expressed in *polynomial form.* The value of each digit in a whole number is the indicated product of that digit and a power of 10. The power is equal to the number of places a digit is removed from ones' place. The value of any number (except 0) to the zero power is one, as $(10)^0 = 1$. Therefore, the 9 in 349 may be expressed as $9 \times (10)^0$. The numeral $9 \times (10)^0$ names the same number as 9×1, or 9. The product of a number and 10^0 will always be that number.

Complete place value without the use of a supplementary aid is the distinguishing characteristic of the Hindu-Arabic numeration system. This feature was made possible by using 0, which was introduced into this numeration system in about A.D. 600.[2] Although 0 was first introduced into our numeration system about 1300 years ago, its effective use as a digit dates back less than a thousand years.

The introduction of 0 ranks as one of man's great intellectual achievements. Why did the invention of 0 take so long? There are two plausible reasons. First, man could not conceive of the need of

[2]Vera Sanford, "Hindu Arabic Numerals," *The Arithmetic Teacher*, December 1955, 2:115.

a symbol or digit to represent the absence of a quantity. It is logical to have one stroke to represent 1. When there is no quantity to be represented, it is equally logical to disregard this fact. Second, the use of an abacus served most of the functions of 0 as a place holder. It should be remembered that several thousand years ago man had a limited use for number. Priests and scribes performed most of the needed computations, and this restricted group was able to deal with number. In addition, until the fifteenth century printing was unknown and hence it was seldom necessary for most people either to read or write numerals. It was not necessary then to have the digit 0 to hold a vacant place in a numeral and to show the frequency of that place.

Essentials of a numeration system

We have discussed the characteristics of the Hindu-Arabic system of numeration and evaluated the systems used by the ancient Egyptians and Romans. In the light of these discussions, we should be able to list the essential elements of an effective numeration system. First, the system must have the property of completeness. This means that the system must function without the use of supplementary aids. A system of this kind must have (1) a base; (2) symbols; (3) a fixed value for a place; (4) 0; and (5) a decimal point.

Man invented all five of these items in order to deal effectively with numbers.

Ten is the base of the decimal system, hence there are 10 symbols needed. The number of digits used in any system of numeration is the same as the base. Most ancient numeration systems had a base that is a multiple of five, as 5, 10, or 20. The Babylonian numeration system used a base of sixty. We measure time on that scale today.

A symbol as used in a numeration system must have a *name*, as well as *design* or *form*. The names of the digits, which vary in different languages, are used in enumeration. Table 5.2 gives the number names of the first 10 counting numbers in three different languages.

Although the names for the digits are different in the different languages of Western cultures, the symbol used to represent a number is the same except for 7. The symbol for "seven" in English-speaking countries is 7, but in many European countries it is ⁊.

The third and fourth elements of the set of essentials of a system of numeration are designated place value and 0, respectively, which we have already described. The fifth element is a point, such as the decimal point in the decimal system used to locate ones' place. The use of a decimal point extends place value to the right of ones' place in the decimal system. Any digit, except 0, written to the right of that point

TABLE 5.2
Number Names in Different Languages

English:	one	two	three	four	five	six	seven	eight	nine	ten
German:	*ein*	*zwei*	*drei*	*vier*	*funf*	*sex*	*siben*	*acht*	*neun*	*zehn*
Spanish:	*una*	*dos*	*tres*	*cuatro*	*cinco*	*seis*	*siete*	*octo*	*nueve*	*diez*

TABLE 5.3
Use of Periods in Grouping Numerals

Billions	*Millions*	*Thousands*	*Units*
Hundreds-Tens-Ones	Hundreds-Tens-Ones	Hundreds-Tens-Ones	Hundreds-Tens-Ones

has a total value less than 1 but greater than 0.

Periods in numerals

It is difficult to read the numeral 371495 when written in the form shown. In order to facilitate the reading of large numerals, we use *periods*. Beginning on the right, three consecutive places constitute a period. A comma is used to separate periods in numerals containing five or more digits. A comma may be used in a four-place numeral provided that numeral does not represent a date or a period of time. The first period in our way of grouping numerals is designated *units*. This period and each succeeding full period to the left contain three places, designated ones, tens, and hundreds. Table 5.3 lists the first four periods in grouping numerals. Each period to the left has a value 1000 times the value of the period to the right. There are 1000 thousands in a million and 1000 millions in a billion. It is possible, therefore, to represent a million as 1000 thousands. Similarly, 1000 millions represents a billion.

The value of a billion is not the same in all countries. In our country a billion is equal to 1000 millions, while in England a billion is equal to a million millions. A billion of this value is 1000 times as large as our billion.

There are other period names that may be used to designate a period to the left of the billions' group. For a numeral to represent a value greater than a numeral in the billions' period, that numeral must have at least 13 places.

Scientific notation

In order to facilitate the reading of numerals that name either very large or very small numbers, we frequently use *scientific notation*. Scientific notation is another way of naming numbers. It eliminates the use of many different periods that are needed to express billions or other large numbers. To express a number in scientific notation, place a decimal point immediately after the first nonzero digit on the left and then indicate a multiplication by the appropriate power of 10. In a whole number, the power of 10 is the same as the number of places in the given numeral to the left of ones' place. For example, the numeral 3,245,000,000 may be expressed in scientific notation as 3.245×10^9. Similarly, the number named by 2.9×10^7 may be expressed in our numeration system as 29,000,000. We use negative exponents to express very small numbers in scientific notation. The numeral 2.5×10^{-8} is another name for .000000025.

DIFFERENT NUMBER BASES

The decimal system is the most widely used and best understood of all systems of numeration. Other numeration systems are used under certain conditions. Many electronic computers use the *binary system* of numeration, in

which numbers are expressed as powers of two. A study of different number bases is important for two reasons. First, electronic computers use systems of numeration with bases other than base ten, hence a knowledge of these systems has a practical application. Second, knowledge of several different systems of numeration should increase the pupil's understanding of the decimal system. This is the chief reason for devoting some time in the elementary school to different systems. Modern elementary mathematics textbooks include work with bases other than base ten in grades 5 and/or 6. The teacher should understand and know how to present subject matter of this kind, as explained in the following pages.

Base five

The teacher should have the class review the equivalence of certain coins in our coinage system. The pupils know that 5 pennies have the same value as 1 nickel and that 5 nickels have the same value as 1 quarter dollar. If our coinage system had one coin with the same value as 5 quarter dollars, that coin would further illustrate how five of one group are equal to one of the next greater group.

The teacher can show the meaning of a number base by having the class package or group cards or markers of some kind. If the number of cards in a group is 5, then the base of the system of numeration will be five. As soon as a pupil has 5 cards in a group, he makes one group of 5. In a numeration system, the 5 ones overload the ones' place, so he changes the 5 ones to 1 five, which is the next greater group than the ones. The class readily understands how cards can be grouped into bundles according to a given base. The part of the operation that causes the difficulty is the symbolic representation of the grouping. A review of the procedure in the decimal base should be helpful. A pupil counts 10 cards to form a group. If he represents a card by a bead on an abacus, the 10th bead on the ones' rod overloads that place and the 10 ones make 1 ten on the next rod to the left, as shown in (A) and (B) of Figure 5.2. We record the number as 10, which is the first two-place numeral in the decimal system. In the same way, the number needed to form a group in any base overloads a place, just as 5 ones overloads the ones' place in base five. This numeral is written as 10_5. The subscript indicates the base.

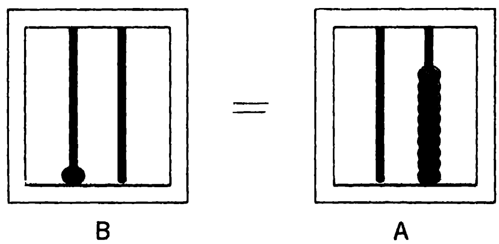


Figure 5.2

We say that "5 ones make 1 five" and we write this fact as 10_5. We read this numeral as "1, 0, base five." There are three steps in the procedure for expressing a number in a base different from the decimal base. They are:

1. *Stating the fact:* We give the decimal equivalent of the quantitative situation.

2. *Writing the numeral:* We write the numeral for expressing the number in the given base.

3. *Reading the numeral:* We read the numeral in the new base.

We used an abacus to show how to represent numbers in base ten. An abacus may also be used to represent numbers in any base, as base five. Five beads on a rod overload the place that

rod represents. From right to left, the first place is ones' place. Then fives' place, twenty-fives' place, and successively higher powers of five. A modified abacus may be used to represent the grouping of cards in base five. As soon as the 5th bead is indicated on a rod, that place is overloaded. The 5 beads on the ones' rod are exchanged for 1 bead on the fives' rod as in (C) and (D) of Figure 5.3.

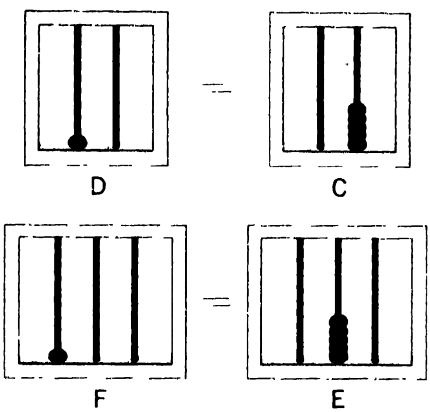


Figure 5.3

After a pupil forms 5 bundles or packages of 5 cards each, he forms a new pile. He then makes the corresponding representation on an abacus. The fives' rod contains 5 beads, hence it is overloaded. These 5 beads are exchanged for 1 bead on the twenty-fives' rod, as in (E) and (F) of Figure 5.3. Now the interpretation of the representation is patterned along the three steps mentioned. We say: "5 fives make 1 twenty-five." We write: "100_5." We read: "1, 0, 0, base five."

In a similar manner, other groups are formed. When a pile contains 5 bundles of 25 each, a new pile is formed. The pupil states that 5 twenty-fives are the same as 125 and writes this fact as 1000_5.

To read a numeral expressed in a base other than base ten, give each digit its cardinal value. Read 32_5 as, "3, 2, base five." There is no standard way to name the values of the places held by the digits in numerals written in a base other than base ten. The base is sometimes indicated by a word; for example, 10_8 may be written as 10_{eight}. Neither "8" nor "eight" is a numeral in base eight. It is easier to use 10_8 than to write the form 10_{eight}. The choice of a symbol to use should depend upon the ease of usage. The authors will use a subscript to designate a base other than base ten. Numerals in base ten will have no subscripts.

Same number—different numerals and bases

Figure 5.4 illustrates the use of four different bases to represent the number 18. Although the numerals are different in each base, they represent the same number (p. 64).

In row (a) there are 18 tally marks to be grouped. In row (b) these marks are grouped by tens. There is 1 group of ten and 8 ones, hence the base is ten. The numeral in base ten is 18. In expanded form 18 is equal to $1 \times 10^1 + 8$.

Row (c) shows how to represent 18 in base eight. There are 2 groups of eight and 2 ones. Using the threefold plan of describing the activity of grouping, we have the following: We say: "In 18 there are 2 groups of eight and 2 ones." We write: "22_8." We read: "2, 2, base eight."

$$\text{I} \quad 22_8 = 2 \times 10_8 + 2$$
$$\text{II} \quad 22_8 = 2 \times 8 + 2$$

Equation (I) shows the expanded notation for 22_8. Equation (II) uses numerals in base ten to show the equivalent of equation (I). Digits that name a number less than 8 are the same in both

bases. The numeral 10_8 names the same number as 8 in base ten.

Row (d) shows the 18 tally marks grouped as fives. We say: "In eighteen there are 3 fives and 3 ones." We write: "33_5." We read: "3, 3, base five."

$$\text{I} \quad 33_5 = 3 \times 10_5 + 3$$
$$\text{II} \quad 33_5 = 3 \times 5 + 3$$

Equation (I) shows the expanded notation for 33_5. Equation (II) uses numerals in base ten to show the equivalent of equation (I). The numeral 10_5 names the same number as 5 in base ten.

Figure 5.4 shows that we can express 18 as a two-place numeral in a base that is five and less than eighteen. A two-place numeral will not express this number when the base is less than five. Just as 3 ones makes a group of 1 three and is expressed as 10_3, so 3 groups of threes makes 1 group of 3×3, or 9, and is expressed as 100_3. In base ten the numeral is 3^2; in base three the numeral is 10^2_3. Each place value in base ten may be expressed as a power of ten. In the same way, each place value in base three may be expressed as a power of three. Therefore, 10_3^2 represents the same number as 3^2 in the decimal base.

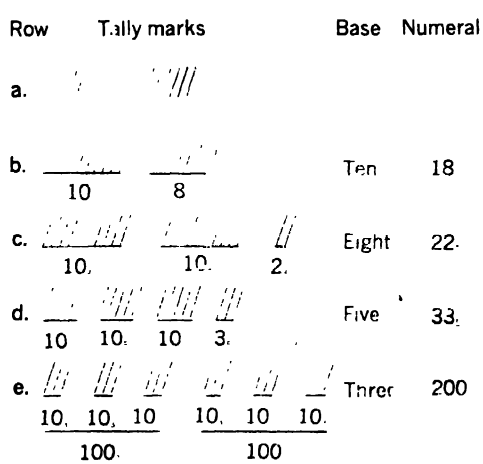


Figure 5.4

Refer to row (e) of Figure 5.4 and proceed as follows: We read: "In 18 there are 6 groups of threes, but 3 groups of threes makes 1 group of 3×3, or 9, hence there must be 2 groups of nines." We write: "200_3." We read: "2, 0, 0, base three."

$$\text{I.} \quad 200_3 = 2 \times (10)_3^2 + 0 \times 10_3 + 0$$
$$\text{II.} \quad 200_3 = 2 \times 3^2 + 0 \times 3 + 0, \text{ or } 2 \times 3^2$$

Equation (I) represents the expanded notation for 200_3. Equation (II) uses numerals in base ten to show the equivalent of equation (I). The numeral 10_3 names the same number as 3 in base ten.

The column on the right in Figure 5.4 shows four different numerals for the number 18. They are 18, 22_8, 33_5, and 200_3. The groupings in each of the rows show that the first two-place numeral in any base is 10. Similarly, the first three-place numeral in any base is 100.

The number of digits in a system of numeration that has complete place value is the same as the base. Thus, in base five there must be 5 digits. They can be 0, 1, 2, 3, and 4. Table 5.4 gives the numerals from 1 to 100 in base five. The numerals from 1 to 100 in any base can be written in equal rows and columns so as to form an *array* that is a square. The pattern shown in the table applies to the sequence of the numerals from 1 to 100 in any base. The reader can determine if he has discovered the pattern by using the digits 0, 1, 2, and 3 to write the numerals from 1 to 100 in base four.

TABLE 5.4

Numerals from 1–100 in Base Five

1	2	3	4	10
11	12	13	14	20
21	22	23	24	30
31	32	33	34	40
41	42	43	44	100

Duodecimal base

It is reasonable to assume that the base of our numeration system would be *duodecimal* (twelve) instead of decimal if man had 12 fingers instead of 10. A duodecimal base would have some advantages over a decimal base. It would be easier to express some fractions as decimals in base twelve than in base ten because 12 has more factors than 10. The only factors of 10 (excluding the number itself and 1) are 2 and 5, but the factors of 12 (excluding 12 and 1) are 2, 3, 4, and 6. Many fractions in the decimal base that are nonterminating, such as $\frac{1}{3}$, would be terminating in base twelve. Thus the value of the fraction $\frac{1}{3}$ in base twelve could be expressed as .4. Many units of certain familiar measures, such as foot, dozen, and months in a year could also be expressed in the duodecimal system as 10 inches in a foot, 10 things in a dozen, and 10 months in a year.

In order to have a duodecimal system of numeration, 12 symbols are needed. These would include the 10 digits used in the decimal system plus two more symbols. The two additional symbols could be represented by *T* and *E*, the first letters of words "ten" and "eleven."

Binary base

A numeration system that uses only two digits, 0 and 1, is called a *binary*, or base two, system. The array at the right shows the numerals from 1 to 100 written in base two so as to form a square. The binary scale for representing numbers has great practical application in certain types of electronic computers.

1	10
11	100

Table 5.5 gives the place values of the first six places in the binary scale. These values are expressed in both base two and in the decimal base. It is clear from the table that it requires more places to represent all numbers except 1 in base two than in the decimal base.

Since 0 and 1 are the only two digits used in the binary scale, it is possible to represent any whole number in this scale by use of electric bulbs arranged in a row. Figure 5.5 shows how these bulbs may be arranged.

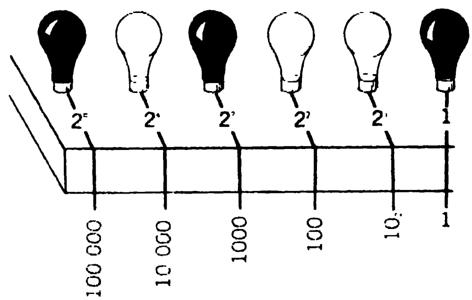


Figure 5.5

Each bulb holds a place in a circuit wired to represent a number in base two. A colored bulb shows that a 1 is represented on that place. A clear bulb indicates that a 1 is not represented,

TABLE 5.5

Place Values in Binary Scale

two^5	two^4	two^3	two^2	two^1	one	Binary place value
10_2^5	10_2^4	10_2^3	10_2^2	10_2^1	1	Binary notation
100000_2	10000_2	1000_2	100_2	10_2	1	
2^5	2^4	2^3	2^2	2^1	1	Decimal value
32	16	8	4	2	1	

hence 0 holds that place in the binary numeral. The numeral in the binary scale represented by the lighted bulbs is 101001_2. The decimal value of the number represented by this binary numeral is the sum of the indicated values of the places held by the lighted bulbs. The value of these places is $32 + 8 + 1$, or 41. Therefore, 101001_2 and 41 are different names for the same number.

Numerals in different bases

After dealing with different bases, the pupil should discover the pattern for writing the numerals in any base. Table 5.4 gives the numerals for the first 25 numbers expressed in base five. Table 5.6 gives the numerals for the first 20 numbers in base ten expressed in bases two, four, six, eight, nine, and twelve.

In Table 5.6 the numerals in each row except the first are different, but the number represented in any row is the same. The reader who does not discover the pattern of the sequence of the numerals for a given base can readily find the correct numeral by making a tally diagram, as in Figure 5.4.

From the preceding discussion of different number bases it is clear that the numeration system in each base is patterned after the decimal system. Each numeration system must have a base, and the number of symbols or digits used must be the same as the base. In order to have completeness for expressing whole numbers, one of the digits must be 0. Regardless of the base, every numeral may be written in expanded notation. Thus the numeral 402_5 may be written in expanded form as $4 \times 100_5 + 0 \times 10_5 + 2$. This numeral may be

TABLE 5.6

Numerals in Different Bases

Ten	*Two*	*Four*	*Six*	*Eight*	*Nine*	*Twelve*
1	1	1	1	1	1	1
2	10	2	2	2	2	2
3	11	3	3	3	3	3
4	100	10	4	4	4	4
5	101	11	5	5	5	5
6	110	12	10	6	6	6
7	111	13	11	7	7	7
8	1000	20	12	10	8	8
9	1001	21	13	11	10	9
10	1010	22	14	12	11	T
11	1011	23	15	13	12	E
12	1100	30	20	14	13	10
13	1101	31	21	15	14	11
14	1110	32	22	16	15	12
15	1111	33	23	17	16	13
16	10000	100	24	20	17	14
17	10001	101	25	21	18	15
18	10010	102	30	22	20	16
19	10011	103	31	23	21	17
20	10100	110	32	24	22	18

written by using powers of the base. The numeral would then be $4 \times 10_5^2 + 0 \times 10_5^1 + 2$, or in shorter form, $4 \times 10_5^2 + 2$.

CHANGING FROM BASE *b* TO BASE TEN

It is possible to express a number in any base, for example, base *b*, provided the base is a whole number greater than 1. The pupil should know how to express a number in base *b* in the decimal scale in order to enrich his understanding of a base. The teacher should have the pupil discover the procedure for changing from a given base to a decimal base. It is very easy to show a pupil how to make the change, but then computation may be the major achievement in the learning situation. The three steps in learning to change from base *b* to a decimal base are as follows:

1. Understand that the decimal value of 10_b is *b*.
2. Write a numeral in any base in expanded notation in the polynomial form.
3. Replace each symbol in base *b* with the correct symbol in base ten. In base twelve, replace *T* and *E* with 10 and 11, respectively.

First, the teacher should review the meaning of forming a group in establishing a base. The pupil learned from grouping cards or markers that 5 ones are equal to 1 five, expressed as 10_5. The decimal equivalent of 10_5 is 5. Similarly, 8 ones are equal 1 eight, expressed as 10_8, with the decimal equivalent as 8. Other illustrations should be given until the pupil discovers that the decimal equivalent of 10_b is b. The pupil should generalize that the subscript of the numeral indicating the base is the decimal value of the given base. If necessary, have the class form groups with markers and then make a written record of the experience.

Second, to be sure that all pupils understand the procedure of writing numerals in expanded form in base ten, have the class write a numeral, such as 248, in expanded notation without the use of exponents and then with exponents, as follows:

$$\begin{aligned} 248 &= 2 \cdot 100 + 4 \cdot 10 + 8 \\ &= 2 \cdot 10^2 + 4 \cdot 10^1 + 8 \end{aligned}$$

The same pattern applies to numerals expressed in any base. Thus, 215_6 may be written in expanded notation as follows:

$$\begin{aligned} 215_6 &= 2 \cdot 100_6 + 1 \cdot 10_6 + 5 \\ &= 2 \cdot 10_6^2 + 1 \cdot 10_6^1 + 5 \end{aligned}$$

If a numeral includes one or more zeros, use the long form, as in (a) below, until the pupil discovers that the product of 0 and a number is 0 and that the sum of 0 and a number is that number. He should then use the form shown in (b).

a $405_5 = 4 \cdot 10_5^2 + 0 \cdot 10_5^1 + 5$
b $405_5 = 4 \cdot 10_5^2 + 5$

The third step in transforming from base *b* to base ten consists in replacing 10_b with its decimal equal, *b*, and then performing the indicated computations. Since 10_8 in base eight names the same number as 8 in base ten, replace 10_8 with its decimal equivalent and then perform the indicated computations as shown for the numeral 256_8.

$$\begin{aligned} 256_8 &= 2 \cdot 10_8^2 + 5 \cdot 10_8^1 + 6 \\ &= 2 \cdot 8^2 + 5 \cdot 8^1 + 6 \\ &= 2 \cdot 64 + 40 + 6 \text{ or } 174 \end{aligned}$$

The reader can test his understanding of the procedure for changing from base *b* to a decimal base by expressing each of the following in base ten:

a 305	d 2*TE*
b 456	e 4003
c 10011	f 3312

CHANGING FROM BASE TEN TO BASE *b*

The pupils have learned how to change a number expressed in base b to a decimal scale. Next it is valuable for them to learn to change a number expressed in base ten to base b. The pupil should understand that for every operation with numbers there is always an undoing or inverse operation. The teacher should be sure to point out that the inverse relationship between changing from base ten to base b and vice versa is the same as the addition-subtraction and multiplication-division relationships.

The teacher should review the following characteristics of numerals that express numbers in different bases:

1. The number of digits used is the same as the base, and one of these digits must be 0.
2. The digit of greatest value is one less than the base.
3. A place is overloaded if its frequency is equal to or more than the base.

Changing from a decimal base to base b involves forming groups of b and expressing the number of groups with the digits in that base. To be sure that the class understands the procedure, the teacher should have each pupil participate in the following three activities in the sequence given:

1. Grouping of markers
2. Grouping by subtraction
3. Grouping by division.

As a pupil forms groups, he must know how to record that activity. The first group formed in base b is recorded as 10_b, and each succeeding place in a numeral is recorded as the next power of 10_b. If a place is overloaded, the number expressed in that place is regrouped as the next power of the base. Note the pattern in the following:

Decimal base

$$10 \times 10 = 10^2, \text{ or } 1 \times 10^2$$
$$10 \times 10^2 = 10^3, \text{ or } 1 \times 10^3$$

Base four

$$4 \times 10_4^0 = 10_4, \text{ or } 1 \times 10_4$$
$$4 \times 10_4 = 10_4^2, \text{ or } 1 \times 10_4^2$$

Base b

$$b \times 10_b^0 = 10_b, \text{ or } 1 \times 10_b$$
$$b \times 10_b^n = 10_b^{n+1}, \text{ or } 1 \times 10_b^{n+1}$$

The teacher has the class group markers in base b. To illustrate, each pupil has 14 markers and forms groups of fours. The following tally diagram illustrates grouping with markers. The pupil expresses the result in base four as shown:

14 = |||| |||| |||| ||
$= 3 \times 4 + 2$ (Decimal representation)
$= 3 \times 10_4 + 2$, or 32

Next, select a number, such as named by 23, so that the corresponding numeral in base b, as base four, will contain more than two places. The pupil forms 5 groups of 4 markers, with 3 markers remaining, as shown in the tally diagram. The fours' place is overloaded, so the 5 fours are regrouped as 1 group of 16 and 1 group of 4. The written record of the work is as follows:

23 = |||| |||| |||| |||| |||| |||
$= 5 \times 4 + 3$ (Decimal representation)
$= 5 \times 10_4 + 3$
$= (4 + 1) \times 10_4 + 3$ (Rename 5 as 4 + 1)
$= 4 \times 10_4 + 1 \times 10_4 + 3$
$= 1 \times 10_4^2 + 1 \times 10_4 + 3$
$= 100_4 + 10_4 + 3$ or 113_4

The pupils separate markers into different base groups until they discover that the same result can be achieved by subtraction. Instead of grouping cards or some other kind of marker to express a number in a different base, as 37 in base five, the pupil should see that he can make repeated subtractions of fives.

In 37, there will be 7 groups of fives and 2 ones remaining. The 2 ones represent the number of ones in ones' place in the new numeral. The fives' place is overloaded, so the pupil subtracts 5 fives from 7 fives to form 1 group of 25 and 2 fives. The written record of the activity is as follows:

$$
\begin{aligned}
37 &= 7 \times 5 + 2 \\
&= 7 \cdot 10 + 2 \\
&= (5 + 2) \cdot 10 + 2 \qquad \text{(Rename 7 as 5 + 2)} \\
&= 5 \cdot 10 + 2 \cdot 10 + 2 \\
&= 1 \cdot 10^2 + 2 \cdot 10 + 2 \\
&= 100 + 20 + 2, \text{ or } 122
\end{aligned}
$$

The class should easily discover that division may be used instead of subtraction to find the number of groups of base b in a number in the decimal scale. The pupil learned that division is a quick way of performing repeated subtraction, hence division provides a quicker way to change from a decimal scale to base b than subtraction. We may illustrate the use of division by expressing 47 in base five.

a $5\overline{)47}$ = 9 r 2 — The remainder 2 represents the number of ones in ones' place in base five. The 9 shows that fives' place is overloaded

b $5\overline{)9}$ = 1 r 4 — The quotient 1 shows the number of twenty-fives. The remainder 4 shows the number of fives in fives' place.

The record of the division is as follows:

$$
\begin{aligned}
47 &= 9 \times 5 + 2 \\
&= (5 + 4) \cdot 5 + 2 \qquad \text{(Rename 9 as 5 + 4)} \\
&= 5 \cdot 5 + 4 \cdot 5 + 2 \\
&= 1 \cdot 10^2 + 4 \cdot 10 + 2 \text{ or } 142
\end{aligned}
$$

The work in (a) and (b) can be shortened as follows:

```
5)47
5) 9 r 2
   1 r 4
           1 4 2 = 47
```

The quotient of the first division shows the number of fives and the second quotient shows the number of twenty-fives. In the same way, each succeeding quotient shows the number of groups of the next power of the divisor.

We shall give another illustration of the division method by expressing 48 in base three.

```
3)48
3) 16 r 0
3)  5 r 1
    1 r 2
           1 2 1 0 = 47
```

Check $1210 = 1 \cdot 10^3 + 2 \cdot 10^2 + 1 \cdot 10$
$= 1 \cdot 3^3 + 2 \cdot 3^2 + 3$, or 48

It is important to indicate a remainder of 0 in a division, such as $48 \div 3 = 16$ r 0. The 0 shows the number of ones in ones' place in base three for the given numeral. The pupil should check each answer by performing the inverse operation, as shown in expressing 48 in base three.

ADDITION AND SUBTRACTION IN BASE *b*

The four operations are performed in base *b* in the same manner as in base ten. If a place in the sum in base ten is overloaded, the number in that place is regrouped. The same situation applies in base *b*.

Regrouping in addition

Base ten	*Expanded notation*	*Conventional algorism*
46	40 + 6	46
+ 18	10 + 8	+ 18
	50 + 14 = 64	64

Base five	*Expanded notation*	*Conventional algorism*
24	= 20 + 4	24 (carry 1)
+ 13	= 10 + 3	+ 13
	30 + 12 = 42	42

The thought pattern used to find the sum in base five in the example at the right above is as follows: "4 + 3 = 12 in base five. Write 2 in ones' place. The 1 five added to the other 3 fives (2 + 1) is 4 fives. Write 4 in fives' place. The sum is 42_5."

The reader should be able to supply the missing numerals in the following:

$$\text{a.}\ \begin{array}{r} 4\ 5_8 \\ +\ 1\ 6_8 \\ \hline \_\ 3_8 \end{array} \qquad \text{b.}\ \begin{array}{r} 5\ 0\ 4_6 \\ +\ 3\ 2\ 4_6 \\ \hline \_\ \_\ \_\ 2_6 \end{array}$$

$$\text{c.}\ \begin{array}{r} 3\ 0\ 1_4 \\ +\ 1\ 3\ 3_4 \\ \hline 1\ 1\ \_\ \_{}_4 \end{array} \qquad \text{d.}\ \begin{array}{r} 1\ 0\ 1\ 0_2 \\ +\ 1\ 1\ 1\ 1_2 \\ \hline \_\ \_\ 0\ \_{}_2 \end{array}$$

To be sure that each sum is correct, express the numbers named in expanded notation and then add.

The pattern for regrouping in subtraction in base ten applies in base b.

Base ten		*Expanded notation*	*Conventional algorism*
42	=	40 + 2 = 30 + 12	42
− 17	=	10 + 7 = 10 + 7	− 17
		20 + 5	25
		= 25	

Base five		*Expanded notation*	*Conventional algorism*
32_5	=	$30_5 + 2 = 20_5 + 12_5$	32_5
-14_5	=	$10_5 + 4 = 10_5 + 4$	-14_5
		$10_5 + 3$	13_5
		$= 13_5$	

The thought pattern for subtracting in base five in the example at the right above is as follows: "Since 4 ones are greater than 2 ones, change 1 five to 5 ones to make 7 ones and then subtract 4 ones. Write 3 ones in ones' place. Now subtract 1 five from 2 fives. Write 1 five in fives' place."

The reader should apply the pattern shown and supply the missing numerals in the following examples:

$$\text{a}\ \begin{array}{r} 7\ 2_8 \\ -\ 2\ 6_8 \\ \hline \_\ 4_8 \end{array} \qquad \text{b}\ \begin{array}{r} 4\ 0_5 \\ -\ 1\ 3_5 \\ \hline 2\ \_{}_5 \end{array}$$

$$\text{c}\ \begin{array}{r} 4\ 3\ 2_6 \\ -\ 1\ 2\ 5_6 \\ \hline \_\ \_\ 3_6 \end{array} \qquad \text{d}\ \begin{array}{r} 4\ 2\ 0_5 \\ -\ 1\ 3\ 2_5 \\ \hline 2\ \_\ \_{}_5 \end{array}$$

Check the solutions by expressing each number named in expanded notation and then subtract.

EXERCISES

1. What are the two basic plans for systems of numeration? Evaluate each.
2. Express the following in both the Egyptian and Roman numeration systems. Use the symbols of each system.
   a. 38 c. 54
   b. 207 d. 136
3. What were the deficiencies in the ancient numeration systems that the Hindu-Arabic system overcame?
4. Give the names of the first six number periods in a numeral.
5. Write 725 billion in scientific notation.
6. What are the essentials for a system of numeration to be self-contained?
7. Express the following in the base indicated:
   a. 37 in base four
   b. 245 in base eight
   c. 560 in base five
8. Find the decimal numeral that expresses the same number as:
   a. 321_6 c. 11011_2
   b. 4056_8 d. $310T_{12}$

9. Use the following symbols as digits for a system of numeration:
   a. ○ = 0   d. △ = 3
   b. | = 1   e. □ = 4
   c. ∧ = 2
   Write the decimal numeral that represents the same number as the following:
   a. ∧□   c. △ ∧○○
   b. |△○   d. □□| ∧

10. Perform the operations indicated. Check each solution of expressing the numbers named in expanded notation.

a. $\begin{array}{r} 34_7 \\ +45_7 \\ \hline \end{array}$ b. $\begin{array}{r} 744_8 \\ +56_8 \\ \hline \end{array}$ c. $\begin{array}{r} 45_8 \\ -27_8 \\ \hline \end{array}$

d. $\begin{array}{r} 402_5 \\ -123_5 \\ \hline \end{array}$ e. $\begin{array}{r} 1001_2 \\ -11_2 \\ \hline \end{array}$ f. $\begin{array}{r} 4003_6 \\ -2015_6 \\ \hline \end{array}$

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NUMBER SYSTEMS AND THEIR PROPERTIES

Chapter 5 dealt with systems of numeration (sets of symbols that represent numbers). The present chapter is concerned with systems of numbers and their properties. A *number system* is a set of numbers and one or more operations.

A number is an idea. A number cannot be seen, touched, or erased. The number 2 may be described as an idea common to all pairs and couples. The number 2 cannot readily be defined on an elementary level. A written numeral, on the other hand, has physical properties and can be seen, touched, and erased.

The topics included in this chapter are as follows: number and numeral; properties of number systems, logical development of number systems, genetic development of number systems.

NUMBER AND NUMERAL

Traditional arithmetic programs make little or no distinction between number and numeral. Modern programs, however, give more emphasis to this distinction, although different programs vary in the degree of importance they assign to the topic; indeed, there is controversy as to how much attention should be given to teaching the difference be-

tween a number and its numeral. Fehr asserts that it is nonsense to make a fine distinction between number and numeral.[1] A common rule for divisibility by 3 states that a number is divisible by 3 if the sum of its digits is divisible by 3. Dubisch has pointed out that if strict attention is paid to the difference between number and numeral, the rule for divisibility by 3 must be restated as follows: "A number is divisible by 3 if, in the decimal numeral representing the number, the sum of the numbers represented by the digits in that numeral is divisible by 3."[2]

Probably the most useful way to distinguish between number and numeral is to help pupils recognize that each number has many numerals. Just as the same boy may be called Bob and Robert, the number one half may be named as $\frac{1}{2}$ or .5. One of the most common activities in mathematics involves replacing one numeral by another without changing the number. This process is now commonly referred to as *renaming* a number. Many arithmetic lessons can profitably begin by asking pupils to write as many numerals (names or symbols) as possible for a given number. Skillful teacher guidance can then direct this effort in desirable directions (see p. 12). Another useful way to point out the difference between number and numeral is to ask pupils to consider several numerals for the same number, such as $\frac{1}{2}$, .5, and 50% for one half, and discuss which symbol is most useful in various situations.

When the pupil recognizes that each number has many numerals, he must learn which numeral is best for each specific situation. Skill in recognizing when to replace one numeral by another (without changing the number) is essential for learning and understanding computation and fundamental mathematical ideas.

Language that is overprecise may be more confusing than helpful. Teachers must learn when the distinction between number and numeral helps to clarify an idea. For example, the authors believe that the "inaccurate" expressions "one-digit number" and "one-place number" are preferable to the more precise phrase "number represented by a numeral containing a single digit" (see p. 115).

PROPERTIES OF NUMBER SYSTEMS

Many rules in traditional arithmetic, such as invert and multiply and move the decimal point, involve numerals rather than numbers. A number cannot be inverted. Inverting the numeral will not produce the desired result in modular arithmetic (see p. 218). Moving the decimal point is impossible in the Roman system because there is no decimal point. Modern mathematics programs distinguish between properties of number systems and properties of numeration systems. A number property is independent of the set of numerals used. A *prime* number is prime when represented by Roman or Arabic numerals.

The commutative property[3]

A system is commutative for the operation of addition if the sum of any two numbers in the system is the same re-

[1] Howard F. Fehr, "Sense and Nonesense in Modern Mathematics Programs," *The Arithmetic Teacher*, February 1966, 13:83–91.

[2] Roy Dubisch, "On Numbers and Numerals," *The Mathematics Teacher*, December 1963, 56:647.

[3] Texts more than 10 years old usually refer to the "commutative law." More modern practice is to use the term "commutative property."

gardless of the order in which they are added. This property can be stated more concisely with the use of algebraic language: For every x and y in system S, $x + y = y + x$.

Comparison of the two statements above shows why properties of number systems are frequently stated with the help of algebraic symbolism rather than in pure verbal form. Modern programs use more algebraic language than traditional programs, although the difference between the newer and older programs in this respect is not as apparent at the elementary level as at the secondary level.

A system S is commutative with respect to multiplication if for all x and y in S, $xy = yx$.

All the number systems of elementary mathematics are commutative for addition and multiplication but not for subtraction and division. It seems evident that whole numbers are commutative for addition, but this feature is not easy to prove at the elementary level. Commutativity, when it exists, is usually accepted as a postulate (assumption) even in early college work. On the other hand, it is easy to prove that whole numbers are not commutative for subtraction. A single example which shows that $3 - 1$ is not equal to $1 - 3$ is sufficient to prove the point. Such an example is called a *counterexample*. Any mathematical statement can be disproved by a single counterexample.

The commutative property of number systems is simple enough to be understood by pupils in the lower elementary grades and important enough to be referred to at all levels of mathematics. Other important properties will be discussed in the following sections.

The associative property

The associative property of a system S for addition and for multiplication may be stated as follows: *For all x and y in S:*

$(x + y) + z = x + (y + z)$	Addition
$x(yz) = (xy)z$	Multiplication

The associative property is important because of the binary nature of addition and multiplication.[4]

When more than two numbers are to be added, the binary nature of addition demands that the operation first be performed on two numbers. The associative property of addition indicates that when three numbers are to be added the result of adding the sum of the first and second numbers to the third number is the same as adding the first number to the sum of the second and third numbers.[5] A similar statement may be made for multiplication.

Some useful activities involving the associative and commutative properties are the following:

1. Ask the pupil to rename $2 + 3$. A response of 5 is natural and correct at any grade level, but with teacher guidance, the response of $3 + 2$ should be common as early as grade 1. Pupils should be able to rename $76 + 234$ as $234 + 76$ well before they can deter-

[4]Subtraction and division are also binary operations but do not have the associative property, as illustrated by the following counterexamples:

$$6 - 5 \neq 5 - 6$$
$$(6 - 3) - 2 \neq 6 - (3 - 2)$$
$$6 \div 2 \neq 2 \div 6$$
$$(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$$

[5]It is true that $(x + y) + z = (x + z) + y$, but this is a consequence of both the associative and commutative properties because y and z occur in different order in the two expressions. The associative property indicates that in the sum of three numbers the correct answer will be obtained when addition is first performed on the first and second numbers or on the second and third numbers. The combined effect of the associative and commutative properties is that the correct sum will be obtained no matter which pair of numbers is added first. In a similar manner, it is a consequence of the associative property alone that $(3 \times 4) \times 5 = 3 \times (4 \times 5)$, but it is a consequence of both the associative and commutative properties that $(3 \times 4) \times 5 = (3 \times 5) \times 4$.

mine the sum of 310. Answers of $(1 + 1) + 3$ and $2 + (2 + 1)$ should also be obtained by first-grade pupils. Exercises of this nature help the pupils learn the basic pattern of the commutative property long before a name is attached to it. When a pattern is understood, it is much easier to attach a name to it.

2. Ask the class to rename $4 + (6 + 1)$. Again, $4 + 7$ and 11 are correct, but examples of this type should be given often enough so that $(4 + 6) + 1$ will also be given. In a renaming activity, any correct answer should be accepted, but the pupils should be encouraged to give others until the name or numeral desired by the teacher is obtained. If the desired answer is not given in a reasonable time, the teacher should give the answer and then immediately give several similar examples to see if the class understands the idea involved. Renaming sessions should be frequent and brief and should usually have a specific goal.

3. Both of the above activities can be repeated with examples such as 3×4 and $4 \times (5 \times 3)$ when the multiplication concept has been introduced.

4. When the patterns have been named as the commutative and associative properties, pupils may be asked to rename $4 + 5$ with the use of the commutative property or $3 + (7 + 5)$ with the use of the associative property.

5. The following exercise involves pupil recognition of an equation as true or false. The correct answers are given in parentheses.

$5 - 3 = 3 - 5$	(False)
$3 \times 5 = 5 \times 3$	(True)
$3(4 + 5) = 3 \times 4 + 5$	(False)
$32 \times 25 = 8 \times (4 \times 25)$	(True)
$4 \times (7 \times 25) = (4 \times 25) \times 7$	(True)
$89 + 43 = (89 + 11) + 32$	(True)

6. Use examples similar to those above and discuss why they are true or not true in terms of the properties of number systems under discussion.

The distributive property[6]

The associative and commutative properties are defined in terms of a set of numbers and *one* binary operation. A system may be commutative with respect to one binary operation (as addition) and not for another (as subtraction). The distributive property is defined in terms of a set of numbers and *two* operations. The distributive property is the only property discussed in this chapter defined in terms of two operations.

A precise definition of the distributive property (of multiplication over addition) for a system S may be given as follows: For all x, y, and z in system S, $x(y + z) = xy + xz$.

The distributive property indicates that $3(4 + 5)$ may be renamed as $3 \times 4 + 3 \times 5$. There are many other names for the number represented by $3(4 + 5)$, including 3×9. The distributive property indicates that if multiplication is to be performed before addition in $3(4 + 5)$, the *multiplication must be distributed over both addends*.

Some of the following activities are typical for helping pupils obtain a better understanding of the distributive property:

1. Ask pupils to rename $4(5 + 10)$. Continue the activity until $4 \times 5 + 4 \times 10$ is obtained. If it is necessary to give the answer, use several other examples to be certain that the basic pattern is understood. Talk about multiplying each addend or distributing the multiplication over both addends.

[6] "Distributive property" is a commonly accepted abbreviation for the longer name, "the distributive property of multiplication over [or with respect to] addition," and will be so used in this text.

2. The activity in (1) may be performed before the pupils know the name of the distributive property. After pupils have learned this property by name, the following activity and similar ones may be introduced. Ask the pupils to rename the following using the distributive property. Correct answers are given in parentheses.

$4(10 + 7)$	$(4 \times 10 + 4 \times 7)$	
4×23	$(4 \times 20 + 4 \times 3)$	Other answers possible
5×8	$(5 \times 5 + 5 \times 3)$	Other answers possible

3. When pupils learn that multiplication is also distributive with respect to subtraction, they can rename $7(30 - 2)$ as $7 \times 30 - 7 \times 2$. This can be profitably done both before and after the pupils have learned the meaning of the word "distributive."

4. Pupils should learn that the expression $7(3 + 7)$ can be evaluated most readily as 7×10, while for most people $7(20 - 1)$ can be evaluated most readily if it is renamed $7 \times 20 - 7 \times 1$. A list similar to the following may be given and the pupils asked whether to add (subtract) or multiply first. It should be recognized that this is a subjective question and the answer may be different for different pupils in light of their attitudes and skills. Pupil disagreements that provoke discussions should be encouraged.

$10(8 + 9)$	Adding first is easier (comparing two choices) but multiplying first is not much more difficult.
$17(4 + 6)$	Adding first is shorter
$11(40 - 1)$	Multiplying first is probably easier for most pupils

4. If $a = b$ is a true statement, then the renaming concept allows a to be renamed as b or b to be renamed as a. Therefore, since $3(4 + 5) = 3 \times 4 + 3 \times 5$, the left-hand member of the equation may be renamed as $3 \times 4 + 3 \times 5$, or the right-hand member may be renamed as $3(4 + 5)$. The latter renaming is much less likely to occur if the pupils are not given guidance in this direction. The following situations illustrate the advantage of this renaming:

12		Rename as $7(12 + 8)$ or 7×20
49	51	Rename as $\frac{1}{4}(49 + 51)$ or $\frac{1}{4} \times 100$

The value of this activity cannot be overemphasized.

The identity property

The identity property may be introduced by tables of the type that follow:

a.	b.
$1 + 0 = 1$	$1 \times 1 = 1$
$2 + 0 = 2$	$2 \times 1 = 2$
$3 + 0 = 3$	$3 \times 1 = 3$
$4 + 0 = 4$	$4 \times 1 = 4$

The table in (a) should enable the pupil to see that the sum of 0 and a number is that number. The table in (b) should enable him to see that the product of a number and 1 is that number. The pattern in (a) shows that 0 is the *identity element* for addition. The precise statement of this property of 0 is given as: For all x in S, $x + 0 = 0 + x = x$. In a similar manner, the number 1 is the *identity element* for multiplication, which is indicated in precise mathematical language as: For all x in S, $x \cdot 1 = 1 \cdot x = x$.

Elementary pupils should become familiar with the identity concept through the renaming process. They should learn to rename 2 as $2 + 0$ or $0 + 2$ and, when multiplication has been introduced, 2×1 and 1×2. Renaming 5 as $5 + 0$ can be useful in helping pupils to learn that $4 + 1 = 5$ and $3 + 2 = 5$ (see p. 165). Renaming $\frac{1}{2}$ as $\frac{1}{2} \times 1$ is useful in helping pupils recognize that one half and two fourths name the same number.

The inverse property

The inverse property is probably the most complex of the fundamental ideas under discussion. It can be illustrated by the following patterns:

a.	$0 + 0 = 0$	b	$1 \times 1 = 1$
	$^-1 + {}^+1 = 0$		$2 \times \frac{1}{2} = 1$
	$^-2 + {}^+2 = 0$		$3 \times \frac{1}{3} = 1$
	$^-3 + {}^+3 = 0$		$4 \times \frac{1}{4} = 1$
	$^-4 + {}^+4 = 0$		

The pattern in (a) shows pairs of *inverse numbers* for addition. A pair of inverse numbers in addition has a sum of 0 (the identity element for addition). The pattern in (b) shows pairs of inverse numbers in multiplication. A pair of inverse numbers in multiplication has a product of 1 (the identity element for multiplication).

A system is said to have the *inverse property for addition* if for every member x in S there is a number ^-x such that

$$x + {}^-x = {}^-x + x = 0$$

A system is said to have the *inverse property for multiplication* if for every[7] number x in S there is a number $\frac{1}{x}$ such that

It is helpful to remember that inverse elements come in pairs and that the proper operation performed on such a pair gives the identity element. Finally, if a is the inverse of b, then b is the inverse of a.

Inverse elements are important in understanding inverse operations.[8] The most efficient way to divide by $\frac{1}{2}$ is probably to multiply by 2 (the inverse of $\frac{1}{2}$ for multiplication).[9]

Subtraction of $^-2$ can be accomplished by adding $^+2$ (see p. 215). The pair of numbers $^-2$ and $^+2$ form a pair of inverse elements for addition because their sum is 0 (the identity element for addition). It is a significant mathematical fact that the rules "invert and multiply" and "change the sign and add" are both examples of the inverse concept.

Closure

The final property to be discussed is closure. The set of whole numbers is closed with respect to addition because the sum of two whole numbers is always a whole number. The set of whole numbers is not closed with respect to subtraction because the difference $6 - 8$ is not a whole number.

A set of numbers is *closed with respect to addition* if for all x and y in S the sum $x + y$ is also in S.

A set is *closed for multiplication* if for all x and y in S the product xy is also in S.

The set of whole numbers is closed with respect to multiplication but not with respect to division, since $2 \div 3$ is not a whole number. A single counterexample is sufficient to prove that a set is not closed for a given operation.

The closure property has very few direct applications, but an understanding of this concept helps one gain a knowledge of the development of number systems, discussed later in this chapter.

[7] Some systems, as a number field, allow [illegible] as an exception since 0 does not have an inverse for multiplication (see p. 81).

[8] Division is the inverse operation of multiplication because division by 2 can be undone by multiplication by 2, and conversely. In the same way, subtraction is the inverse operation of addition (see p. 150).

[9] The pair of numbers 2 and $\frac{1}{2}$ form a pair of inverse elements because their product is 1 (the identity element for multiplication). This product is 1 no matter what numerals are used to represent 2 and $\frac{1}{2}$ (including 2 and .5). It may be helpful for pupils to recognize that .5 is not obtained by inverting 2.

Understanding of the commutative, associative, distributive, identity, inverse, and closure properties is the key to mastery of arithmetic and algebra. A more detailed treatment of the manner in which these concepts contribute to the understanding of arithmetic is discussed in the next section.

EXERCISES

1. Identify the property used to rename the numbers as indicated:
   a. $3+6=6+3$
   b. $4=4+0$
   c. $5\times 3=3\times 5$
   d. $3=3\times 1$
   e. $23=3+20$
   f. $\frac{3}{5}=\frac{3}{5}\times\frac{2}{2}$
   g. $x+y=y+x$
   h. $27+4=20+(7+4)$
   i. $4\times\frac{1}{4}=1$
   j. $3(30+2)=3\times 30+3\times 2$
   k. $50=10\times 5$
   l. $\frac{1}{4}\times 17+\frac{1}{4}\times 23=\frac{1}{4}(17+23)$
   m. $99+18=(99+1)+17$
   n. $28\times 25=7(4\times 25)$
   o. $^{-}3+3=0$
   p. $3\times 21=3\times 20+3\times 1$
   q. $1=\frac{1}{2}\times 2$
   r. $3(a+5)=3a+3\times 5$
2. Which of the following sets are not closed with respect to addition? subtraction? multiplication? division? (a) the set of whole numbers; (b) the set of even numbers; (c) the set of odd numbers; (d) the set $\{0, 1\}$
3. Two numbers are inverse elements for multiplication. What is the product of these two numbers?
4. What is the sum of two numbers if they are inverse elements for addition?
5. The terms "axiom" and "postulate" are used interchangeably in modern mathematics. Which of the following words is the best description for these terms? (a) theorem; (b) definition; (c) structure; (d) assumption
6. Undefined terms are essential in mathematics for which of the following reasons: (a) to make everything clear; (b) to avoid circular reasoning; (c) to be more precise
7. Which of the following sentences uses the word "numeral" or "number" correctly? Reword the incorrect statement. (a) erase the number on the board; (b) write the numeral on your paper; (c) invert the number

A number field

A mathematical structure involves undefined terms, definitions, and axioms or postulates (assumptions). Logical consequences called theorems are then developed from the postulates and definitions. No serious treatment of mathematics can ignore the nature of a mathematical structure.

Traditional high school plane geometry provides the best-known example of a mathematical structure. One of the major reasons for the important place that geometry has held in the secondary curriculum for many years is the manner in which it demonstrates how a large body of mathematics can be developed from a relatively small set of definitions and assumptions with the help of logical deductions.

In traditional mathematics, the first theorem that a student proved occurred in plane geometry. In a modern program, theorems are usually proved in early work in algebra. A theorem cannot be proved without definitions and assumptions. Theorems are proved in

TABLE 6.1

Eleven Postulates in a Number Field

Addition	*Multiplication*
1. Set S is commutative for addition.	6. Set S is commutative for multiplication.
2. Set S is associative for addition.	7. Set S is associative for multiplication.
3. Set S has an identity element for addition (the number 0).	8. Set S has an identity element for multiplication (the number 1).
4. Every number in S has an inverse for addition (the sum of a number and its inverse is 0).	9. Every number *except 0* has an inverse for multiplication (the product of a number and its inverse is 1).
5. The set S is closed with respect to addition.	10. The set S is closed with respect to multiplication.
Addition and Multiplication	
11. Multiplication is distributive over addition in the set S.	

conjunction with a mathematical structure. The mathematical structure most frequently associated with arithmetic and algebra is a number field. A number field includes the following:

1. A set of numbers, S
2. Two binary operations, addition and multiplication
3. An equals relation which states that $a = b$ if and only if a and b are symbols for the same number
4. Eleven postulates that include five for addition, five for multiplication, and one for multiplication with respect to addition (see Table 6.1).

At the elementary and secondary levels, number and the operations of addition and multiplication are undefined terms. A number field does not tell what a number is but indicates how numbers behave with respect to addition and multiplication.

The number field is important because its postulates and definitions provide the basis for proving fundamental rules of operation in arithmetic and in algebra which were accepted without proof in traditional approaches to these subjects. The mathematical importance of the number field lies in the fact that every correct procedure in arithmetic and algebra can be deduced as a logical consequence of its postulates and definitions.

The concept of a number field is less than one hundred years old[10] and it has been only recently that educators and mathematicians realized that the basic concepts of this structure are simple enough to be understood at elementary levels and can serve as correct and useful guides in dealing with and understanding numbers.

Much of the remainder of this text is devoted to demonstrating how to introduce the ideas of the number field and how to use them effectively in helping pupils to learn and understand elementary mathematics (including arithmetic). The elementary school teacher should understand that when a pupil uses the distributive property as a guide in learning a particular topic, he is not only being helped with that topic but also is developing a deeper understanding of this property which will be useful in future situations.

[10]The concepts involved in the number field, as commutative and associative, are much older.

Of course, the elementary pupil is not expected to prove theorems. However, when a pupil discovers that a number cannot be divided by 6 because it does not have a factor of 2, he is performing the kind of reasoning that is the basis for proof. The teacher should be alert for opportunities to encourage pupils to draw conclusions from known or accepted facts. Informal proofs may be desirable in the upper elementary grades (see p. 206).

LOGICAL DEVELOPMENT OF NUMBER SYSTEMS

Number systems do not spring into being as full-grown entities but evolve slowly over a long period of time to meet the needs of civilization. Our number system is the culmination of thousands of years of experience. The growth of our number system may be examined from a logical or genetic point of view. This section will deal with the logical development of the system and the next section with the genetic development.

A logical development starts with the simplest set of numbers (which must be described mathematically) and then proceeds, step by step, to extend this system into the one that we now know. The natural numbers are usually used as the foundation for this logical development. Some treatments begin with the whole (cardinal) numbers.

The discussion of the logical development will be carried out in terms of the properties (postulates) of a number field and how these properties or the lack of them affect the capabilities of the system.

The number field is defined in terms of two operations. Arithmetic for everyday purposes has four operations. This apparent contradiction is readily resolved by defining subtraction in terms of addition (and inverses for addition) and division in terms of multiplication and its inverse elements.[11]

The existence of inverse elements (numbers) in a number system determines if it is always possible to perform the operation of subtraction (closure with respect to subtraction). In the set of natural numbers (counting numbers) in which there are no inverses for addition, subtraction may be performed for $9 - 3$ but not for $3 - 9$. Subtraction can always be performed in the set of *integers* (see p. 83), where every number has an inverse for addition.

The natural numbers

The natural numbers were the first to be discovered by any civilization and are frequently called the counting numbers.[12]

The natural numbers are: $\{1, 2, 3, 4, 5, \cdots\}$. A graph of the natural numbers is given in Figure 6.1.

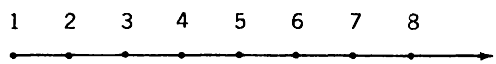


Figure 6.1

The properties of natural numbers given in Table 6.2 indicate that addition and multiplication can always be performed with the set of natural numbers

[11] Subtraction may be defined in terms of addition without the use of inverses and must be in systems, as the set of whole numbers, where inverses do not exist. The difference $a - b$ may be defined as a number c such that $b + c = a$. This definition is the basis for checking subtraction. When inverses do exist, the difference $a - b$ is then usually defined as the sum of a and the inverse of b (inverse for addition). The two definitions do not conflict and either can be proved from the other in a number field.

In the same way, $a \div b$ may be defined as a number c such that $a = bc$, or if inverses do exist, then $a \div b$ may be defined as a multiplied by the inverse element of b (for multiplication).

[12] Some authors include 0 as a natural number, but historically 0 has not usually been considered a natural number.

TABLE 6.2
Properties of the Natural Numbers

Operation	*Closure*	*Associative*	*Identity*	*Inverse*	*Commutative*	*Distributive*
Addition	Yes	Yes	No	No	Yes	Yes (multiplication over addition)
Multiplication	Yes	Yes	Yes	No	Yes	

but that subtraction and division cannot always be performed. The difference $2 - 6$ and the quotient $2 \div 6$ do not exist in the set of natural numbers. In traditional programs, pupils were frequently told that it was impossible to subtract 6 from 2. In a modern program, pupils are told that 6 cannot be subtracted from 2 in the set of natural numbers (or whatever set is involved at the time).

The natural numbers are more than adequate to meet the needs of a primitive civilization. These numbers even suffice for the everyday needs of somewhat advanced civilizations. The Roman system of numeration is a set of numerals representing the natural numbers, since 0 is not involved. This system met the needs of Roman merchants (with the help of an abacus or similar device).

When a civilization begins to advance in a technical sense, the natural numbers quickly become inadequate. When this stage is reached, a resourceful civilization extends the number system. In the logical development, the first extension yields the set of integers.

The integers

The set of integers is: $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$. A graph of the integers is given in Figure 6.2. The properties of the integers are listed in Table 6.3.

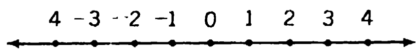


Figure 6.2

Every integer has an inverse element for addition. Pairs of inverse numbers are indicated on the graph in Figure 6.3.

The mathematical consequence of the existence of inverses for addition is that the set of integers is now closed

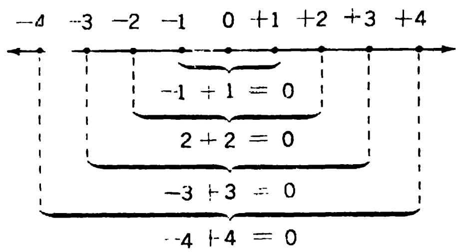


Figure 6.3

TABLE 6.3
Properties of the Integers

Operation	*Closure*	*Associative*	*Identity*	*Inverse*	*Commutative*	*Distributive*
Addition	Yes	Yes	Yes	Yes	Yes	Yes (multiplication over addition)
Multiplication	Yes	Yes	Yes	No	Yes	

TABLE 6.4
Properties of the Rational Numbers

Operation	*Closure*	*Associative*	*Identity*	*Inverse*	*Commutative*	*Distributive*
Addition	Yes	Yes	Yes	Yes	Yes	Yes (multiplication over addition)
Multiplication	Yes	Yes	Yes	Yes	Yes	

with respect to subtraction. The difference $2 - 6$ may be translated into the sum $2 + {}^-6$ or ${}^-4$. The difference $2 - {}^-6$ may be transformed into the sum $2 + 6$, or 8 (see p. 216).

The integers are not much more useful for everyday purposes than the natural numbers. Negative numbers have relatively few everyday applications.

From a mathematical point of view, the major deficiency of the integers is that division cannot always be performed, since $2 \div 6$ is not an integer. This fact provides the basis for the next extension to the rational numbers.

The rational numbers

Typical rational numbers are: $\{-\frac{3}{4}, -\frac{1}{2}, 0, \frac{2}{3}, 1, \frac{17}{3}, 203\}$. A graph of the set of rational numbers is given in Figure 6.4.[13]

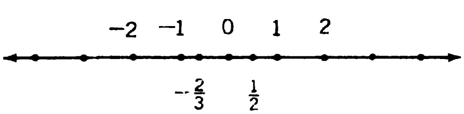


Figure 6.4

It is not possible to give definitions of the natural numbers or the integers suitable for the elementary level A *rational number* may be defined as the quotient of two integers with the divisor not equal to 0. In everyday language rational numbers are referred to as fractions.

The set designation given for both the natural numbers and the integers makes clear which numbers are and which are not members of each set. It is not possible to do this in the same manner with the rationals. In listing some typical rational numbers, as above, the listing of $\frac{17}{3}$ does not make it clear that $\frac{231}{418}$ is also a member of the set.

The graph of the set of rational numbers is deceptive. It appears to be a complete line because rational numbers are so "close" together (dense) that the physical points representing these numbers merge together to make an apparently continuous line. The graph of the rational numbers is actually full of "holes." One "hole" is at the $\sqrt{2}$. The proof that the $\sqrt{2}$ is not rational is now found in many secondary textbooks as well as in those designed for elementary school teachers.[14]

Table 6.4 indicates that the rationals form a number field. Every rational number except 0 has an inverse for multiplication. Therefore the rational numbers are closed with respect to division except for division by 0.

The graph in Figure 6.5 indicates some pairs of inverses for multiplication.

[13]It is not possible to label the points on the graph to make clear which numbers do and which do not belong to the graph.

[14]See F. E. Grossnickle, J. Reckzeh, and H. Bernhardt, *Discovering Structure in Algebra* (New York: Holt, Rinehart and Winston, Inc., 1962) p. 478.

TABLE 6.5
Properties of the Real Numbers

Operation	*Closure*	*Associative*	*Identity*	*Inverse*	*Commutative*	*Distributive*
Addition	Yes	Yes	Yes	Yes	Yes	Yes (multiplication over addition)
Multiplication	Yes	Yes	Yes	Yes	Yes	

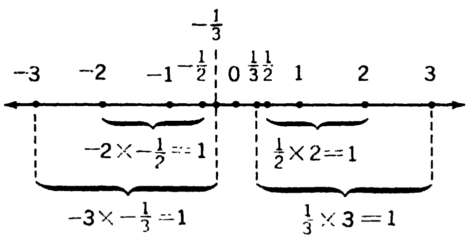


Figure 6.5

The rational numbers are closed with respect to all four binary operations except for division by 0. This must be true for any set of numbers meeting the conditions of a number field.

The rational numbers are adequate to meet almost all everyday needs of individuals and business in today's complex society, but the set does not suffice for many scientific and technical needs. Since the $\sqrt{2}$ is not rational, it is not possible to find a rational number to describe the length of the hypotenuse of a right triangle whose other two sides are of length (measure) 1, as shown in Figure 6.6. This fact leads to the next extension, the real numbers.

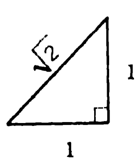


Figure 6.6

The real numbers

The real numbers include all the rational numbers (and therefore all of the integers and natural numbers). A real number that is not rational, as the $\sqrt{2}$, is an *irrational number*. The real numbers are obtained when the set operation of union is applied to the set of rational numbers and the set of irrational numbers. The real numbers may be partitioned into the set of rational numbers and the set of irrational numbers. The rationals and the irrationals are disjoint. Typical real numbers are $\{-3, -\sqrt{2}, -\frac{1}{2}, 0, 1, \sqrt{2}, 7, \frac{18}{7}\}$. A graph of the set of real numbers is given in Figure 6.7.[15]

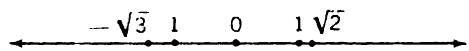


Figure 6.7

As Table 6.5 indicates, the real numbers also form a number field. Therefore the reals are closed with respect to addition, subtraction, multiplication, and division (except for division by 0). The operation of square root is always possible for all positive reals while it is not always possible for the rationals. The $\sqrt{-2}$ is not a real number, however, so some operations are still not possible in the reals.

While the set of real numbers is adequate for most technical and industrial needs today, it is not for some scientific purposes.

[15]This graph is without "holes." Every point corresponds to one real number and every real number corresponds to exactly one point.

The complex numbers

Typical complex numbers are: $\{-13, \sqrt{-1}, 2 + \sqrt{-1}, 0, \sqrt{7}, -5 + \sqrt{-13}\}$. A graph of the set of complex numbers is given in Figure 6.8.

Complex numbers are of the form $a + bi$ where $i = \sqrt{-1}$ and a and b are real numbers. The graph of the complex numbers covers the entire plane. The complex numbers include the real numbers (and therefore the rationals, integers, and naturals).

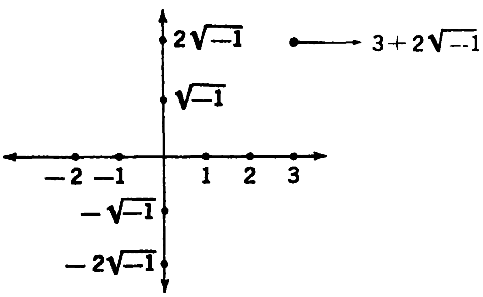


Figure 6.8

As Table 6.6 indicates, the complex numbers also form a number field. It is possible to find the square root of any complex number, but inequalities no longer always make sense in this system. It is not possible to decide in a consistent manner whether $3 - 2i$ is more or less than $2 + 3i$.

The complex numbers form the most complete system that mankind has been able to devise. Mathematicians have proved that any further extension of it will no longer be a number field. On the other hand, efforts to extend it have led to interesting mathematical discoveries.

Set diagrams

The set of natural numbers is a subset of the set of whole numbers, and the set of whole numbers is a subset of the set of integers. The set of integers is a subset of the set of real numbers, and the set of real numbers is a subset of the set of complex numbers. These facts can be effectively illustrated in set diagrams, frequently called Venn or Euler diagrams.

The set diagram in Figure 6.9(A) makes it clear that the natural numbers form a subset of the whole numbers, and so on. It is not clear in the diagram that rational and irrational numbers are disjoint. It should be noted that the irrationals are real numbers that are not rational and so are in the circle representing reals and outside the circle representing rationals.

It is clear in Figure 6.9(B) that the rationals and irrationals are disjoint, but shading in the space between the circles representing rationals and irrationals is necessary to indicate that no numbers exist that are real but neither rational nor irrational.

GENETIC DEVELOPMENT OF THE NUMBER SYSTEM

The genetic development of the number system is different from the logical development because the former is

TABLE 6.6
Properties of Complex Numbers

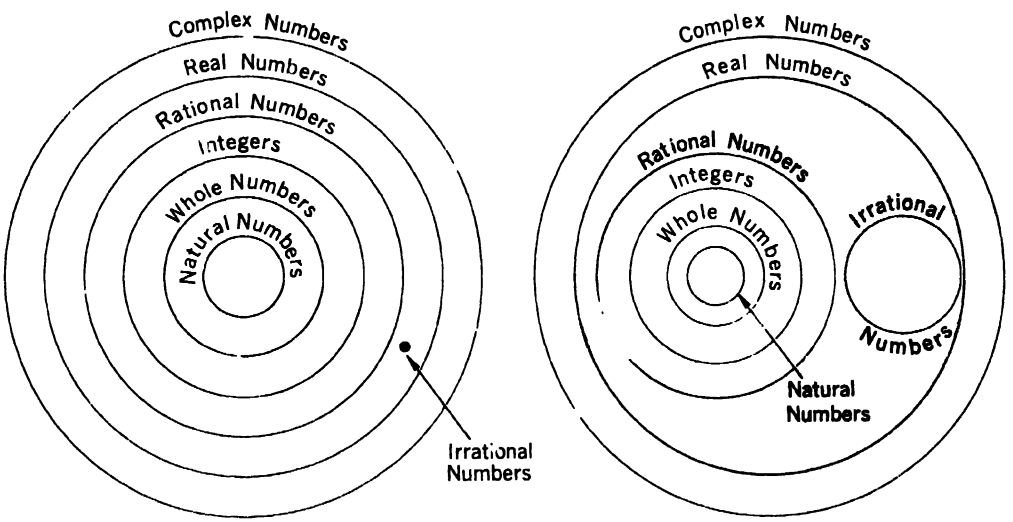
Operation	*Closure*	*Associative*	*Identity*	*Inverse*	*Commutative*	*Distributive*
Addition	Yes	Yes	Yes	Yes	Yes	Yes (multiplication over addition)
Multiplication	Yes	Yes	Yes	Yes	Yes	

guided by the everyday needs of society.

History indicates that developing civilizations have a need for fractions (rational numbers) long before they need negative numbers. The Greeks used both rationals and irrationals (positive) without using 0. Negative numbers were not accepted by all mathematicians as recently as 1800.[16]

The genetic approach may be different among different civilizations. The tendency seems first to extend the natural numbers to obtain the positive rationals (fractions) and then the positive reals. When the number 0 is annexed to the system, then these sets become nonnegative. The number 0 is neither positive nor negative. The annexation of 0 to the system is essential if a place system of numeration is to be used. Zero was used by Hindu mathematicians about 600 years after the birth of Christ, making possible the formation of the numeration system that traveled from India through Arabia to Europe in the Middle Ages. The extension from nonnegative reals to the entire set of real numbers occurs in the latter stages of a genetic development. The final extension to the set of complex numbers takes place only when a society becomes sophisticated in its scientific and mathematical needs.

The development of systems of numbers for pupils moving from the kindergarten to the senior high school follows the genetic development much more than the logical one. The following list



A Figure 6.9 B

indicates the approximate grade levels at which the various number systems predominate, although it must be clear that there is much variation from program to program as well as within any single program where attention is paid to individual differences.

[16] J. L. Kelly, *Introduction to Modern Algebra* (Princeton, N.J.: D. Van Nostrand Company, Inc., 1960), p. 51, footnote.

TABLE 6.7
Properties of Number Systems

	Closure	*Associative*	*Addition Identity*	*Inverse*	*Commutative*
Natural numbers	Yes	Yes	No	No	Yes
Whole numbers	Yes	Yes	Yes	No	Yes
Integers	Yes	Yes	Yes	Yes	Yes
Rational numbers	Yes	Yes	Yes	Yes	Yes
Real numbers	Yes	Yes	Yes	Yes	Yes
Complex numbers	Yes	Yes	Yes	Yes	Yes

[a] Distributive property of multiplication over addition.

Kindergarten	**Natural numbers**
Grades 1, 2, 3	Whole numbers
Grades 4, 5, 6	Nonnegative rationals[17]
Grades 7, 8	Integers and positive reals
Grades 9 10	Real numbers
Grades 11, 12	Complex numbers[18]

[17] The nonnegative reals include all real numbers that are not negative and therefore include 0, which is neither positive nor negative. This set is sometimes called the numbers of arithmetic, since it is this set of numbers that traditional arithmetic has been concerned with for hundreds of years.

[18] The complex numbers will be used as the universe mostly in courses designed for college-bound students.

It must also be recognized that even though a given set of numbers may predominate at a given stage, much readiness activity for the next extension may be taking place.

Table 6.7 summarizes the properties of the number systems under discussion. The last three sets in the table form number fields.

A modern approach to learning arithmetic and algebra demands a thorough knowledge of number systems and their properties.

EXERCISES

1. What are the two basic operations for dealing with numbers?
2. Enumerate the basic properties of systems of numbers.
3. Give several applications for the identity property for multiplication.
4. Which subset of the real numbers is closed for all four operations (except division by 0).
5. What is the intersection of the set of natural numbers and the set of whole numbers? the set of rational numbers and the set of integers?
6. If W = whole numbers, Ra = rational numbers, and Ir = irrationals, name the following sets:

   A: $Ra \cup Ir$ $\quad$ C: $Ir \cap Ra$

   B: $R - Ir$ $\quad$ D: $W \cap R$

		Multiplication			*Distributive*[a]	*Field*
Closure	*Associative*	*Identity*	*Inverse*	*Commutative*		
Yes	Yes	Yes	No	Yes	Yes	No
Yes	Yes	Yes	No	Yes	Yes	No
Yes	Yes	Yes	No	Yes	Yes	No
Yes	Yes	Yes	Yes	Yes	Yes	Yes
Yes	Yes	Yes	Yes	Yes	Yes	Yes
Yes	Yes	Yes	Yes	Yes	Yes	Yes

7. Which sets of numbers form a number field?
8. Give a precise statement that the system S has the inverse property for addition.
9. What conclusion can be drawn when it is known that a number system does not have the inverse property for multiplication?
10. Which set of numbers has all the properties for a number field except one?
11. Which number property is defined in terms of two operations?
12. Which of the following are binary operations:

    a. Addition c. Squaring

    b. Square root d. Multiplication

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MATHEMATICS IN THE KINDERGARTEN AND PRIMARY GRADES[1]

A number of changes are taking place at the present time in the mathematics curriculum of the kindergarten and primary grades. These changes reflect in large measure the pressures that are being placed on the elementary school by the secondary schools and colleges. Perhaps the chief reason for these changes is the realization that the work in the primary grades must prepare students for future work in mathematics as well as for dealing effectively with the quantitative situations that arise both in school and outside school.

The teacher faces the problem of how to merge the best of well-established practices with the new ideas that are being introduced into the mathematics program. In the past many of the topics now taught in the kindergarten and grade 1 were deferred to grades 2 or 3. Recent investigations have indicated, however, that the child in the beginning grades can not only learn new mathe-

[1]The authors are grateful to Dr. Mary Grau, Superviser of Elementary Education, Montgomery County, Maryland, for her helpful suggestions concerning this chapter.

matical ideas but can also apply these ideas to increase his understanding of mathematics.

Many of the new ideas are connected with mathematical structure, as indicated in Chapter 6. The idea of sets, which is fundamental at all educational levels, is first developed in the kindergarten and then expanded at higher levels. The properties that deal with operations with whole numbers are later shown to apply in operations with fractional numbers and decimals. Geometric ideas of a very elementary kind are being introduced in the kindergarten, and new approaches are explored as the child progresses through school. A spiral organization of the curriculum insures that these ideas recur at frequent intervals. Thus the pupils come to have an understanding of the structure of mathematics and learn mathematical procedures of increasing complexity and difficulty.

This chapter considers the following topics: the mathematics program, K–2; sets, learning to count, read, and write numerals, place value, readiness for the basic facts; beginning geometry.

THE MATHEMATICS PROGRAM, K–2

During the years 1920–1955 many schools eliminated arithmetic from the curriculum of the kindergarten and grades 1 and 2. This policy resulted from the false notion that young children could not learn arithmetic. It has since been proved that children enter school with a considerable background of number concepts gained through experiences in their environment. These experiences are derived from play, from asking questions and listening, and from experimenting. Many of the first ideas of size and quantity, however, are vague and sometimes inaccurate.

A strong foundation in mathematics can and should be laid in the kindergarten. This section presents an outline that the authors regard as a tentative, workable program for grades K–2 that is in line with the most recent changes that are being made in the mathematics curriculum at the primary level. The authors have taken into consideration such experimental programs as the SMSG, the GCMP, and the Stanford experiments, as well as the most recent and up-to-date courses of study and textbooks. While these programs differ in some respects, the outline that follows is a conservative consensus of these practices and represents the view of a large number of curriculum-makers and classroom teachers what to teach.

Number experiences[2]

Kindergarten

*1. Observing and describing sets of objects, models, and pictures

*2. Comparing sets of objects (more, less, same)

*3 Joining and removing sets of objects

*4. Ordering sets of objects (larger, smaller, same)

5. Recognizing and comparing geometric shapes (line, square, triangle, circle)

*6. Associating numbers with sets of objects and shapes

7. Recognizing and reading numerals as such from 1 to 12

8. Counting to 29, rote and with objects

9. Familiarity with the clock face, calendar, money, liquid measure, ruler.

[2]The items preceded by asterisks may be regarded as the new mathematics.

Grade 1

*1. Set concept reviewed and extended

2. Reading and writing digits to 10

3. Number ray—counting, sequence, and order

4. Equalities and inequalities

5. Preaddition and subtraction experiences—facts having sums through 12; finding sums and differences

6. Numerals and place values to 100

7. Addition of numbers named by two-place numerals—no regrouping

8. Subtraction of corresponding examples in (7)

*9. Commutative property; identity element for addition

10. Problem solving based on pictures, drawings, and classroom situations

11. Measurement—units used on the measuring devices listed under K9

12. Geometric concepts—analyzing and comparing figures (see K5).

Grade 2

*1. Extending set concepts; number ray, equalities, and inequalities

*2. Using commutative and associative properties

3. Addition and subtraction facts having sums through 18

4. Place value—renaming numbers in expanded form

*5. Equation format for problems

6. Addition of numbers named by two-place numerals—regrouping

7. Subtraction in the corresponding examples in (6)

*8. Addition and subtraction as opposite operations

9. Premultiplication experiences

10. Predivision experiences

*11. Geometric concepts

*a. Analyzing and comparing geometric figures

*b. Line segments, rays, angles

*c. Congruent shapes and forms

Informal but planned program

The learning program for the kindergarten and grade 1 should be informal but it should be carefully planned to lead children to successively higher levels of understanding and abstraction. Various levels of development may be observed even though learning is a continuous process. Five levels of growth may be identified as the child learns about number and numeration with the set of whole numbers. These levels, with their characteristic behavioral responses, are as follows:

Level I.	Perceives mathematical ideas and symbolic forms
Responses	Arranges a set of various-sized disks in order Shows four fingers for four years
Level II.	Discovers and relates mathematical meanings
Responses	Sees a common characteristic in a set of objects Associates the number property "four" with any set of four elements and names the number property "four"
Level III.	Translates understandings into mathematical models
Responses.	Recognizes, reads, and writes the symbol 4
Level IV.	Applies mathematical generalizations to problem solving
Responses:	Identifies house number 4 Counts to determine the size of a set Numbers pages
Level V.	Creates, constructs, and uses mathematical models
Responses:	Names "four" in several ways, as $2 + 2, 3 + 1$, or $7 - 3$ Uses numerals to identify places and objects

While these levels describe the sequence of learning, the individual growth patterns of children vary greatly.

Some pupils will need to move in very brief steps with many varied repetitions. Others will gain an insight into complex relationships and will seem to accomplish several learnings simultaneously. To meet these differences, the teacher must carefully observe the pupils' methods of work and their responses. Some of the ways to determine the pupils' number readiness are:

1. Observing the ways the pupils use number in informal situations
2. Observing their performance and interest in number games
3. Recording questions pupils ask and the comments they make about number
4. Recording number vocabulary used by pupils
5. Making an inventory of behavioral responses.

Basic concepts

A significant number of mathematical concepts may be identified as a basis for building an instructional program for beginning pupils. Mathematical ideas would include the concept of sets; the ideas of more than and less than; the concepts of cardinal and ordinal numbers; the concept of order; the idea of subsets, equivalent sets, and empty sets; the idea of joining and removing sets; numerals and number names; counting by ones and twos; the number ten; the concept of grouping by tens; the meaning of two-digit numerals; the concept of place value for numbers less than 100; concepts of inequalities; the concept of addition; equality and equations; geometric concepts of space, point, line, and geometric figures. The sequence of activities that develop concepts and understandings indicates the logical order of their use. The activities for one concept should be used as sequentially as possible even though experiences associated with other concepts may vary from day to day. The teacher must determine which activities are appropriate for kindergarten, while using many incidental experiences to foster growth in mathematical ideas, and emphasizing a structural and sequential approach.

SETS

Young children have prenumber ideas based on their perception of objects in the environment. As has been shown, with very young children, contacts with objects are physical and manipulative, with little regard for their quantitative aspects. Later children begin to play with objects of various kinds, such as blocks and toys, and to separate them into groups. Still later they learn to identify small groups of objects and to name the number in the set. Young children learn how to recognize groups up to 4 without counting them by the time they enter school. The mathematics program should first build on physical activity, such as the manipulating and grouping of objects. As children mature, the class work should deal with pictures of objects and sets of abstract geometric designs. A suitable sequence of activities related to sets in the mathematics laboratory is as follows:

1. Observing and describing sets of objects; for example, the child can *describe* a doll, a ball, and a top as a set of toys.
2. Comparing sets of objects. Comparison of sets involves the concepts of more, less, and the same, with and without counting. When the child is shown three sets of different sizes, he can learn to identify the largest, the smallest, and the in-between set. No counting is involved except of an intuitive kind. Matching elements is a good way of making comparisons of sets.

3. Joining and removing sets. The young child joins two small sets and sees intuitively that the new set is larger than either of the given subsets. The basic idea here is addition. Likewise the pupil can be led to remove some (a subset) of a set of objects. The basic idea is subtraction.

4. Ordering sets. The child arranges three given sets of 1 to 4 objects in order of size, without counting any of the groups directly (see Fig. 7.1).

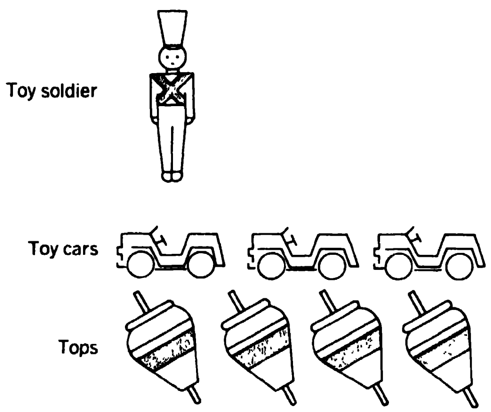

Figure 7.1

5. Associating numbers with sets of objects. To this point the child has not been expected to count to tell how many. Now he learns to count elements one at a time. This is not difficult for a majority of children in a typical kindergarten class. It has been found that the average child on entering grade 1 can count up to 30 objects.[3]

6. Associating a line with the idea of a set of points. A point has location but no dimensions. A line or a line segment is an infinite set of points.

7. Learning how to show a pattern of a set (dominoes).

Many classroom objects can be used to give pupils experience in counting or grouping sets, for example:

blocks	erasers
books	marbles
buttons	milk bottle tops
chairs	pebbles
children	pencils
clothespins	seeds
crayons	small geometric patterns
desks	soda straws
disks	toys (animal)[4]

A flannel board can be used to good advantage in working with sets.

Knowledge of mathematical concepts of preschoolers

The following references are valuable in determining the extent to which preschool and kindergarten children have knowledge of mathematical concepts. Some striking information is reported in these studies. Young children have been found to know much more about mathematics than is usually believed to be true.

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[4]Elda Merton and Lola May, *Mathematics Background for the Primary Teacher* (Wilmette, Ill.: John Colburn and Associates, 1966).

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The development of number concepts in the kindergarten begins with the idea of sets. The concept of sets is familiar to most pupils at this age level as they play with sets of blocks, choose sets of pictures, arrange their toys into sets, and form other sets in a similar manner. Many experiences must be provided, however, in order to clarify number ideas. One of the most important is matching sets.

Matching sets

The pupil may show the one-to-one correspondence of the members of two sets, as illustrated, in Figure 7.2, or he may reproduce a number of markers equal to the number of elements in a given set. Often the pupil must match the members of a set by one-to-one correspondence by stating, "One for you, one for me," by matching objects, as in placing a straw for each carton of milk and then answering the question, "Do you have enough straws or too many?" Through these one-to-one matching activities pupils discover the idea that the number of elements in two matching sets is the same (see Chap. 4).

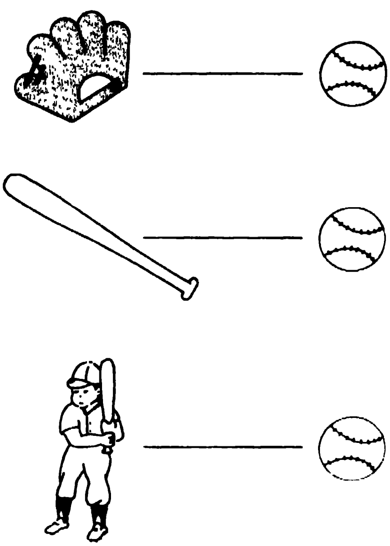


Figure 7.2

When counters of different-sized sets are distributed to each pupil the matching game readily leads to more or less and greater than and less than. Pupils recognize that sets differ in size. When a pupil states that one set has more objects than another, he is recognizing that the cardinal number of one set is greater than the cardinal number of another set. The comparison terms introduced in these early experiences refer to relations. Pupils are not expected to master the terminology; the most important aspect is a clear understanding of the ideas and concepts. Sets of counters of various types, flannel board cutouts, and magnetic cutouts are excellent materials to provide experiences in matching sets. Each child should have his

own set of counters with which to explore ideas. Children should be given the opportunity to demonstrate sets that contain more or less than the sets the teacher describes or demonstrates; for example:

Make a set that is more.
Make a set that is less.
Make a set that is 1 more; 1 less.

Many games provide an opportunity for working with sets. A collection of toys may be placed on the floor. Send one child away. Remove some toys. When the child comes back say, "Toys, toys on the floor, are there less, are there more?" The first-stage answer is "more" or "less"; later the child tells how many toys were taken away and how many are left. The number of toys in the set would vary from time to time.

Number of a set

After the pupil uses one-to-one correspondence to match sets, he should advance to the next step in exploring sets. He should discover that sets *A*, *B*, and *C* all have something in common:

A: {hat, coat, shoe, dress}
B: {book, candle, desk, basket}
C {John, Mary, Ruth, Bill}

The sets can be matched with each other, as sets *A* and *B*, *A* and *C*, and *B* and *C*. These sets are equivalent (not equal) and they can be matched with other sets that contain the same number of elements. Every corresponding set, such as the set of wheels on a car, the set of legs on animals, and the set of sides of a square, has the same property of fourness. Therefore we designate this property as the number 4.

Just as there are sets that have the property of fourness, there are sets that have a common property of fiveness, threeness, or any other number. In order to express this property, we need names and symbols. The name for the characteristic of fourness is four and the symbol is 4. (See p. 100 for activities that help the pupil understand the meaning of four.)

Order

A teacher may have a set of name cards for each pupil of the class. These cards may be arranged at random on the teacher's desk. If the teacher plans to refer to these cards, they should be arranged in some orderly sequence. The cards may be given a number or they may be arranged alphabetically. The teacher should have some distinguishing way of identifying the position of the cards in the set for future reference.

In the same way that a set of name cards is arranged in sequence, numbers are arranged in order—each succeeding whole number increases by 1. Thus any set in one-to-one correspondence with a set containing a single tally stroke has the number 1, the first natural number. Similarly, any set in one-to-one correspondence with a set having a pair of tally marks has the number 2, the second natural number. In the same way each succeeding number designates a set that has one more element than the previous set. The pairing for the first five numbers is shown at the right. Such matching is of value to show the symbols and sequence for the first 10 numbers. These symbols or numerals are basic in our decimal system of numeration.

I	1
II	2
III	3
IIII	4
~~IIII~~	5

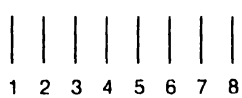
LEARNING TO COUNT, READ, AND WRITE NUMERALS

What counting means

When a pupil is able to match sets, identify a number name, and give the order of numbers, he has the background for counting. He may identify the number of elements in a small set without counting, as in recognizing a set of 4 or 5. A pupil would need to count to identify sets having more than 4 or 5 elements.

Counting is a means of describing a quantity. Primitive man had to represent the number of sheep in his flock by showing a set of pebbles or other objects. One pebble or object matched or corresponded to each sheep. If he had been able to count, he could have stated that his flock contained 16 sheep or *n* sheep. Instead of pairing a sheep with a pebble, he would have paired a sheep with a number. The basic concept conveyed by counting consists in pairing or matching an object with a number.

We can find the number of squares in the set by pairing a number with each square:



The counting numbers are ordered, and therefore we begin with 1 and continue counting in sequence. It is significant to note that the number name of the last element of a set is the name of the number in that set. The order in which the elements are counted does not affect the number in the set. Regardless of the sequence of counting objects, the name of the last number counted will always designate the number of the set.

Classroom activities involving counting

Many classroom activities afford opportunities for counting. These activities should be directed at helping the pupil develop the essential understandings for rational counting, such as matching names in a one-to-one correspondence with objects counted and learning the order of number names.

Number games and singing games aid pupils in memorizing number names in order or sequence. Examples of such games are "One little, two little, three little Indians" and "One, two, buckle my shoe." The teacher should understand that games of this type help the pupil to remember number names and order and not what the numbers mean. Counting to find how many involves knowing the order of the number names and matching the names in a one-to-one correspondence with the object counted, as in counting the number of beads on a string: "One, two, three. I have three beads on my string." Only sets of 1 to 10 are used at this stage of learning.

Picture line

A picture line aids pupils in learning to count and in understanding cardinal and ordinal concepts. A picture line consists of a set of pictures, each about 9 by 12 inches in size, which can be arranged in a line on the floor. Rules are established for working with the picture line. Pupils are familiar with a starting point when playing games on the playground. A starting point is established for the picture line, which may be indicated with a piece of paper or any suitable device. Activities with the picture line are related to the starting point (see Fig. 7.3). Activities that provide counting experience include following directions such as:

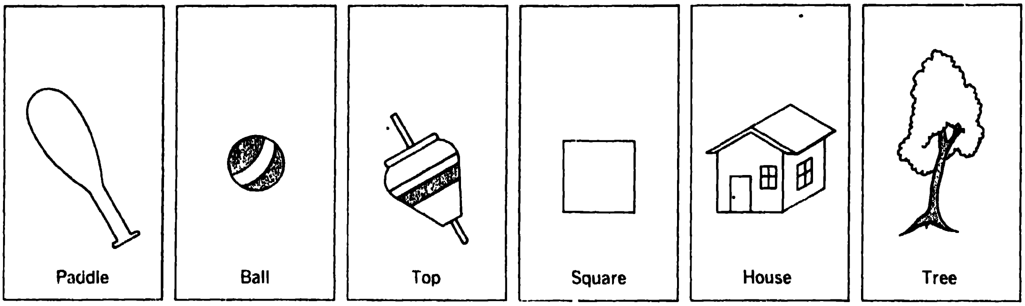


Figure 7.3

Take three steps on the picture line. Where are you?

Take four steps. Where are you?

How many steps must you take to reach the house?

Ordinal and cardinal numbers

Numbers are used in two ways, designated by the terms "ordinal" and "cardinal." A number has a position or order in the number scale and also a frequency. The term "ordinal" refers to the position or order of a number in relation to other numbers. Thus 4 is the fourth number in the number series. It comes after 3 and before 5. When time is expressed as 4 o'clock, the 4 represents a usage of ordinal number. The question, Which one? calls for the use of an ordinal number. The cardinal value of a number represents the number of a set, as 4 names a set of 4 objects. A cardinal number answers the question, "How many? If an object is fourth in line, the set must contain at least 4 elements and there must be 3 elements ahead of the fourth. The 3 is a cardinal number. The cardinal value of the *empty set* is 0. The empty set is the only set that has no members, and the number naming the set is 0 and should be so designated.

A picture line offers an effective means of showing the cardinal and ordinal values of a number. These values are developed by having the pupil dramatize or show the answers to such questions as the following:

Which is the first picture?

Which is the last picture?

Walk to the second picture.

Walk to the last picture.

Take eight steps forward and three steps back. Where are you?

Take three steps two times. Where are you? How many steps did you take altogether?

John, take two steps. Fred, take three steps. Who is farther from the starting point? by how many steps?

Activities of the type described help the pupil to become familiar with the following:

1. Both cardinal and ordinal value of a number
2. Counting
3. A reference point on a line
4. Movement (direction) on a line.

Names and symbols for numbers

If children have had a rich background of experiences in working with sets of physical objects and can name the number of a set, they will have little difficulty with the symbols (numerals) that name the numbers. Many children can recognize a numeral without understanding the number concept that it names. The introduction of the numerals, 1 to 10, should be closely associated

with sets of counters. A set of 10 digits in flannel or magnetized cutouts may be arranged on a flannel board or chalkboard (see Fig. 7.4). As a child demonstrates a set and names the number associated with it, have him find the numeral that names his set.

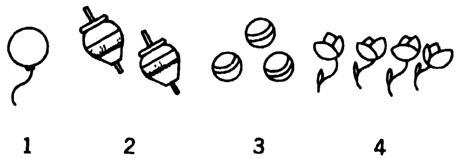


Figure 7.4

Each pupil should demonstrate the number of a set with markers. The teacher should then have the class make a representation of that set on an 8 by 10 inch card. The card should contain pictures of objects or geometric designs to show the cardinal value of the number, the number named, and the symbol or numeral (see Fig. 7.5).

The 10 cards designating the digits should be included in a chart to be displayed on the bulletin board or arranged in sequence in some other prominent place in the classroom.

A second chart should be used to enable the pupil to recognize the number in a set without counting each object. The pupil should recognize the number of a set by the arrangement of the objects. Figure 7.6 shows the type of chart to use to aid the pupil in identifying the number in a set. The pattern of the arrangement of the symbols should enable the pupil to name the number of the set without counting by ones. Thus the child can learn quickly to recognize 4 as 2 twos; 6 as 3 twos or 2 threes; 8 as 4 twos or 2 fours; 9 as 3 threes; and 10 as 2 fives.

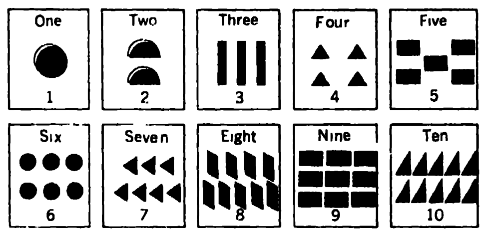


Figure 7.6

An interesting matching game can be used to provide practice in recognizing names, groupings, and symbols of the

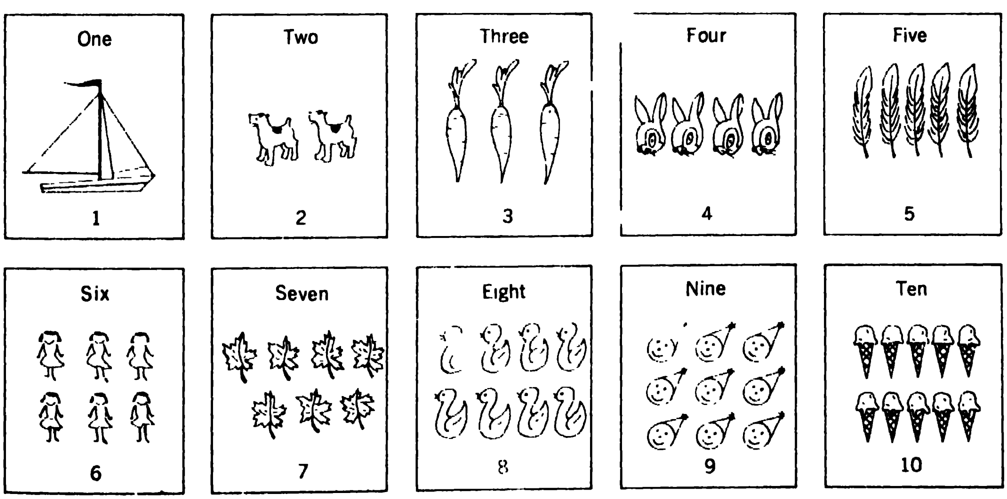


Figure 7.5

numbers from 1 to 10. The teacher should prepare three sets of 4 by 6 inch oaktag cards. On one set of 10 cards the 10 number names should be printed, on the second set the 10 number symbols, and on the third set pictures representing each of the numbers. The children should then match the three sets of cards to see how well they understand the three ways of representing the numbers from 1 to 10.

The children may also play games in which they use the cards to compare numbers. For example, two children may draw single cards from one of the sets. Each child reads his number. Then one child says, "My number is 4. John's number is 2. My number is larger than John's." Later he may be able to say how much larger (or smaller). In another game three children may play. Each draws a single card from one of the three packs. One child tells what the number is on his card. The child who has the largest number is "it" and places the other cards in his pile.

Varied activities reinforce the recognition, reading, and matching of numerals with a set. Activities of the type that follow are effective in achieving this goal:

1. Reading a numeral and arranging a set of counters to match the numerals.

2. Playing the games of matching picture cards with numeral cards, as described before.

3. Drawing sets of objects to match the numeral (see Fig. 7.7).

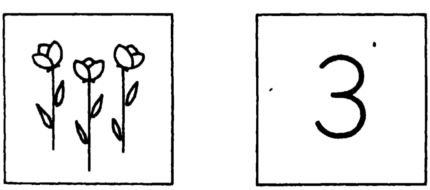


Figure 7.7

4. A set of numbered cards may be arranged in order on the demonstration board.

$\boxed{0}\ \boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8}\ \boxed{9}$

a. Name the number that comes before and after a given number:

$4\ \boxed{}\ 6\ \boxed{}$

b. Name the number that comes between two numbers:

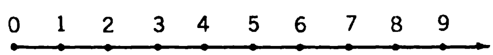
$5\ \boxed{}\ 7$

c. Arrange the cards in order:

$\boxed{5}\ \boxed{3}\ \boxed{4} \rightarrow \boxed{3}\ \boxed{4}\ \boxed{5}$

5. A review of the walk-on picture line provides a good basis for introducing the number ray. The teacher may draw a ray on the chalkboard with points equally spaced and marked. The children decide upon a starting point and relate it to the starting point on the picture line. The starting point is named zero (0) because we have taken no steps on the line. The numerals are then matched with the points on the line. Emphasis must be placed on the idea that we are counting the steps from zero.

6. A wide variety of exercises may be used with the number ray.



a. Decide which direction to move on the number ray when counting forward or backward.

b. Name a number that comes before or after a given number.

c. The numerals on the number ray may be matched with pictures of sets (see Fig. 7.8).

7. Children have experience in counting by following the numbered dots to draw a picture (see Fig. 7.9).

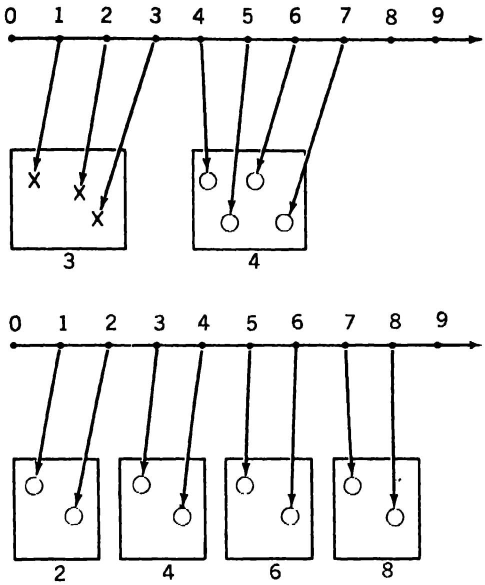


Figure 7.8

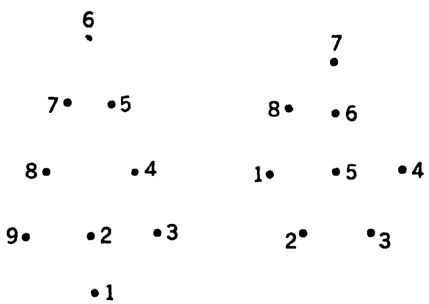


Figure 7.9

After children have gained facility in reading the numerals, writing the numerals should be included in handwriting lessons. At this stage only the numerals 0 to 9 are presented.

Learning to write the number symbols

Since almost all schools now teach manuscript rather than cursive writing in grades 1 and 2, this method should be used at the start in writing number symbols. The child is not ready for the finer muscular movement that is required in cursive writing. Learning to write numerals requires careful teaching and guidance, and much practice. The teacher should prepare large cards on which the number symbols to be written are placed before the children as models. The characteristics of each number symbol should be discussed when it is presented. The children should note where to start and stop their writing and the general size and shape of the different parts of the more difficult digits, especially 2, 3, 4, 5, and 8. The children can trace these models to get the feeling of the digits presented. The teacher should also prepare a special set of numerals $\frac{1}{2}$-inch wide made of emery paper for children who have difficulty in getting a correct mental image. The children can trace these numerals with their forefingers. This kinesthetic experience is very helpful in such cases.

An adaptation of the Fretsaw figures is also an effective means of teaching the slow-learning pupil to form the different digits. The Fretsaw figures, which are made of bakelite, fit into a frame or mold. Each figure may be removed from the matrix and the pupil may trace the outline of the digit with his finger. In this way both the kinesthetic and visual senses are used to learn the format of a given digit.

The authors have had many of their students who are preparing to teach in the kindergarten or primary grades make adaptations of the Fretsaw figures by cutting the digits 1–10 from $\frac{1}{2}$-inch or $\frac{3}{4}$-inch boards. These models, about 6 inches high, are suspended from hooks along a 1-inch-square beam, as in Figure 7.10. Each model corresponds to a number. The teacher should remove the models that identify the digits as soon as the pupils have learned the sequence of the first 10 numbers. Pupils

who are slow in learning how to write the numerals are usually helped by using manipulative materials of this kind.

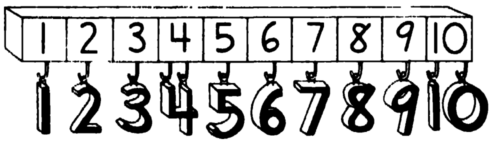


Figure 7.10

Tracing numerals

The children should first trace several dotted numerals on a sheet of paper with their fingers, then with their pencils, following the directions stated by the teacher. The numerals should appear on the paper with an "x" showing where to begin each stroke, and arrows should show the direction in which the pencil is to move (see Fig. 7.11).

Several children may be called on to write a numeral on the chalkboard. They should write the numeral several times, first aided then unaided by broken lines. Only one new numeral should be presented at a time. After sufficient practice in writing the new numeral, the child should write all of the numerals he has learned. The teacher should help any child who appears to be having difficulty by guiding his forefinger as he traces over the kinesthetic emery numeral.

Difficulties in writing numerals

The teacher should bear in mind that children may not distinguish the different number patterns and may not even see them correctly. For example, if a child cannot distinguish between 3 and 5 or 6 and 9, he is certain to have incorrect mental images that confuse him when he tries to read or write these number symbols. When the teacher suspects there is confusion, the child should be asked to point to or write the numerals that the teacher names. Prompt attention will clarify the pupil's understanding of the meaning of numbers and correct the errors he makes in writing the numerals.

Certain number symbols seem to give special difficulty. Children sometimes write a 5 as a capital S and an 8 as two circles that are not connected. Sometimes children start the 7 and the 9 on the base line and use an upward movement. Sometimes they start the numerals at the wrong point or construct them backwards. Children often write certain numerals in reverse form. Thus 3 is often written like a capital E. The numerals 2, 5, 6, and 7 are also often reversed. This tendency is similar to reversals in reading. The treatment of such difficulty consists largely in making certain that the child acquires a correct sense perception by careful analysis of the numeral and guided tracing of the numeral made with emery paper. In severe cases the child may even be shown how to make only part of a numeral at a time and practice it, such as ɔ, ∠, ʖ, ?, and °, and then to write the whole numeral.

The steps in learning to read and write numerals are summarized in Figure 7.12.

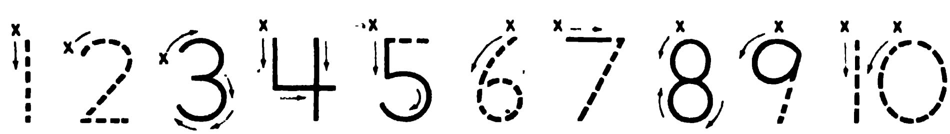


Figure 7.11

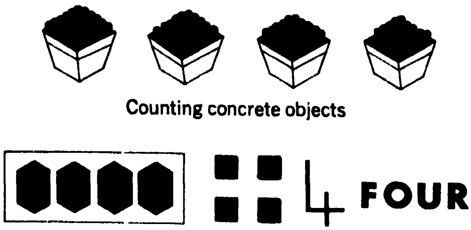


Figure 7.12

Counting to 20 and writing the numerals

The children are now ready for controlled counting to 20 and for writing the numerals. First the teacher should provide 20 markers to be counted and if possible a fact finder frame with 20 buttons. The children should then count the markers by ones to 20 with the teacher to be sure that the correct words are used in the right sequence. Several children may then be asked to count the objects. As much counting as is necessary both individually and in unison should be done until the children can say the numbers in correct sequence without reference to the object.

To guide the work in the writing of the numerals from 11 to 20, the teacher should use the models printed in a pupil's workbook or prepare a strip showing the numbers which can be placed on the chalkboard for reference. Patterns may also be prepared by the teacher for the children to trace when there is difficulty in writing.

The teacher should note that the number names for part of the teens do not follow the regular pattern for naming numbers. Thus the names eleven and twelve have no identifying relationship with the other number names in the teens. If the regular pattern for number names were followed, the name for 11 would be "one-teen" and for 12 "two-teen." The regular pattern for number names in the teens is not followed until 16. In the teens the names of the units' digit precedes the name that identifies the decade, as in *six teen*. In all of the higher decades, the name designating the decade precedes the name of the digit in units' place, as twenty-four for the numeral 24.

With the help of the teacher the children can construct a number table such as that of Table 7.1. The children will discover the orderliness in the sequential arrangement of the numbers. This arrangement is a characteristic of our system of numeration.

The number table can be used in many ways to give practice in counting and in comparing numbers. Another form of a hundredboard is illustrated in Figure 7.13. Each printed numeral is covered with a cardboard disk. The pupil does not see the numerals as he counts. When he finds a given number by counting the blank disks on the chart, he should remove the last disk counted and check with the printed numeral given beneath it. In this way he can verify his counting.

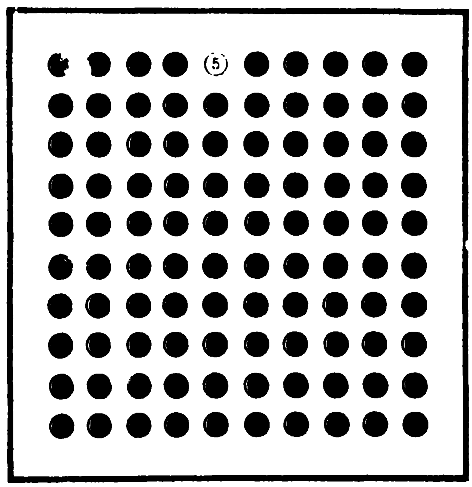


Figure 7.13

TABLE 7.1
A Number Table

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

If the pupil discovers the pattern of the sequence of numbers to 100, he should find it easy to count beyond that number. He should also be able to appreciate the poem by Eleanor Farjeon:

Numbers[5]

There are hundreds of Numbers. They
  mount up so high,
That if you could count every star in the
  sky
From the Tail of the Bear to the Water-
  man's hat,
There still would be even more Num-
  bers than that!

There are thousands of Numbers. So
  many there be,
That if you could count every drop in
  the sea
From the Mexican Gulf to the Lincoln-
  shire Flat,
There still would be even more Numbers
  than that!

There are millions of Numbers. So many
  to spare,
That if you could count every insect in
  the air,
The moth, the mosquito, the bee and the
  gnat,
There still would be even more Numbers
  than that!

There's no end to Numbers. But don't
  be afraid!
There only are ten out of which they are
  made,
Learn from Nought up to Nine, and the
  rest will come pat,
For the number of Numbers all come
  out of that!

[5]"Numbers," from *Poems for Children* by Eleanor Farjeon. Copyright, 1938, by Eleanor Farjeon Published by J. B. Lippincott Company. Reprinted by permission of Harold Ober Associates, Inc. Copyright 1938 by Eleanor Farjeon, copyright renewed.

PLACE VALUE

The abacus

When the child can count by ones to 20 and read and write the numerals, he is familiar with the sequence of the numbers. Not only must he know the sequence of the numbers, however, he must also know what each digit in a two-or-more-digit numeral represents. An abacus is an effective instructional aid for teaching *place value* in a numeral.

An abacus for teaching place value with two-place numerals should contain two rods with 10 beads on each rod. Nine beads on the rod in ones' place should be the same color. The tenth bead should be the color of the first nine beads on the rod in tens' place. If the first 9 beads on the rod in ones' place are white and the tenth bead is red, the first 9 beads on the rod to the left should be red with the tenth bead a different color. A color scheme of this kind is effective for showing that 10 ones have the same value as 1 ten. The tenth bead on a rod has a different color from the other beads on that rod. When that bead is represented on a rod, the place that rod holds is overloaded. If the ones' place is overloaded, the 10 beads are removed from the rod and 1 bead is shown on the tens' rod. Since

the color of the tenth bead on the ones' rod is the same as the color of the first 9 beads in tens' place, the color pattern suggests that 10 ones should be regrouped as 1 ten.

The pupil would show 1 bead on the rod in tens' place to enable him to represent the other numbers in the teens. He would indicate 1 bead on the rod in tens' place to represent 10 and 4 beads on the rod in ones' place to represent 14, as shown in Figure 7.14. For each succeeding number he would indicate a bead in ones' place and continue that pattern until all 10 beads are shown on the rod in ones' place. This place would be overloaded, so he would regroup as before and show two beads on the rod in tens' place. He would follow that pattern to represent any two-place numeral.

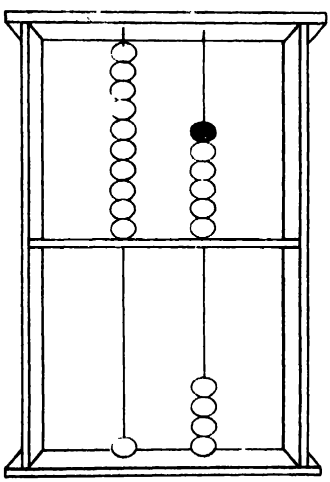


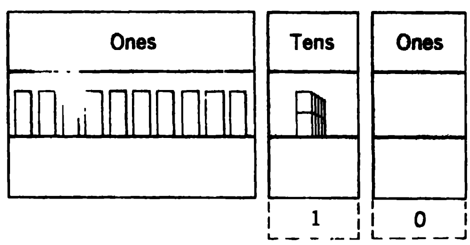
Figure 7.14

Place-value charts and other teaching aids

A place-value chart is an effective instructional aid for showing the meaning of a two-place numeral as well as for demonstrating regrouping in the operations of addition and subtraction.

To show the meaning of the number 10, the teacher should use tickets to show that the 1 means 1 group of ten. Then the teacher should tell the class that the 1 is written in tens' place to show 1 ten and that there are not any ones to write in ones' place. The 0 holds ones' place and keeps the 1 in tens' place. We therefore call 0 a *place holder.* Figures 7.15–7.18 show how to teach the meaning of the number 10.

First the teacher places 10 tickets side by side in the ones' pocket (Fig. 7.15, A), then removes the tickets one by one from the pocket, fastens them with a band to make a single group of 10 cards, and then places this bundle in the tens' pocket, which is at the left of the ones' pocket (Fig. 7.15, B). There are then no tickets remaining in the ones' pocket. Similarly, the teacher can use tickets to show that 11 means 1 group of ten and 1 one, that 14 means 1 group of ten and 4 ones, and that 20 means 2 groups of tens and no ones. With bundles of tens and single tickets, the children can learn to show any number from 1 to 99.



A B

Figure 7.15

The concept of 0 as a place holder is not an easy one for children to grasp. They must have many experiences with numbers in which there are zeros before they understand this function of 0. Other ways of using markers to show the meanings of the number 24 are illustrated on page 106.

1. Markers arranged in rows

2. Bundles of 10 dowel sticks and single sticks (see Fig. 7.16)

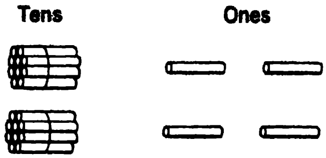


Figure 7.16

3. Squares—strips of 10 and single squares (see Fig. 7.17)

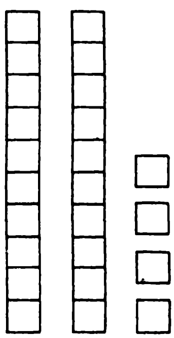


Figure 7.17

4. A hundredboard (see Fig. 7.18).

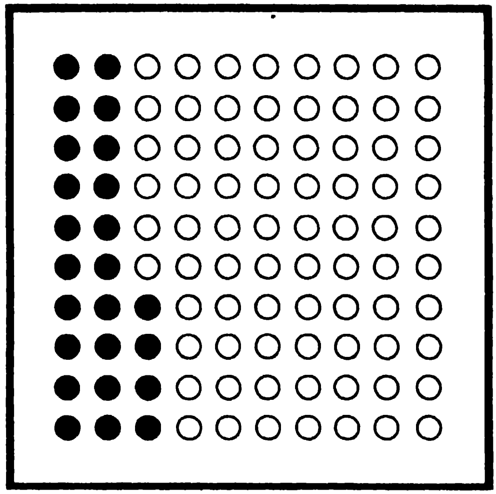


Figure 7.18

Using the number table to make discoveries about place value

The teacher should use the number table (see page 104) to help the children discover relations among the numbers. Some of the generalizations the children can make are the following:

1. The digit in ones' place is the same in each column.
2. The tens' digit in each line is the same, except in the last number in the line.
3. Each succeeding number in a line increases by 1.
4. Each succeeding number in a column increases by 10.
5. The smallest two-place number is 10, and the largest two-place number is 99.
6. 30 is 3 tens; 40 is 4 tens; 100 is 10 tens.
7. 25 is 2 tens and 5 ones; 30 is 3 tens and no ones.
8. 29 is 1 less than 30.

The teacher can use the table to teach the children to count by tens, by twos, and by fives.

READINESS FOR THE BASIC FACTS

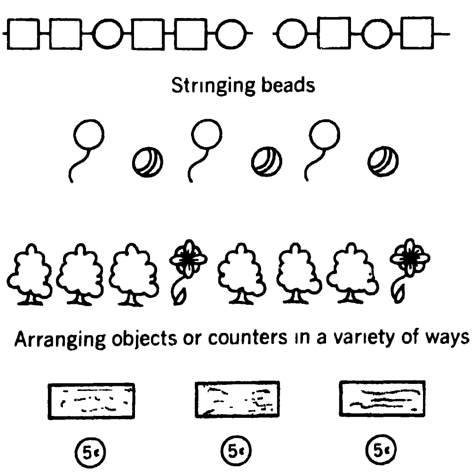
Activities leading to readiness

It is debatable whether the activities described in this section should be classified as part of the learning experiences for counting or for readiness for the basic facts. It is of little significance under which classification the activities are placed, however. What is important is for the pupil to participate in these activities. Some suggested activities include the following:

1. Discovering number patterns
a. Arranging objects
b. Discovering odd and even numbers
c. Showing patterns with a fixed number of objects
d. Playing games involving number patterns
2. Describing a pattern verbally. We shall explore each of these activities.

Discovering number patterns

Arranging objects Patterns play an important role in mathematics, and many mathematical patterns or models are related to the physical environment. When a scientist observes a particular pattern in the physical world he looks for a mathematical pattern with which to describe the situation. Patterns in mathematics lead to the discovery of basic ideas. For example, a foundation for the understanding of ratio in a later grade is developed through arranging or matching objects (see Fig. 7.19).



Stringing beads

Arranging objects or counters in a variety of ways

Cost of candy: matching a bar with coins

Figure 7.19

Discovering odd and even numbers Pupils can use their counters to form sets of twos. From this activity the class should discover that some sets have an extra or an odd counter left from forming sets of twos. This activity leads to the discovery of *odd* and ***even numbers***. The concepts of order can be discovered when the counters are arranged in sequence, in Figure 7.20.

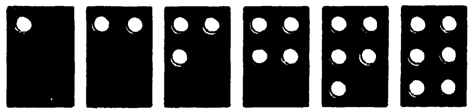


Figure 7.20

The teacher has the class find the answers to questions such as the following:

Name the even numbers.

Name the odd numbers.

Name two odd numbers. Which even number(s) comes between the two odd numbers?

Name an odd number. Which odd number comes before? after?

Name two even numbers. Which odd number(s) comes between the two even numbers? Counting by twos leads readily from working with even numbers. Pointing to sets of two counters and naming the even numbers in the process, rather than rote recitation, provides a meaningful experience. The more able pupils may learn the pattern of odd numbers. Basic concepts of multiplication and division are developed by having the pupils use their sets of counters to answer the following questions:

Who can show four with two even numbers?

Who can show another number with sets of two even numbers?

What numbers can you show with sets of two?

Can you show four with two odd numbers?

What other numbers can you show with two odd numbers?

Showing patterns with a fixed number of objects In the activity described on page 107, the pupil discovered a pattern for an infinite set. He now makes a pattern with a finite set of objects, such as a set of 3 counters. Each child is given 3 counters or objects to show different ways to place them in two spaces. He may fold a sheet of paper or use two paper plates or two paper cups as containers in which to place the counters. Figure 7.21 shows the plan to follow when using a folded sheet of paper.

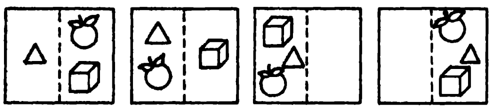


Figure 7.21

How many different ways are there to arrange the objects?

Can we have the same number of objects in each space?

Can we have twice as many in one space as in the other?

Let us make a record of these ways on the counting frame (see Fig. 7.22). (A paper clip or a clothes pin serve as a marker.)

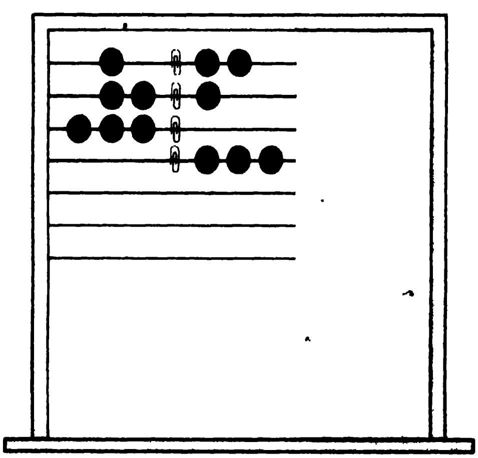


Figure 7.22

Use four objects to make patterns with two sets (see Fig. 7.23).

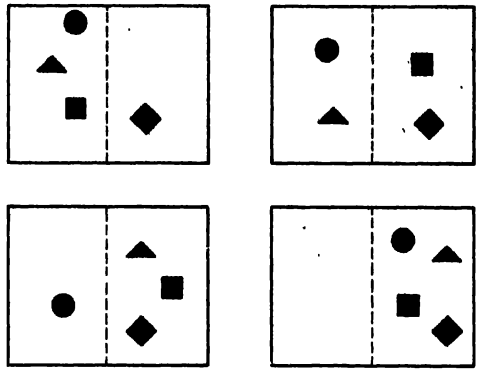


Figure 7.23

Are there any other ways to arrange the objects?

Who put the most objects in the first space?

Who put the next largest number in the first space?

Did anyone have the same number of objects in each space?

Let us make a chart record for the bulletin board (see Fig. 7.24).

Can we make a different order of patterns for the chart?

Which patterns can we make in pairs?

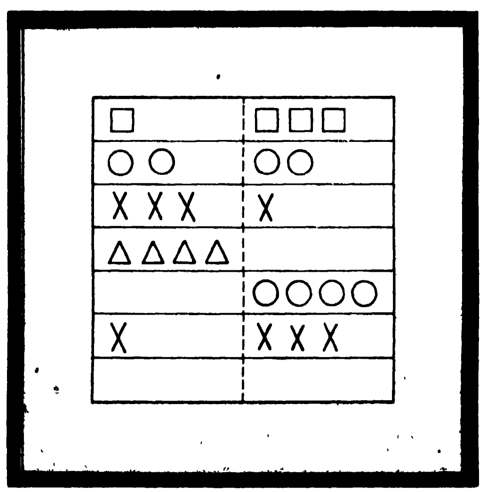
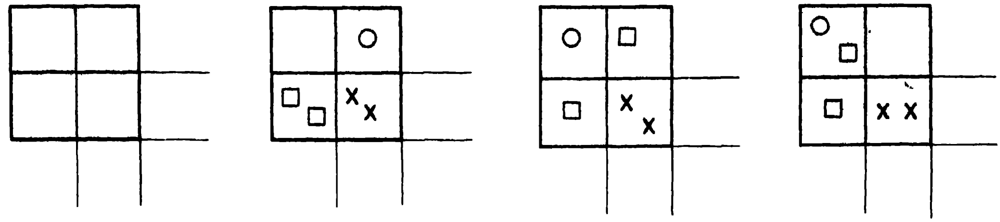
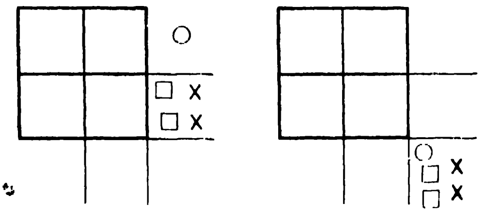


Figure 7.24



(1) The puzzle (2) Place the counters



(3) Move the counters (4) Move the counters again

Figure 7.25

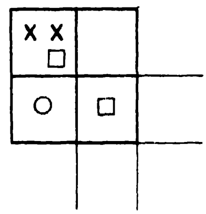


Figure 7.26

Playing games involving number patterns Concepts of addition may be extended through the use of cross-number puzzles. Place on the floor a large piece of chart paper that contains patterns of cross-number puzzles. Have the pupils place counters or objects on the squares. The pupil tells the arrangement of the counters and the number on each square. Figure 7.25 shows the plan to follow for the number 5. Can the counters be arranged in a different pattern or patterns? What are the results? (Several charts should be placed side by side so that children can observe the different patterns for the same number, as in Fig. 7.26.)

The diagrams in Figure 7.27 show how readiness may be developed for the subtraction facts involving the number 5. Give each child in a group a different number of counters, such as 1, 2, 3, or 4. Ask the children to place a counter on the first space of the chart. Then have them tell how many more counters are needed to make 5. Follow a similar plan for the first 10 numbers. Have the pupils select the counters they need from the box to complete their pattern (see Fig. 7.28). Ask the children to do the following:

Take one counter from your pattern. How many do you have?

Take the counters from one side. Tell how many you took and how many you have on the other side.

Describing a pattern verbally

After the class has had many experiences in discovering and demonstrating number patterns, pupils should de-

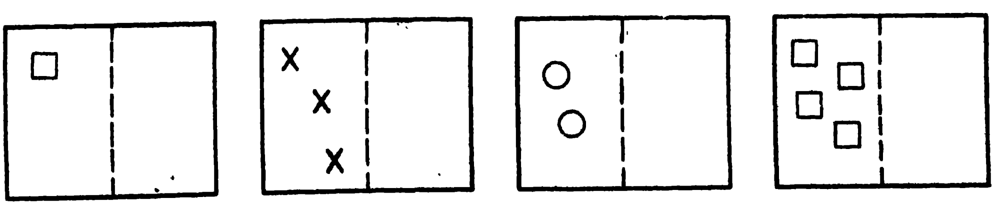


Figure 7.27

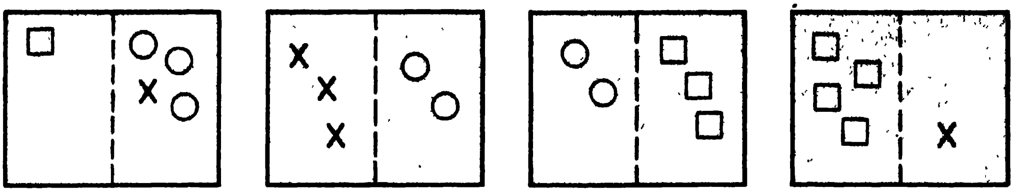


Figure 7.28

scribe patterns in verbal statements. Exercises of this kind are the foundation for the basic facts in addition and subtraction. Most kindergarten children will learn the basic facts in addition for the facts having sums less than 10 by dealing with the patterns. The work is verbal. The symbolic representation of the facts comes at a later stage in grade 1. Chapter 8 describes the plan for introducing the symbolic form.

The class may have sets of cards, with one set of cards containing the same number of pictures as the combined number of two cards. The pupil matches the cards. Figure 7.29 shows the plan for the number 5. The pupil combines a set of 2 pictures and a set of 3 pictures to form a set of 5 pictures.

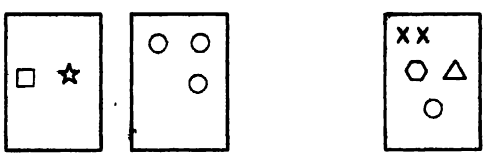


Figure 7.29

The teacher records on the chalkboard the numerals that name the sets. The pupil reads the statement to show that numerals may represent the joining of sets. The pupil reads the representation in the diagram as, "Two and three are five." The symbolization of the operation is thus the only part of the operation to be learned in order to understand addition.

BEGINNING GEOMETRY

Kindergarten and first grade

Young children find it a new and exciting experience to learn to recognize and identify geometric shapes in familiar objects. The teacher can help children look at the world from the point of view of geometry, an experience that has significance for their mathematical education in later grades. Experiences with geometric concepts should be informal and intuitive.

One of the earliest learnings for many pupils results from the need to orient themselves in space. Many informal activities provide opportunity for this experience. Plans for arranging the room and work centers include the following activities and discussions:

My desk faces the chalkboard.

The teacher's desk is near the door.

I can walk around the room.

We will keep the paper on the top shelf.

We walk through the doorway and turn right to go to the cafeteria.

The library table is in the back of our room. Positional terms are introduced when pupils describe, without pointing, the location of various objects in the room. They may play a game in which one pupil describes the position of an object and the other pupils guess what object he has in mind:

The object is in the front of the room.

It is above the chalkboard.

The object is on the bottom shelf under the window box. Dramatizing stories provides orientation to space and locations (points):

We will play the "Three Bears."

Mark a point on the floor where the house will be.

Mark a point where Goldilocks lives.

Show the path where Goldilocks walks to the bears' house. Pupils are familiar with the term "point." They can see and feel the point of a pin or the point of a pencil. In geometry, however, a *point* is described as an exact location. We represent points by drawing dots. After pupils have learned to read and write numerals, they develop great interest in connecting dots that follow numerical order to make shapes.

When geometric shapes are introduced early in the kindergarten, it is important that correct concepts be developed from the beginning. Traditionally pupils are asked to cut out a circle when a circular shape is what is wanted. The circular disk is not a circle. Only the boundary is the circle. The interior enclosed by the true circle and the circle is called the *region.* One excellent way to represent circles, squares, triangles, and rectangles is with cutout forms or taped forms on the floor. Pupils enjoy games that involve the following activities:

Stepping inside the square

Walking all the way on the square

Standing outside the square.

Many singing games include similar experiences with inside, outside, and on the figure:

"Round and Round the Circle"

"Farmer in the Dell"

"In and Out the Circle"

Models of different-sized geometric shapes may be made of wire, strips of wood, and the like. Pupils may be asked to find the set of squares or the set of triangles. Discuss the properties that distinguish a triangle from a square:

A triangle has three sides and three corners.

A square has four sides and four corners.

The comparison of the sizes of geometric figures is an extension of the study of order in geometry:

Find all the squares. Arrange the squares in order of size, with the smallest first.

Arrange the rectangles in order of size, with the largest first.

Find all the squares that are the same size; be sure they match exactly. Pupils will have many opportunities to see that some figures match in both size and shape and that some have the same shape but are of a different size. Pupils are learning to recognize congruent figures as a foundation for understanding congruence at a later time, although at this point they do not learn the terms "similar" and "congruent."

Bulletin board display

An interesting activity to reinforce geometric concepts and relate them to the physical world is to make a bulletin board display. Children may find pictures of any object in their environment that represents a shape. These pictures may be arranged according to an agreed-upon plan (see p. 112).

The concept of shapes may be used effectively in constructions in art activities. The teacher may have available cutout shapes of colored construction paper in various sizes and colors. Pupils may make original pictures using one, two, or more shapes (see p. 98).

When children have participated in a program rich in experiences in recognizing and identifying geometric shapes,

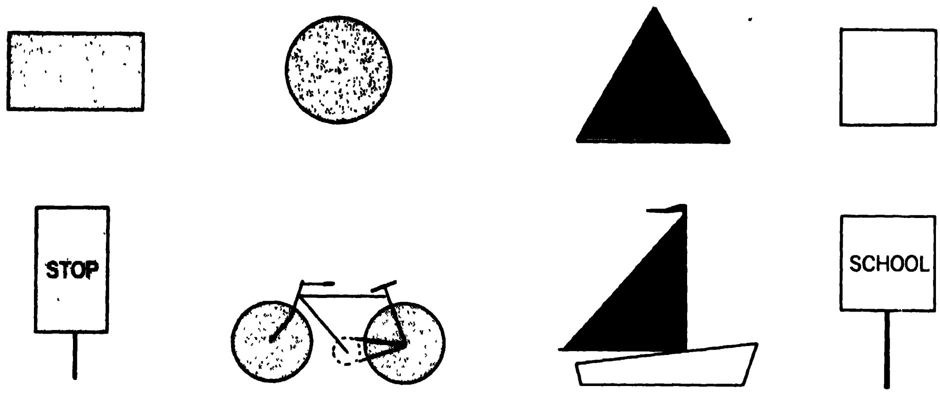


Figure 7.30

they learn many basic properties of geometric figures. Some of these basic concepts are closed and open curves, interior and exterior of plane figures, points, paths, triangles, rectangles, squares, circles, points, and shapes.

Lines and curves

Lines and curves may be introduced through various exercises. The teacher may draw two dots on the floor with chalk. Directions guide the children to discover that many paths may be made between two points and that a straight path is the shortest.

Who can walk from one dot to the other?

Can anyone walk in a different path?

Can anyone walk in another path?

Let's draw the paths John and Sue made.

Can anyone walk on another path?

Are all the paths the same length?

Which path is the shortest?

Pieces of string illustrate paths and curves. The teacher may give each child a piece of string as a model. Children are guided to experiment with patterns using the string.

Show the pattern of a path with your string.

Are any paths alike?

How does Jerry's path differ from Jane's?

Does anyone have a path whose ends meet? (a closed curve)

What kinds of figures can you make when the ends meet?

Young children have many experiences with geometric solids. A collection of boxes and other containers is a useful set for exploring shapes. The sensory experience of handling various solids enriches the perception of relationships between plane figures and solids. Children may count the number of faces of a square box and the number of corners. They may match a square shape (or rectangular shape) with one face of a box. They may classify all the solids that have a circular shape at one side or end. They learn that many sizes and shapes are used in constructing doll-house furniture. Building blocks may be arranged according to shapes. Constructions with building blocks may be made with triangular- or square-shaped blocks or both. Counting experiences may be provided by asking children to count the number of triangular- or square-shaped blocks in the construction.

EXERCISES

1. Why is a systematic, planned program of arithmetic necessary in the primary grades?
2. Be ready to defend or to criticize the outline of the contents of the curriculum for grades K–two presented in this chapter. Compare the contents with the materials contained in several primary arithmetic textbooks or workbooks used in local courses of study.
3. Illustrate the five levels of growth described on page 92.
4. Why is it important that children be taught to recognize groups of 1 to 4 at a glance? How will this help them to identify groups of 6 to 10?
5. What charts do you think are necessary to guide the experiences of children in learning to read numbers?
6. Observe the performance of some young children who are beginning to write numerals and diagnose any difficulties they have with the formation of numerals. If possible, use the kinesthetic cards described on page 102 to help correct the faults of some pupil.
7. Evaluate the use of a picture line for teaching counting.
8. Evaluate the program given in this chapter for building readiness for the basic facts.
9. List the geometric concepts you would introduce in grade 1.
10. Discuss the place of the textbook or workbook in number work in the kindergarten; in grade 1.

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CHAPTER 8

PATTERNS FOR TEACHING THE BASIC FACTS IN ADDITION AND SUBTRACTION

Chapter 3 identified three aspects of learning elementary mathematics: the acquisition of new knowledge, the search for structure and a pattern, and the application of the pattern. These activities apply to learning the basic facts in addition and subtraction. Before beginning a systematic presentation of the basic facts, the teacher must be sure that the class has mastered certain number concepts that constitute the activity listed as the acquisition of knowledge. An understanding of these concepts may be considered readiness for the new learning. Page 106 lists the kinds of activities that are designed as preparation for learning the basic facts in addition and subtraction. This background constitutes the mathematical readiness for the new work.

Chapter 8 deals with the following topics. programs for teaching the basic

facts; teaching the facts in addition and subtraction; subtraction situations; practicing the facts.

PROGRAMS FOR TEACHING THE BASIC FACTS

Definition of a basic fact

A basic fact in addition or multiplication is an equation *that shows the sum or product of any pair of one-digit numbers.*[1] The corresponding equations form the basic facts in subtraction and division provided 0 is not used as a divisor. There are 100 basic facts (10 × 10) in each operation except division, which contains only 90 basic facts.

The addition facts

Table 8.1 gives the sums of the pairs of one-digit numbers. To find the sum of a number pair such as (3, 5), locate the row headed by 3 in color and the column headed by 5 in color and read the numeral 8 that is in both the row and the column. The sum of 3 and 5 is 8, hence the basic fact is $3 + 5 = 8$. By reversing the order in the number pair the fact is $5 + 3 = 8$.

From the table it is possible to derive the basic facts in subtraction. To find the fact $8 - 3 = 5$, locate 8 in the row headed by 3. The difference between these numbers will be named by the 5 at the head of the column containing 8. Similarly, identify the fact $8 - 5 = 3$.

The 100 facts in addition may be grouped into sets, as follows:

Set I: The 45 number pairs having sums of 10 or less, excluding set II.

Set II: The 19 number pairs in which 0 is a member of a number pair.

Set III: The 36 number pairs having sums greater than 10.

TABLE 8.1

Basic Facts in Addition and Subtraction

+	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
0	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	**1**	2	3	4	5	6	7	8	9	10
2	**2**	3	4	5	6	7	8	9	10	11
3	**3**	4	5	6	7	8	9	10	11	12
4	**4**	5	6	7	8	9	10	11	12	13
5	**5**	6	7	8	9	10	11	12	13	14
6	**6**	7	8	9	10	11	12	13	14	15
7	**7**	8	9	10	11	12	13	14	15	16
8	**8**	9	10	11	12	13	14	15	16	17
9	**9**	10	11	12	13	14	15	16	17	18

The sums of the number pairs in set I are found in the shaded part of the table; the sums of the number pairs in set II are given in boldface type; and the sums of the number pairs in set III are found in the clear part of the table.

Chapter 2 indicated that the arithmetic curriculum in kindergarten and grades 1 and 2 is in a state of flux. In these grades there is no fixed pattern to the planned and sequential arithmetic program. Spitzer has stated that by the completion of grade 2 the pupil should have mastered the basic facts in addition and subtraction.[2] He recommends that introduction of the basic facts be delayed until the second half of grade 2. The authors, on the other hand, suggest that the basic facts be presented throughout the first two grades. In most schools the average pupil

[1] The more precise phraseology would be given as "any pair of numbers named by one-digit numerals." It is conventional to speak of a one-digit or a one-place number. We shall use that terminology

[2] Herbert Spitzer, *Instruction in Arithmetic*, Twenty-fifth Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: The Council, 1960), Chap. 5.

should be expected to have mastered the 100 facts in addition and subtraction by the end of grade 2.

TEACHING THE FACTS IN ADDITION AND SUBTRACTION

Two different plans may be followed in introducing the basic facts in addition and subtraction. The facts may be introduced separately for each operation or simultaneously for the two operations. At present most textbooks present the facts in the two operations together. We shall follow the latter plan in this chapter.

The method of presenting the facts together is effective because it helps the pupil to discover the relationship between the facts in the two operations. Bruner has pointed out that ideas in number work that are difficult for a pupil to grasp are *invariance* and *reversibility*.[3] Invariance refers to the fact that regrouping a number does not change its value. Reversibility means that there is an inverse or undoing operation for a given operation. The pupil should begin early in his number work to discover that the operations of addition and subtraction, as well as those of multiplication and division, are reversible.

Order of teaching the facts

There is no established order for teaching the basic facts. It has been traditional to teach the set of facts having sums of 10 or less before teaching the set having sums greater than 10. Any order is satisfactory if the pupil is able to discover patterns and relationships. The facts should never be taught as specifics and in isolation from other facts. The following principles should determine the order of teaching the facts in addition and subtraction:

1. The learner should be able to discover a pattern among the facts.
2. The pattern should enable the learner to derive new facts.
3. The learner should be able to generalize about the pattern.

The application of these three principles makes it impossible to teach the facts in a random sequence. Rather, the facts must be presented in a sequential order or selected in such a way that some pattern will apply to a given set of them.

There may be different orders for teaching the facts that would meet the criteria listed. We shall be concerned with the sequence consisting of the *sets of related facts*. A set of related facts is derived from the number pairs that have the same sum. This set is the union of the set of number pairs that have the same sum in addition and the corresponding set of number pairs in subtraction. The facts for the set of related facts for the fours include the following:

$2 + 2 = 4$	$4 - 2 = 2$
$1 + 3 = 4$	$4 - 1 = 3$
$3 + 1 = 4$	$4 - 3 = 1$
$0 + 4 = 4$	$4 - 0 = 4$
$4 + 0 = 4$	$4 - 4 = 0$

Properties of addition

The following properties apply to addition of whole numbers:

Commutative property, as $a + b = b + c$

Associative property, as $(a + b) + c = a + (b + c)$

Identity element, as $a + 0 = a$

Closure

The property of closure makes it possible to add any two whole numbers. If a and b are any two whole numbers and their sum is a whole number, c, as

[3]Jerome Bruner, *The Process of Education* (Cambridge, Mass.: Harvard University Press, 1963), p. 41.

$a + b = c$, then the set that contains a, b, and c is closed with respect to addition. From the standpoint of the pupil in beginning number work, closure is a property of addition that is of academic value.

Properties of subtraction

Subtraction is the inverse, or undoing, operation. Therefore the properties that apply to addition do not apply to subtraction. The order in which two numbers are subtracted affects the difference. The numbers named by the numerals $5 - 2$ and $2 - 5$ are different. Subtraction, therefore, is not commutative. Similarly, the way in which three numbers are grouped affects the answer. The numbers named by the numerals $(9 - 5) - 4$ and $9 - (5 - 4)$ are different. Therefore the associative property does not apply to subtraction.

Since subtraction is not commutative, there is no identity element for this operation. Although 0 may be subtracted from a whole number and the answer will be that number, it is not possible to subtract a whole number greater than 0 from 0 in the set of whole numbers. The illustration involving 0 in subtraction shows that the set of whole numbers is not closed with respect to this operation. It is possible to subtract in the set of whole numbers only in examples of the type $a - b = c$ when $a \geqq b$ (read $\geqq$ as "equal to or greater than").

Classroom materials

The classroom should be equipped with certain materials that may be used to represent an operation and its effect on a pair of numbers. The representation of an operation on a pair of numbers is often known as a *model*. A model for showing the sum of 2 and 3 may be a group of 2 disks or counters and 3 disks or counters to form a group of 5 disks or counters.

There is no standard list of items for equipping the elementary mathematics classroom. It is better to have on hand a few materials that are used efficiently than many materials that are used infrequently and ineffectively. Among desirable classroom materials are a *flannel board* and an *abacus* or a *place value chart,* preferably all three. (See Appendix for a description of these materials.)

Many modern classrooms are equipped with chalkboards on which magnetic disks may be effectively used for modeling the basic facts. A magnetic disk may be made of wood about $\frac{3}{8}$ inch in thickness and 2 inches in diameter. The center of the disk contains a small magnet that will adhere to the chalkboard or to a flat surface of iron or steel. The classroom should contain at least 10 of these magnetic disks for demonstrating the basic facts having sums of 10 or less.

The teacher can make magnetic disks by attaching a piece of magnetic tape to the face of a plastic disk or some other type of marker or counter. The tape will adhere to any surface to which a magnet will adhere. Since the tape is inexpensive, it is possible to equip a classroom with magnetic counters at a low cost.

Pupil materials

It is important for the pupil to have materials to use at his desk. He uses these items to discover number patterns, to model a particular situation, and to show joining and separating of sets. From these activities he should be able to discover the meaning of addition and subtraction.

Small objects as counters or markers are more effective for pupil materials

than large ones. It is important for the pupil to get the kinesthetic sensation by dealing with materials at the introductory phase of a given topic. The pupil uses these materials until he can apply a pattern that enables him to find the sum or difference when adding or subtracting a pair of numbers.

The resourceful teacher will always find objects in the classroom, such as pencils, crayons, or books, that may be used to model a given situation. At the same time, the pupil should have his own materials that he may use specifically to discover number facts and relationships. These materials include *cylindrical disks*, *fact finders*, and *rectangular strips* and *squares*. Each pupil should have at least two sets of these materials. (See Appendix for a description of these learning aids.)

Readiness for the basic facts

Chapter 7 demonstrated that sets may be joined, separated, and compared. Addition describes the union or joining of two disjoint sets. Subtraction describes the separation of a set into two subsets or shows the comparison of two sets. Sometimes a teacher introduces the basic facts in addition or subtraction before the pupil is ready for this type of work. Lack of readiness results from an inadequate background for the work. In the absence of this background a pupil learns the facts by rote. Page 106 lists the kinds of activities that help to prepare the pupil for learning the number facts in symbolic form.

The following section explains how to introduce one or more sets of related facts having sums less than 10 as well as those having sums greater than 10. The facts for operations addition and subtraction will be presented together.

The set of related facts for the threes

Principle 3 (page 30) states that a wide variety of meaningful experiences enriches learning. These experiences are twofold in nature. First, they include the activities that are basic in the readiness program: joining sets and making verbal statements about the groupings. Second, they include the activities that are effective for presenting the facts in symbolic form. A suggested list of activities under the second classification follows for introducing the set of related facts for the threes.

1. The pupil joins 1 counter and 2 counters, forming a group of 3 counters. He then interchanges the two groups. He also demonstrates that if no counters are joined to 3 counters, there will still be 3 counters. Next, he separates 3 counters into two groups of 1 counter and 2 counters. If no counters are removed, the group contains 3 counters.

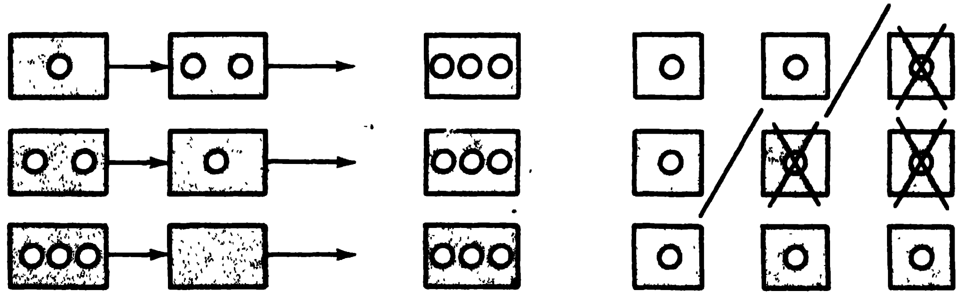


Figure 8.1

2. The teacher demonstrates the activity described in (1) on the flannel board or makes a drawing to model the grouping (Fig. 8.1).

3. The pupil identifies the set of related facts on a *number ray* drawn on the chalkboard. In contrast to a number line, which extends in opposite directions from a starting point, a number ray has an end point marked zero and a set of points extending in one direction. Some classrooms have a number ray marked off on the floor so that a pupil can step off a number fact in the set of related facts having sums of 10 or less.

Figure 8.2 shows the facts for the threes. The teacher directs one or more pupils to trace on the ray with their fingers the facts represented. This procedure enables the teacher to determine whether the class knows how to read the ray. The pupil should discover that the movement on the ray is to the right for addition and to the left for subtraction.

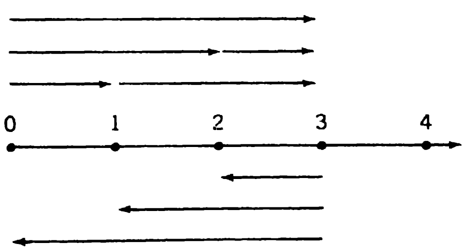


Figure 8.2

4. The teacher writes the following number sentences or equations for the threes:

$$\begin{array}{ll} 1+2=3 & 3-1=2 \\ 2+1=3 & 3-2=1 \\ 0+3=3 & 3-0=3 \\ 3+0=3 & 3-3=0 \end{array}$$

The pupil reads the number sentence $1 + 2 = 3$ as, "1 plus 2 equals 3" and the sentence $3 - 1 = 2$ as, "3 minus 1 equals 2." A number sentence has the property of *order*. The terms of the number system must be considered in sequence. Each numeral or sign is read as it appears in the equation. The sign + indicates that we are to add. Just as we can join two groups, we can add two numbers. The sign − indicates that we are to subtract. Just as we can separate a group from a given group, we can subtract one number from another. The sign = indicates that two numbers are equal or that they name the same number.

5. Finally, the teacher has the class compare the set of related facts for the twos as well as that for the threes. The set of facts for the twos contains the following:

$$\begin{array}{ll} 1+1=2 & 2-1=1 \\ 0+2=2 & 2-0=2 \\ 2+0=2 & 2-2=0 \end{array}$$

The pupil tells why the sum of the number pair (1, 2) is greater than the sum of the pair (1, 1).

The set of related facts for the fours

In order to discover patterns and relationships among facts, we shall consider the next set of related facts, the set for the fours. The pupils should experience the same kinds of activities in dealing with the fours as they did with the threes. These include joining and separating each of the number pairs with markers, representing the facts on a number ray, and writing the facts in an equation. The pupil should write these pairs in tabular form, as shown in either (a) or (b). As soon as he discovers the pattern for writing the number pairs for a given sum, he does not need markers to derive the facts for that sum.

a.

4	
2	2
1	3
0	4

b.

4	
0	4
1	3
2	2

A number pair that enables a pupil to derive the facts in a set of related facts is the key number pair. The key pair in (a) is the pair of equal numbers (2, 2) and in (b), the pair (0, 4). The key number pair may be a pair of equal numbers, as in (a), a pair that differs by 1 for a set of facts in which the sum is an odd number, for example, the pair (2, 3), or a pair in which one of the numbers is 0, as in (b). The number pair to be used as a key number pair is of minor importance if the pupil discovers a pattern for deriving the other related facts. Any number pair may then be used as a starting point for deriving the set of related facts. Most pupils find it easy to discover the pattern for writing the number pairs in tabular form if the first pair involves a double, as $3 + 3$, or the identity element for addition, as $0 + 3$.

Discovering relationships in a set of related facts

The class should compare the sets of the threes and fours and discover some or all of the following relationships or facts:

1. The order of adding two numbers does not change the sum.
2. Adding 1 to a number gives the next number.
3. Adding 1 to one number of a number pair and subtracting 1 from the other number of the pair does not change the sum. This procedure forms a new number pair in the set of related facts. (Only one-digit numbers are used for number pairs.)
4. One number sentence can be formed in addition when the numbers in a pair are the same, and two sentences can be formed when the numbers are different.
5. Every equation in addition can be written as an equation in subtraction; for example, $2 + 1 = 3$ in addition and $3 - 2 = 1$ or $3 - 1 = 2$ in subtraction.
6. The sum of 0 and a number is the same as that number.
7. The remainder from subtracting 0 from a number is that number.
8. The number of facts in either addition or subtraction in a set of related facts is one more than the sum of each fact.
9. If the sum in a set of related facts is an even number, one number pair in each operation will have equal numbers; if the sum is an odd number, each number pair will have unequal numbers.

The number of discoveries the pupils will make about the facts in a set of related facts depends upon the experience they have had in dealing with numbers. The class should discover the following four items pertaining to the set of related facts:

1 The commutative property of addition
2. The identity element of addition
3. The patterns for deriving new facts in a set of related facts when a key fact is given.
4. The pattern for writing the corresponding fact in subtraction for a fact in addition.

If the pupil does not make these discoveries, the teacher should help him by asking leading questions pertaining to a given item. In order to help him discover the commutative property of addition, the teacher has the pupil tell what happens to the sum when two numbers are interchanged. The pupil compares the sum with the nonzero number when 0 is added to that number to discover the identity element for addition.

In order to discover the pattern for finding a new fact from a key fact, such as $0 + 4$, the teacher should have the

pupil write the set of related facts, as follows:

$$\begin{array}{ll} 0 + 4 = 4 & 4 + 0 = 4 \\ 1 + 3 = 4 & 3 + 1 = 4 \\ 2 + 2 = 4 & \end{array}$$

The left-hand column shows that one number of a number pair is one more in the next fact and that the other number of the pair is one less. The right-hand column shows the facts made by changing the order of the numbers of a pair of unequal numbers.

The fourth item enumerated above is concerned with deriving the corresponding fact in subtraction from a fact in addition. Many pupils find it difficult to write the corresponding fact in the inverse operation. In order to gain an understanding of the relationship between the facts in the two operations, the pupil identifies each number in an equation, as follows:

$$\begin{array}{ccccc} 1 & + & 3 & = & 4 \\ \swarrow & & \downarrow & & \searrow \\ \text{Addend} & & \text{Addend} & & \text{Sum} \end{array} \qquad \begin{array}{ccccc} 4 & - & 1 & = & 3 \\ \swarrow & & \downarrow & & \searrow \\ \text{Sum} & & \text{Addend} & & \text{Addend} \end{array}$$

The equation in subtraction to correspond to the equation $1 + 3 = 4$ is $1 + \square = 4$. The pupil finds the missing addend to make the equation true. His thought pattern is, "addend + addend = sum, or $1 + 3 = 4$." The equation $1 + \square = 4$ may be written as $4 - 1 = \square$. Now the thought pattern is, "sum − addend = addend, or $4 - 1 = 3$." The pupil should be familiar with the following equations:

Addend + addend = sum	(addition)
Sum − addend = addend	(subtraction)

The pupil identifies each term of an equation involving the basic facts until he understands the relationship between the facts in the two operations.

The pupil should find the number to make a number sentence true. A number sentence of the kind that follows is an equation. The pupil should *solve* the equation. Each equation involves a number pair that has a sum of 4 or less.

$$\begin{array}{ll} 2 + 2 = \square & 3 - \square = 2 \\ \square + 3 = 4 & \square - 2 = 2 \\ 3 + \square = 3 & 4 - \square = 3 \\ 1 + \square = 3 & 2 - \square = 2 \end{array}$$

As each new set of related facts is introduced the pupil should solve equations involving that set as well as any of the sets previously introduced.

The teacher makes certain that a pupil discovers a pattern that applies to the facts in a set of related facts. After the pupil discovers a pattern, he should make verbal statements or generalizations about the number facts. The teacher needs to remember that *a pupil discovers a number pattern before he describes it verbally.* Some pupils are able to discover a pattern that applies to a set of numbers but may not be able to generalize about the pattern. This level of achievement is satisfactory, since the ability to generalize about a number situation represents a very high level of learning. The formation of a concise statement that applies to a number situation is a challenging activity for the more able pupil.

Renaming numbers in a number pair

A pupil should discover quickly that adding 1 to a number gives the next number. He can then use that knowledge to discover other number facts. A more mature mathematical basis for finding a new fact than adding 1 to a number is renaming a number and applying the associative property of addition. If a pupil knows the sum of a number pair, such as (3, 3), he can use the derived fact $3 + 3 = 6$ to find the sums of another number pair such as (3, 4) by regrouping 4 as $3 + 1$. Then $3 + (3 + 1)$ is another numeral for $3 + 4$, and 3

+ (3 + 1) is another numeral for (3 + 3) + 1, or 6 + 1. By applying the generalization governing the addition of 1 to a number, the sum of 6 + 1 is one more than 6, or 7.

The new learning in this case involves the application of the associative property of addition. This property implies that the way three numbers are grouped does not affect the sum. Figure 8.3 shows that changing the grouping of 2 counters, 1 counter, and 3 counters does not change the total number of counters. The pupil must have many experiences involving the joining of three sets to enable him to discover that the way the sets are joined does not affect the number in the union of the sets. It is assumed that the order of the sets is not changed.

If a pupil knows the sum of a key number pair, such as (4, 4), (0, 4), or (3, 4), he should be able to do two things. First, he should be able to derive all the related facts in the set containing the key fact; and second, he should be able to find a key fact for the next number pair. The application of these learnings should enable the pupil to discover all the sets of facts in addition and the corresponding facts in subtraction. The last set will be the set of eighteen, which includes the two subsets of facts 9 + 9 = 18 and 18 − 9 = 9.

The plan described does not imply that the pupil is never to use models to represent a fact. The use of materials to demonstrate a fact should be minimized, however, since it is possible for the pupil to discover facts from the known facts. The change from dealing with sets of things to sets of numbers represents the principle of growth, which is essential in good learning situations. On the other hand, the pupil who deals with sets of numbers when he should be dealing with sets of objects is engaging in rote learning. The teacher must decide when a pupil is ready to work at the higher level of dealing with sets of numbers rather than sets of objects. An oral interview in which the pupil reveals his thought pattern in a number situation usually enables the teacher to determine the level of understanding the pupil has acquired.

Reading and writing the facts in vertical form

The pupil writes the facts in the first few sets of related facts in the equation form. Then he also writes the facts in vertical form, as shown for the number pair (2, 3):

He reads the fact in the equation form as "2 plus 3 equals 5" and in the vertical form as "2 and 3 are 5." A pupil should always add downward when the numbers named are in a column.

There is no standard phraseology for reading a fact in subtraction; some of the ways of reading the fact shown are as follows:

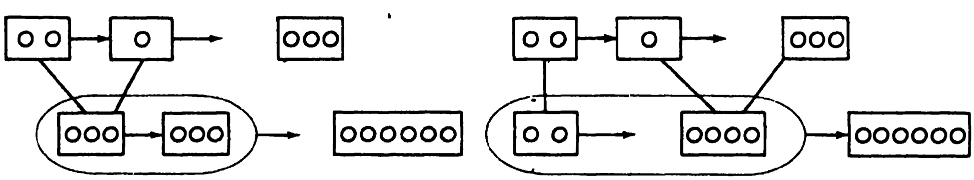


Figure 8.3

a. "Two from 5 is 3."
b. "Five less 2 is 3."
c. "Five take away 2 is 3."
d. "Two and 3 are 5."
e. "Five minus 2 equals 3."

The pupil is familiar with the phraseology in (e) from reading a number sentence. However, the use of the thought pattern in (e) leads to downward subtraction, which is not recommended. Most people subtract upward. If the pupil adds downward and subtracts upward, the opposite procedures help to emphasize that the two operations are inverses. The thought pattern for a subtraction fact expressed in vertical notation should correspond to the pattern used in performing the subtraction algorism. In example (a) the thought pattern should be, "8 from 12 is 4; 3 from 6 is 3." Therefore the pupil should read the fact in example (b) as, "2 from 5 is 3." In general, "A from B is C." The thought pattern for the same fact written in equation form, $5 - 2 = 3$, is "5 minus 2 equals 3."

a. $\begin{array}{r} 72 \\ -38 \\ \hline 34 \end{array}$

b. $\begin{array}{r} 5 \\ -2 \\ \hline 3 \end{array}$

Sets of facts having sums greater than 10

There are two acceptable plans for introducing the facts having sums greater than 10. The one just described applies to all the basic number pairs in addition regardless of the sums. The second plan applies to the set of 36 number pairs that have sums greater than 10. Since the base of our number system is 10, the number 10 is used in arriving at the sum of a number pair greater than 10, for example, the pair (6, 7). In order for this plan to be effective, the pupil must be able to give the sum of $10 + 1, 10 + 2, \cdots, 10 + 8$. The teacher presents such examples until the pupil discovers the pattern that governs the sum.

Figure 8.4

The plan of adding a number to 10 may be illustrated by using the number pair (6, 7). Each pupil has rectangular strips of cardboard or wood divided into 10 equal segments, as shown in Figure 8.4. In order to find the sum of $6 + 7$ by using 10, it is necessary to rename 7. Since 4 added to 6 has a sum of 10, rename 7 as $4 + 3$. If the pupil does not know that one of the components of 7 is 4, he can use rectangular strips to find the number that added to 6 makes 10. He can count off 6 squares and then cover them with one hand, or he can place a strip of 6 squares on the long strip. In either case there will be 4 squares showing. The pupil writes the number pair $(6 + 7)$ as follows:

$6 + 7 = 6 + (4 + 3)$	Rename 7 as $4 + 3$
$6 + (4 + 3) = (6 + 4) + 3$	Associative property
$(6 + 4) + 3 = 10 + 3$	Rename $6 + 4$ as 10

$6 + 7 = 13 \qquad 13 - 6 = 7$
$7 + 6 = 13 \qquad 13 - 7 = 6$

The fact $6 + 7 = 13$ is the key fact for writing the other 11 facts in the set of the thirteens (12 facts are in the set).

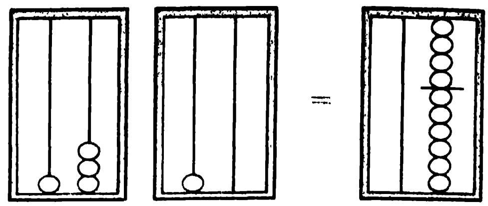
The plan described calls for two specific abilities in dealing with numbers. First, the pupil must be able to add to 10 any number named by the digits 1 to 8. Second, he must be able to separate into two parts a number, for example, b of the number pair (a, b), such that the sum of a and one of the parts of b will be 10. The pupil may use rectangu-

lar strips as described to find how much must be added to the first number (a) of a number pair to have a sum of 10. The number to be added is one of the two parts of b. Therefore the second number of the basic number pair is renamed as the sum of two numbers. Thus, for the number pair (7, 8), 8 is renamed as $3 + 5$, and now the number pair is written as $7 + (3 + 5)$. By applying the associative property, $7 + (3 + 5)$ may be expressed as $(7 + 3) + 5$, or $10 + 5$.

The method described is a long procedure for finding the sum of a number pair. The pupil uses an immature method to discover the facts in a set of related facts. Sometimes teachers assume that a pupil knows the sum of a number pair because he is able to discover a way or pattern for finding that sum. The pupil does not know a fact until he can give the sum of a number pair without hesitation and with assurance. He must have a variety of meaningful experiences with a number pair to give a mature response of that kind.

One new learning involved in dealing with facts having sums greater than 10 that did not apply to those with sums of 10 or less involves regrouping the sum. The pupil joins sets of 6 things and 7 things to form a set of 13 things. The number 13 should be regrouped as 1 ten and 3 ones. The plan of expressing $6 + 7$ as $(6 + 4) + 3$ emphasizes the place-value concept of each digit in the sum.

A modern abacus is an effective teaching aid to demonstrate how a sum greater than 10 is regrouped. To show the grouping of the number pair (6, 7), represent 6 beads on the ones' rod. Then 4 more beads will form 10 beads, which overloads that place. The 10 beads are exchanged for 1 bead on the tens' rod, as shown in (A) and (B) of Figure 8.5. Next, show the 3 beads from the set of 7 on the ones' rod so as to form the number represented in (C).



C B A

Figure 8.5

The pupil should solve open-number sentences that are formed from number pairs having a sum greater than 10. The equations in (a) and (b) are representative of the type the pupil should solve. The equations in (a) are formed from the number pair (5, 6) and those in (b) from the number pair (6, 7).

a $5 + 6 = 5 + (5 + \triangle)$
$5 + (5 + \triangle) = (5 + 5) + \triangle$
$10 + 1 = \square$
$5 + \square = 11$
$\square + 5 = 11$
$11 = \square + 6$
$\square - 6 = 5$

b $6 + 7 = 6 + (\triangle + 3)$
$(6 + \triangle) + 3 = \square + 3$
$10 + 3 = \square$
$6 + \square = 13$
$\square + 6 = 13$
$13 - \square = 7$
$\square - 6 = 7$

SUBTRACTION SITUATIONS

There are two phases of subtraction, which may be designated as *structural* and *functional*. The structural phase refers to the inverse relationship of subtraction to addition. The functional phase refers to the use of subtraction in dealing with a quantitative situation. In order for a pupil to understand subtraction, he must know *when* to subtract as well as *how* to subtract.

A subtraction situation is a verbal statement in a problem that indicates that subtraction may be used in the solution of the problem. Subtraction situations convey the following three distinct meanings:

1. The remainder concept
2. The additive concept
3. The comparison concept.

The three usages may be illustrated by the following problems:

a. Dick had 5 marbles but he lost 2 of them. How many marbles did he have left? (Remainder)

We have a set and have removed a given subset to find the remaining subset. Figure 8.6 shows a model of the situation. Cross off two elements of the set from right to left to show action in the opposite direction of addition.

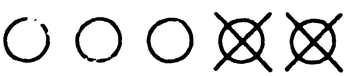


Figure 8.6

b. Dick has 3 marbles but needs 5. How many more marbles does he need? (Additive)

A given set is to be included in a set that is not given. The subset needed to fill the set is to be found. Figure 8.7 shows a model of the situation.

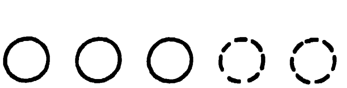


Figure 8.7

c. Dick has 5 marbles and Tom has 3. How many more marbles does Dick have than Tom? (Comparison)

We compare two sets. Two numbers, a and b, must either be equal or not equal, as $a = b$ or $a \neq b$. If a and b are not equal, then $a > b$ or $a < b$. If a and b are not equal, we use subtraction to find how much more one number is than the other. Two equal numbers may be compared by subtraction but their difference is 0, a fact that is readily apparent. Figure 8.8 shows a model for comparing the numbers in problem (c). There is a one-to-one matching of members of set B with some of the members of set A. The unmatched members of set A show how much greater set A is than set B. The same result can be found by subtracting the number of elements of the two sets.

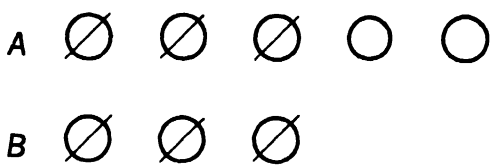


Figure 8.8

The pupil should have little difficulty in fitting the remainder concept and the additive concept of subtraction into the general pattern of subtraction. The general pattern is to find the missing number when the sum of two numbers and one of the numbers are given.

In the remainder concept, a set is separated into two subsets. One of the subsets is removed. The difference between the numbers given is the number in the remaining subset.

In the additive concept, a set is given that consists of two subsets with one subset missing. The number in the missing subset is the difference between the two numbers given.

The comparison concept does not fit into the pattern of the other two usages of subtraction. There is no sum of two numbers. Two sets are given and neither is a subset of a given set. The two sets are to be compared. The comparison may be done by one-to-one matching of the elements of the sets. The number of the unmatched elements is the same as the difference of the numbers given. Therefore, there are two distinct prob-

lematic situations involving subtraction. They are: (1) to find a missing number when the sum of two numbers and one of the numbers are given and (2) to find how much more one number is than another. These two different concepts are a source of difficulty for the pupil in determining when to subtract. It is comparatively easy for a child to understand that subtraction is the undoing operation of addition, but that knowledge is not adequate to enable him to decide when to subtract in problematic situations.

In the first of these subtraction situations, the teacher has the pupil identify the sum. The learner then usually experiences little difficulty in finding the missing addend. In situations in which two numbers are to be compared, the teacher has the pupil identify the numbers to be compared. This activity should enable the pupil to discover that he subtracts to find the difference between the two numbers.

Identifying subtraction situations in problems

The following four statements and questions relating to each typify the procedure to use in dealing with the functional usage of subtraction:

1. Mary picked 4 flowers from a plant that had 6 flowers.

How many flowers were left? How many flowers were in the set? What happened to some of the flowers of the set? How many flowers were left in the set? How do we find the number left? What number is the sum? What is one of the numbers or an addend? What is the missing addend? $6 - 4 = \square$?

2. A bag contained 8 black and white marbles. There were 6 black marbles.

How many of the marbles were white? How many are there in the set of black and white marbles? How many are there in the subset of black marbles? How do we find the number in the subset of white marbles? What is the sum? What is one of the addends? What number is the missing addend? $8 - 6 = \square$? Write the addition fact to correspond to $8 - 6 = 2$.

3. Nancy needs 8 stamps to fill a page in a stamp book. She has 5 stamps.

How many more stamps does she need to fill the page? How many stamps are there in the set needed to fill the page? How many stamps are there in the set Nancy has? What is the sum? one of the addends? What number is the missing addend? $5 + \square = 8$? $8 - 5 = \square$? Write the addition fact to correspond to $8 - 5 = 3$.

4. Rita has 5 dolls and Kathy has 7.

How many more dolls does Kathy have? How many dolls are there in the set Rita has? in the set Kathy has? Which set is the larger? How can we find how many more dolls there are in the larger set than in the smaller set? $7 - 5 = \square$? Write the addition fact to correspond to $7 - 5 = 2$.

It may be necessary for the pupil to use markers or counters to represent the dolls in each set. He may join two sets to show the number of dolls there are designated in the sum or he may separate a set to show the two subsets for each problem.

A table of the facts

After introducing all the basic facts in addition and subtraction, the teacher should have each pupil make a table of the facts. The pupil who writes the numerals in a table is almost certain to discover some of the patterns formed by the numbers. The class should identify as many patterns as possible during a period of discovery to be followed by a period of questioning. The teacher

should ask questions about the table that pertain to the patterns not identified by the class. The use of questions to make a discovery may be termed *guided discovery*. The following questions represent the type to use to enable the pupil to explore the table. If the class discovers a given pattern, the teacher would not give the following questions, which pertain to that pattern. We shall use the term "number" to mean the number named by the numeral. Refer to Table 8.1, which is a table of facts in addition and subtraction.

1. By how much do the numbers increase in each row and column?

2. Why are the sums in the first row and the first column the same as the addends in color?

3. What name is given to each number in color when it is used in a number sentence? to each number within the table in a square or a cell?

4. Show how to find the sum of any basic number pair from the table.

5. Use the table to show that the order of adding a number pair does not change a sum.

6. From the table give all the facts in the set of related facts in addition that have a sum of 12; give the corresponding facts in subtraction.

7. Draw an imaginary line to connect the cell containing 0 in the upper left corner to the cell containing 18 in the lower right corner. What name do we give to the numbers along this line?

8. Describe the numbers in the corresponding lines above and below the imaginary line in problem 7.

9. Describe the numbers in the co. ·-sponding lines above and below the imaginary lines in problem 8.

10. Tell why the numbers in problem 8 are odd and the numbers in problem 9 are even.

11. Tell when the sum of two numbers will be even; will be odd. (It may be necessary for the pupil to have many illustrations of adding odd and even number pairs before he is able to generalize about the sum.)

12. Draw an imaginary line to connect the lower-left cell with the upper-right cell. What are the numbers along this line?

13. Describe the numbers in the corresponding lines above and below the imaginary line in problem 12.

PRACTICING THE FACTS

Practice or drill is an essential part of the arithmetic program. To have mastery of a fact, a pupil must be able to respond to it spontaneously and with assurance. Practice will help in achieving this goal provided the pupil understands the fact. If he participates in the list of activities that are given, the amount of drill needed for mastery of the facts should be at a minimum. Brownell aptly phrased the purpose of drill as being "meaningful habituation."[4] As applied to the basic facts, the pupil should be able to give a habitual response to a fact he understands.

Practice may consist largely of a series of repetitions over a period of time of answers to the sum of a number pair. A program of this kind is not recommended. Learning is more effective when it results from a variety of experiences than from a narrow range of activities. Since practice is part of the learning program, the activities involving practice also should be varied. The teacher should try to provide the kind of practice that uses a fact in as many

[4]William A. Brownell, "Meaning and Skill: Maintaining the Balance," *The Arithmetic Teacher*, October 1956, 3:129–136.

different settings as possible. Some of these activities should provide practice in identifying and discovering the relationship between addition and subtraction. The following activities represent effective means of providing practice in dealing with the basic facts in the two operations mentioned.

1. Use "everybody show" cards for short periods of oral practice. A limited amount of oral practice is desirable. (See Appendix for a description of these cards and how they are to be used.)

2. The pupil writes the missing numerals in cross-number puzzles as shown in (A) of Figure 8.9 and in a matrix as shown in (B) of the same figure.

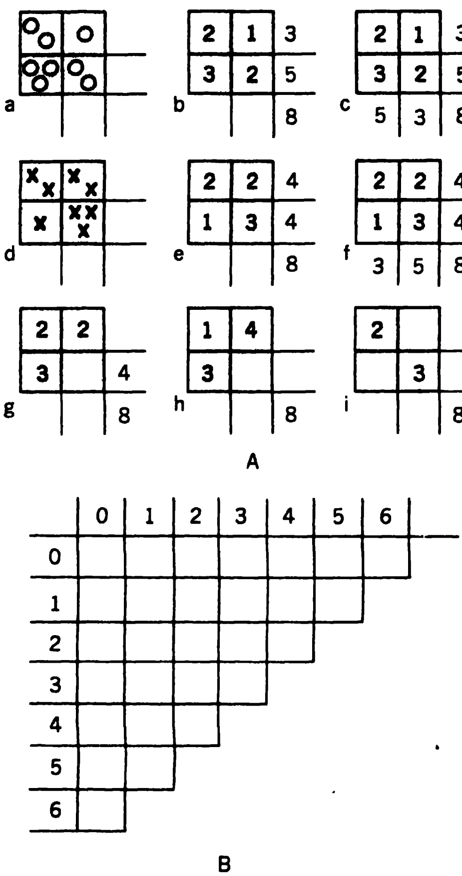


Figure 8.9

The puzzle form shown in (A) may be used after teaching a given number pair, such as (2, 3) as illustrated in a, b, and c. The pupil may not be able to fill in all of the squares in the sum until he has learned the sum of all number pairs given. The pupil writes the numerals in (B) after he has had the number pairs that have a sum of six. A similar form or matrix would be made for other sums. The pupil should be able to use the table to read the sum of a number pair and the corresponding fact in subtraction.

3. The teacher draws on the chalkboard a matrix of the kind shown in Figure 8.10 and writes the numerals in order, beginning with 0 in the first cell and filling each cell. A vertical line drawn between two joining cells indicates the first number pair to be used. In Figure 8.10 this pair is 4 and 2. The pupil writes the number fact $4 + 2 = 6$. He then writes all of the related facts for the sixes in both operations. A similar plan is followed for the set of related facts for each sum.

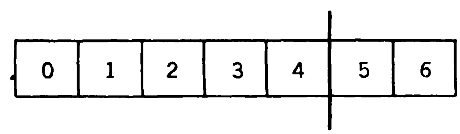


Figure 8.10

4. Have the pupil derive an answer of a grouping from some known fact. Suppose he knows that $5 + 5 = 10$. Have him tell how he can show that $6 + 7 = 13$.

5. Have the pupil write the answers on a folded sheet placed below the number pairs given in the textbook, or ditto the number pairs that are to be practiced. Give each pupil a sheet and have him record the answers. These sheets may be used more than once by the pupil who follows the plan sug-

gested for using the texbook. The teacher writes the answers on the chalkboard in the same order as they should appear on the practice sheet. The pupils exchange papers and compare the answers with those given on the board. Each pupil marks the incorrect answers on the sheet he scores. The teacher then makes a diagnosis to determine if the errors are due to chance or if they result from a faulty knowledge of the facts.

6. Write the fact as an open sentence of the type $3 + \square = 5$.

For the more able pupil, two frames or place holders may be given in one equation, as $\triangle + \square = 5$. Junge found that pupils in grade 2 who developed some facility of understanding of the basic facts in addition and subtraction enjoyed discovering the numbers that would make the equation true.[5]

The work with two place holders in an equation can be challenging to the more able pupil by noting the frames.[6] Thus, the solution in (a) must include a pair of equal numbers because both frames are the same. The solutions in (b) include every basic pair of numbers that has a sum of 8, as $3 + 5 = 8$, $5 + 3 = 8$, or $4 + 4 = 8$.

a. $\square + \square = 8$
b. $\triangle + \square = 8$

7. The pupil identifies the sum and the addends in number sentences of the type $3 + 5 = \square$ and $7 - \square = 2$. The teacher makes sure that the pupil understands the following relationships:

Addend + addend = sum

Sum − addend = addend

8. Have the pupil write a replacement for a numeral in an equation. In the number sentence $4 + 3 = 7$, the pupil replaces 4 with a frame and writes the equation as $\square + 3 = 7$. The number 4 makes the equation true. This number may be renamed with the number pairs (2, 2) and (1, 3). The fact $4 + 3 = 7$ may be written in a different form, as $(2 + 2) + 3 = 7$, $(3 + 1) + 3 = 7$, or $(1 + 3) + 3 = 7$. A place holder in an equation, for example, $\square + 2 = 7$, may be replaced by a set of number pairs. In the equation $\square + 2 = 7$, the number pairs that will make the equation true have a sum of 5. These pairs are (2, 3) and (1, 4). The order of the numbers may be reversed. It is important for the pupil to have a variety of experiences in which a number is expressed by different numerals. The number 6 makes the equation $\square + 1 = 7$ true. This number may be named by the number pairs (3, 3), (2, 4), and (1, 5). The pupil with insight may replace 6 with number pairs in subtraction. With only one-place numerals the number 6 may be named as $(9 - 3)$, $(8 - 2)$, or $(7 - 1)$. Each of these numerals may replace the frame in the equation $\square + 1 = 7$ and make the equation true.

9. Have the pupil change an example in addition to an example in subtraction, and vice versa. The following examples illustrate the procedure to use. If a pupil is given an example of the type shown in the left-hand column, he writes the corresponding example as given at the right.

Addition	*Subtraction*
$3 + \square = 5$	$5 - 3 = \square$
$\square + 2 = 6$	$6 - \square = 2$
$1 + 4 = \square$	$\square - 1 = 4$

[5] Charlotte W. Junge, "Depth Learning in Arithmetic—What Is It?" *The Arithmetic Teacher*, November 1960, 7:341–346.

[6] Mathematically, the number replacements for two different frames may be either equal or unequal. The teacher may arbitrarily set the rules that govern the replacements for the frames.

10. Have the pupil deal with number sentences that illustrate *inequalities.* It is not defensible to present equalities without introducing inequalities. A pupil in grades 1 and 2 should be able to interpret and symbolize the inequality of two numbers. Since 5 is less than 6, this fact may be symbolized as $5 < 6$. Similarly, $8 > 5$ symbolizes the fact that 8 is greater than 5. The sign of inequality always points to the smaller of the two numbers compared.

An exercise of the type that follows shows whether the pupil is able to distinguish between equalities and inequalities. In each circle he supplies the sign, =, >, or <, that will make the mathematical sentence true.

3 + 2 ○ 7	12 − 8 ○ 3
14 ○ 9 + 5	15 ○ 9 + 7
11 − 4 ○ 7	0 + 1 ○ 0

In each of the above examples, the sum or difference of a number pair named by the digits is compared with a given number. After the class learns how to deal with a number sentence of this kind, a more complex sentence should be presented. The pupil must then compare the sum or difference of two sets of number pairs. The following number sentences are of this type:

3 + 2 ○ 6 − 2	5 + 4 ○ 14 − 6
5 − 0 ○ 0 + 5	3 + 7 ○ 15 − 4
6 − 3 ○ 7 − 4	12 − 5 ○ 2 + 4

Each number pair in a number sentence at this level must be an addend or a sum of a basic fact in addition or subtraction.

Ten different activities are listed below for practicing the basic facts in addition and subtraction. The teacher, however, may decide not to have the class participate in all of these activities, although other things being equal, the greater the variety of the activities, the more effective will be the learning that results. In any case, class participation should not be restricted to only one or two of the items listed. In the past in some programs that gave a minimum of attention to meaning and understanding, the sole means of learning the facts was usually repetition. Such a program is entirely inadequate for learning mathematics in today's schools.

Mastery of the facts

The ability to give the answer to a number pair in either addition or subtraction does not necessarily constitute mastery of the basic fact involved. Mastery of a number fact implies a depth of understanding that can be achieved only by having an enriched experience with that fact. A pupil can demonstrate mastery of a fact by doing the following:

1. Demonstrate the fact with markers, with drawings, and on a number ray.
2. Write the fact in both vertical and horizontal forms.
3. Verify the fact from other known facts.
4. Give the four facts for each pair of unequal numbers.
5. Show the pattern for each set of related facts in both addition and subtraction. For the set of the sevens, the pupil can give all 16 facts in this set.
6. Identify addends and sum in an equation involving a basic fact and find any missing term in an equation of this kind.
7. Show how addition and subtraction are related.
8. Make a number sentence true that involves a fact by supplying a missing sign of equality or inequality; for example, 3 + 4 ○ 8.

9. Give the answer to a number pair with assurance and without hesitation.

10. Use a fact in a verbal problem.

The last usage is twofold: First, the pupil should be able to find the answer to a verbal problem in which a number pair is given. He must decide if the problem situation involves addition or subtraction. Second, the pupil should be able to make a verbal problem by using a number pair in either addition or subtraction.

The list of activities implies that mastery of a basic fact reveals growth of power in dealing with number. Many pupils in grade 2 do not achieve complete mastery of the basic facts in addition and subtraction. It is anticipated that the work of the following year will be so organized that most pupils will have mastered the facts before they complete that grade.

Column addition

Columns containing three or more addends involve addition of numbers that are not represented by numerals. If the unseen numeral were visible, it could be either a one- or a two-place numeral. In (a), adding downward involves adding the number named by the numeral 7 to the number 8 which is not named by a numeral. The numeral for 8 would contain one place. Adding upward involves adding 3, which is named by the numeral 3, to 12, which is not named by a numeral.

a	b.
3	2
5	5
7	3

The numeral for 12 would contain two places. For most pupils in grade 1, the work in column addition should be limited to examples in which the unseen addend is named by a one-place numeral. Some pupils have difficulty in adding a column, as in (b), because they are unable to think "7" without seeing the numeral 7 when adding downward or to think "8" without seeing the numeral 8 when adding in the opposite direction.

The pupil should practice adding a column containing three addends in both directions. This is one of the best methods of checking addition. Adding a column of three addends in both directions also illustrates the use of the associative property of addition. If we add downward in (b), the grouping is $(2 + 5) + 3$; if we add upward, the grouping is $(3 + 5) + 2$.

The class should be experienced in dealing with three addends written in horizontal form before attempting to find the sums of three addends written in columns. The way the pupil groups the addends and finds the sum in an equation indicates his readiness for working with numbers that are named by unseen numerals. In the example $3 + 2 + 4$, the numerals should be written as $(3 + 2) + 4$. The pupil should write the example as $5 + 4$. If he can give as the sum of $5 + 4$, he knows the facts. (If the pupil does not know a fact, follow the plan of teaching the fact as given on page 118.) Write a number pair, such as (6, 3), and have the pupil find the sum, or $6 + 3 = 9$. When the number sentence is written with a frame, as $\square + 3 = 9$, the pupil adds $(2 + 4) + 3$. He thinks the sum of $2 + 4$ and adds the number named by the seen 3. Practice of this kind should enable the pupil to think the sum of two numbers represented by seen numerals and to this sum add the number represented by a third seen numeral.

EXERCISES

1. Differentiate between a basic fact and a basic number pair.
2. Illustrate what is meant by teaching the basic facts as specifics. Give a critical evaluation of this plan.
3. List the properties of addition that apply to the basic facts in that operation.
4. What is meant by a subtraction situation? Differentiate between structural and functional subtraction.
5. Show why you would or would not have a pupil read a basic fact in subtraction in the same way when the fact is written in both horizontal and vertical forms.
6. Show why the phraseology for reading a fact in subtraction should or should not correspond to the subtraction situation represented.
7. Joining sets is an operation on sets and addition is an operation on numbers. Show how the application of this statement will affect the teacher's presentation of the facts in addition.
8. Make a list of the essential understandings a pupil should have before he is ready to learn the basic facts in addition.
9. What is the pattern for determining the remaining members of a set of related facts when the key fact is given?
10. Make a list of activities that you consider effective for practicing the basic facts in addition and subtraction.
11. Every basic fact should be the written record of some meaningful experience. Show how the acceptance of this statement will affect the teaching of the basic facts.
12. A teacher presented the view that mastery of the facts is of little significance if the pupil understands the properties of addition and knows how to discover a fact. Criticize this viewpoint and show why you would or would not accept it.

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ADDITION AND SUBTRACTION OF WHOLE NUMBERS

The teacher needs to understand the importance of placing the proper emphasis on structure and computational skill. Until about 1960, most textbooks dealing with the teaching of arithmetic stressed procedures for acquiring skill in performing the four operations and frequently gave no consideration to structure. On the other hand, some recent publications on the subject give little or no consideration to computation. In such cases the operations are treated almost entirely from the point of view of properties and structure. Both types of programs are faulty. The pupil must learn to add and subtract as well as understand the properties of addition and subtraction.

This chapter deals with addition and subtraction of whole numbers and with the relationship between these operations. The topics treated in this chapter are: number sentences in addition and subtraction; adding and subtracting without regrouping; adding with regrouping; subtracting with regrouping; discovering relationships between addition and subtraction.

NUMBER SENTENCES IN ADDITION AND SUBTRACTION

Number sentences (a–c) represent addition situations, while sentences (d–f) are subtraction situations.

Addition	*Subtraction*
a $8 + 9 = n$	d. $17 - 8 = n$
b $n = 8 + 9$	e. $n = 17 - 8$
c $n - 8 = 9$	f. $n + 8 = 17$

Equations (a) and (b) as well as (d) and (e) are the same when read in the reverse order. From these examples it is clear that an equation can be read from left to right or from right to left. The fact that an equation can be read in both directions becomes important when the student is introduced to algebraic equations. Often a student who always reads and solves an equation by proceeding from left to right cannot, for example, solve $24 = 3x$ until it is rewritten as $3x = 24$. In reading a number sentence, neither the order of the numbers nor the order of the signs should be changed. The numerals in equations (a–c) can be interchanged because of the commutative property of addition.

The number sentences in addition show that the sum is missing in each equation. An addition situation therefore involves finding a sum when two addends are given. A problem for equation (c) would be as follows: Jim had 9 cents left after he spent 8 cents for a postage stamp. How much did he have before he bought the stamp? The verbal statement suggests subtraction, but the situation involves addition because the sum of a number pair is to be found. Therefore the pattern for addition is *addend* + *addend* = *sum.*

The subtraction number sentences (d–f) show that the sum is always given but one addend is missing. It was stated earlier that the two different situations involving subtraction are: (1) finding the missing addend when the sum and one addend are given and (2) comparing two numbers. Although two different situations involve subtraction, the pattern for the operation consists of the first usage because addition and subtraction are inverse operations. Since *addend* + *addend* = *sum* expresses the relationship between the addends and the sum in addition, *sum* — *addend* = *addend* expresses the relationship in subtraction between the addends and the sum. The vital issue with respect to writing and solving equations involving addition and subtraction of whole numbers, then, consists in identifying two addends for addition and the sum and one addend for subtraction. The number of digits in the numerals representing the given numbers affects only the degree of difficulty in performing the algorism.

Different levels of solving equations

In beginning work with open sentences, the pupil uses a frame, such as $\square$, or a letter, such as *n*, to hold a place for a numeral in an equation. An equation of this kind may be expressed as $\square + 3 = 5$ or $n + 3 = 5$. At about grade 6 or 7, the pupil will learn that a letter, for example, *n*, which is a place holder for a numeral, is called a *variable.*

The pupil solves an equation to find the number represented by a frame or by a variable. In some equations the solution may include more than one number. In most of the equations that an elementary school pupil will solve, one number will make each equation true. In inequalities usually more than

one number will make the number sentence true.

A pupil may operate at three different levels in solving an equation. These levels may involve the application of the following procedures:

1. An intuitive method
2. The principle that the same number must be named on each side of a true equation
3. An axiom, such as the subtraction axiom, involving the solution of an equation of the type $n + 4 = 7$.

These procedures may be illustrated by solving the equation $n + 8 = 17$.

In grade 2 or 3, the pupil would use an intuitive method, which is based largely on trial and error. To solve the given equation, the pupil thinks of a number that added to 8 has a sum of 17. He may know the answer or may try different numbers to make the number sentence true. He does not necessarily discover that 17 is the sum of two numbers and that one of these numbers is 8.

The pupil solves the given equation at the second level of maturity by recognizing that 17 is the sum of two numbers and that one of the numbers is 8. To find the other number he subtracts $17 - 8$.

At the third level of maturity, the pupil applies an axiom that pertains to a given equation. The solution given applies the subtraction axiom.

$$17$$
$$8 =$$
$$n \quad = 9$$

This axiom states that the same number may be subtracted from each member of a true equation and the resulting equation will be a true equation. Thus, if $a = b$, then $a - c = b - c$. The pupil does not use this method until he begins a systematic study of algebra, which is usually not before grade 6 or 7.

Different levels of solving inequalities

Just as a pupil should grow in maturity in solving equations, he should proceed to a higher level in solving inequalities. To solve an inequality he finds the number or numbers that make the given number sentence true. The pupil may solve an inequality by using an intuitive method or by discovering a pattern that applies.

The pupil uses intuitive methods to solve an inequality in beginning work in this field. In the number sentence $n + 2 > 5$, he may replace n with different numbers, beginning with 1. He should discover that any whole number greater than 3 will make the sentence true. The pupil follows a similar plan in solving an inequality of the type $n - 3 < 5$. He replaces n with different numbers, beginning with 3 that will make the sentence true. Any whole number from 3 to 8 is in the solution set of the inequality. The pupil may replace n with different whole numbers until he discovers a pattern that applies. The application of a pattern for solving an inequality shows greater maturity than the use of an intuitive method.

The pupil can solve an inequality as though it were an equality in order to find a pattern for the solution. In the number sentence $n + 2 > 5$, if the sign $=$ were to replace the sign $>$, the value of n would be 3. Since $n + 2$ names a number greater than 5, the first whole number to make the sentence $n + 2 > 5$ true must be 4, or 1 more than 3. Similarly, substituting the equation $n + 2 = 7$ for the inequality $n + 2 < 7$, the value of n in the equation would be 5. Since $n + 2$ names a number that is less than 7, the value of n must be less than 5. The first whole number to make the sentence true is 4, or 1 less than 5.

In order to apply a pattern for the solution of an inequality of the type $n + 2 > 5$, the pupil does two things. First, he replaces the sign of inequality with the sign of equality and then finds the solution set of the resulting equation. Second, he increases the solution by 1 when the variable is on the side of greater value in the inequality; he decreases the solution by 1 when the variable is on the side of lesser value in the inequality. Thus, if $n + 2 = 5$, n would be equal to 3. Then in the inequality $n + 2 > 5$, the value of n must be at least 1 more than 3, or 4, in the set of whole numbers. If $n - 3 = 7$, the value of n would be 10. Then in the inequality $n - 3 < 7$, the value of n must be at least 1 less than 10, or 9, in the set of whole numbers.

In inequalities of the type $n + 3 < 5$ or $n + 3 > 8$, there usually are several numbers in the set of whole numbers that may replace the variable to make the sentence true. A challenging exercise for the more able learner in grade 5 or 6 is to determine the greatest or smallest whole number that a variable may represent in an inequality to make the sentence true. Thus, in the inequality $n - 5 > 8$, the smallest whole number to replace n to make the sentence true is 14. In the inequality $n + 3 < 9$, the greatest whole number to replace n to make the sentence true is 5.

ADDING AND SUBTRACTING WITHOUT REGROUPING

As soon as the pupil knows the set of facts having sums of 9 or less, he can use these facts in adding two-or-more digit numbers and in the corresponding examples in subtraction. No regrouping is involved in either addition or subtraction in examples of this kind. The new learning in dealing with these examples has to do with the addition or the subtraction of numbers named in like places in numerals.

The teacher can introduce addition of two-place numbers without regrouping by having the class solve the equation $23 + 24 = n$ or a similar equation. A suggested sequence of activities is as follows:

1. Have a pupil demonstrate the addition of 23 and 24 with a place-value chart, as shown in Figure 9.1.

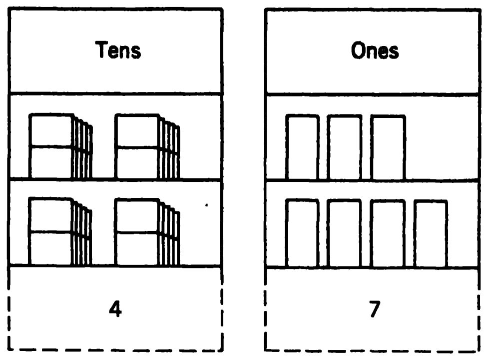

Figure 9.1

2. Write each digit with its place value and then find the sum.

$$\begin{array}{rl} 23 = & 2 \text{ tens} + 3 \text{ ones} \\ +\,24 = & \underline{2 \text{ tens} + 4 \text{ ones}} \\ & 4 \text{ tens} + 7 \text{ ones} = 40 + 7 = 47 \end{array}$$

3. Find the sum when the numerals are written in expanded notation.

$$\begin{array}{rl} 23 = & 20 + 3 \\ +\,24 = & 20 + 4 \\ & 40 + 7 = 47 \end{array}$$

4. Show the standard form of the algorism. Have the pupil identify the place that a digit holds in each addend and in the sum.

$$\begin{array}{r} 23 \\ \underline{+\,24} \\ 47 \end{array}$$

5. Have the pupil write the other example in addition and identify the commutative property of addition.

$$\begin{array}{r} 23 \\ +24 \\ \hline 47 \end{array} \qquad \begin{array}{r} 24 \\ +23 \\ \hline 47 \end{array}$$

6. Write the corresponding examples in subtraction.

$$\begin{array}{r} 47 \\ -23 \\ \hline 24 \end{array} \qquad \begin{array}{r} 47 \\ -24 \\ \hline 23 \end{array}$$

7. Have the class write the following four equations:

$$\begin{array}{ll} 23 + 24 = 47 & 47 - 23 = 24 \\ 24 + 23 = 47 & 47 - 24 = 23 \end{array}$$

8. Use the following equations:

Addend + addend = sum
Sum − addend = addend

Identify the addends and the sum in each equation in (7). Show that addition is a check for subtraction.

After adding and subtracting a few examples in which the addends are two-place numerals, the teacher should give an example in which the addends are three-place numerals, as shown at the right. The class should identify each place in the numerals and generalize as follows: ones are added to ones, tens to tens, hundreds to hundreds, and the like. Similarly, numbers named in like places are subtracted.

$$\begin{array}{r} 425 \\ +132 \\ \hline \end{array}$$

The activities suggested for introducing addition and subtraction of two-place numbers gives emphasis both the relationship between the operations and to computation. The first few lessons dealing with these operations stress computation and place value. Emphasis is then placed on the relationship between the operations. In many arithmetic classrooms the latter type of activity is absent.

ADDING WITH REGROUPING

Regrouping in addition occurs when the sum of the numbers named in any column is 10 or more. That place is overloaded and the number named must be regrouped. The traditional name for the regrouping procedure is *carrying*. The terms "carrying" in addition and "borrowing" in subtraction should be deleted from the vocabulary of the mathematics classroom and replaced by "regrouping."

The model lesson on page 27 shows how to introduce addition involving regrouping of the sum. The sequence of activities for finding the sum of 28 and 26 is as follows:

1. Use rectangular strips and squares to represent the addends and the sum.
2. Find the sum by using a place-value chart.
3. Show the sum on a number ray (Fig. 9.2).

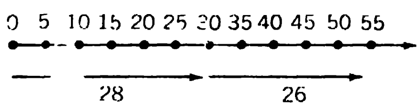

Figure 9.2

4. Write the numerals in expanded form.

$$\begin{array}{rcl} 28 &=& 20 + 8 \\ +26 &=& 20 + 6 \\ \hline && 40 + 14 = 54 \end{array}$$

5. Write the partial sums and then find their sum.

$$\begin{array}{r} 28 \\ +26 \\ 14 \\ 40 \\ \hline 54 \end{array}$$

6. Show the algorism. The pupil writes the 1 ten as shown in initial work.

$$\begin{array}{r} {}^{1}\\ 28 \\ +26 \\ \hline 54 \end{array}$$

7. Identify the new element, which consists in dealing with a place that is overloaded.

8. Read and interpret the presentation given in the textbook.

9. Write the two number sentences that may be formed in addition. They are $28 + 26 = 54$ and $26 + 28 = 54$.

Addition involving regrouping in dealing with two-place numerals should be followed with three-place numerals in which the tens' place in the sum will be overloaded, as in the example at the right. The major objective in the presentation is to enable the pupil to discover the pattern for regrouping a place that is overloaded in the sum. The same procedure applies to regrouping the sum in an overloaded place regardless of the place in the numeral. To help the pupil discover the pattern for regrouping, have the class add in examples of the type shown.

$$\begin{array}{r} 453 \\ +172 \\ \hline \end{array}$$

a. $\begin{array}{r} 48 \\ +27 \\ \hline \end{array}$	b. $\begin{array}{r} 372 \\ +164 \\ \hline \end{array}$	c. $\begin{array}{r} 4731 \\ +2805 \\ \hline \end{array}$
d. $\begin{array}{r} 458 \\ +176 \\ \hline \end{array}$	e. $\begin{array}{r} 54 \\ +71 \\ \hline \end{array}$	f. $\begin{array}{r} 49 \\ +75 \\ \hline \end{array}$

After the pupil discovers the pattern to use for regrouping an overloaded place in the sum, he should add in examples in which two places in the sum will be overloaded. In example (d), the sums in both ones' and tens' places must be regrouped. The teacher has the pupil generalize about the procedure for dealing with overloaded places in examples of multiple regrouping. Whenever the number in a place in the sum is 10 or more, that number must be regrouped. The pupil may not understand how that generalization applies to examples in which the sum of the first column to the left is overloaded. In example (e), the sum in the column on the left is 12, and this numeral is written in the sum. The pupil may think the overloaded place is not regrouped. The grouped value of 12 tens is 1 hundred and 2 tens. The 2 is written in the tens' column and the 1 is written in the hundreds' column. There are no hundreds to be added to the 1 hundred, hence the pupil writes 12 tens in the sum, which is the same as 1 hundred and 2 tens. The example illustrates the fact that adding 0 to a number does not change that number. Example (f) is a combination of types (a) and (e).

Adding by endings

Adding by endings consists in adding a number named by a two-place numeral to a number named by a one-place numeral in one mental response. In the example at the right, if a pupil thinks, "18 + 5 is 23," he uses adding by endings. On the other hand, if he thinks, "8 + 5 is 13; 1 + 1 is 2," he adds by using regrouping.

$$\begin{array}{r} 18 \\ +5 \\ \hline \end{array}$$

The sum of two numbers that illustrate adding by endings may be in the *same* decade as the number named by the two-digit numeral, as in (a), or in the *next* decade, as in (b).

The procedure illustrated in (b) is sometimes designated as "bridging the tens or the decade."

a. $\begin{array}{r} 23 \\ +4 \\ \hline 27 \end{array}$	b. $\begin{array}{r} 27 \\ +4 \\ \hline 31 \end{array}$

Adding by endings is used in two places. First, it is used in column addi-

tion, as shown at the right. Adding downward, the classification of the number pairs is as follows: 7 + 6 = 13, a basic fact; 13 + 8 = 21, adding by endings with bridging; 21 + 4 = 25, adding by endings without bridging.

$$\begin{array}{r} 7 \\ 6 \\ 8 \\ \underline{4} \end{array}$$

Second, adding by endings occurs in multiplication when the number from regrouping a product is added to the next product, provided the next product is named by a two-place numeral. In the example at the right, it is necessary to add 5 to 32 (8 × 4) and 3 to 48 (8 × 6), thus obtaining the product 5176. The sum of 32 + 5 is in the same decade as the product 32, but the sum of 48 + 3 is in the next higher decade than the product 48.

$$\begin{array}{r} 647 \\ \underline{\times\ 8} \\ 5176 \end{array}$$

The maximum number from regrouping that is added to the next product is always one less than the multiplier. If the multiplier is 6, the maximum number from regrouping to be added to the next product is 5. Since 10 × 6 = 60, it is not possible to have a regrouped number of 6 to be added to the next product when 6 is multiplied by a number named by one of the 10 digits.

There are 90 examples that result from adding 1 through 9 to each number from 10 through 19. Half of these examples will have sums in the teens and half will have sums in the twenties. In each of the eight succeeding decades there are 90 such examples, in which half have sums in the same decade and half have sums in the next decade.

Teaching adding by endings The purpose of teaching adding by endings is to enable a pupil to add examples in this group in one mental response. The teacher should begin with examples in which the sum is in the same decade as the two-digit numeral. The following procedures are recommended for grade 3:

1. Have the pupil give the sums when the examples are written in sequence in vertical form as

$$\begin{array}{r} 12 \\ \underline{+3} \end{array} \qquad \begin{array}{r} 22 \\ \underline{+3} \end{array} \qquad \begin{array}{r} 32 \\ \underline{+3} \end{array}$$

The number pair 2 + 3 is the "key fact" of the examples in the given set.

2. Have the pupils give the sums when the examples are written in sequence in horizontal form, as 12 + 3 = □, 22 + 3 = □, 32 + 3 = □.

3. Have the pupil give the sums when the examples using the same key fact are not in sequence, as 12 + 3 = □, 32 + 3 = □, 22 + 3 = □.

4. Have the pupil give the sums when the examples are not in sequence and contain different key facts, as 11 + 4 = □, 13 + 5 = □, 25 + 2 = □.

5. Write a digit, for example, 4, on the chalkboard. Then have the pupil write the sum of 4 and a dictated number, such as 12, 28, or 37. The two-place numeral is unseen, hence the pupil must think this number and add the number named by the seen 4.

The same sequence can be used for examples involving bridging the decade. A variation of item (5) consists in writing on the chalkboard the two-digit numeral in verbal form, as shown. The pupil writes the sums. He must visualize the number named by "sixteen" and then add the numbers named by the seen numerals.

$$\text{sixteen}\left\{\begin{array}{l} 8 \\ 7 \end{array}\right.$$

Long-column addition

Many pupils group numbers out of sequence when finding the sum of the numbers named in a column. The groupings may be formed for two reasons. First, the pupil finds it difficult to add by endings. He makes certain groupings to help overcome this difficulty. Second, he shows insight into numbers by the way he groups the numbers. To show that pupils discover different ways of grouping addends, give the example at the right to pupils in grade 5 or 6 and ask each pupil to give his thought pattern in finding the sum. The variety of responses will show that different procedures are used. The following are some of the methods or thought patterns used in finding the sum of the six addends.

$$\begin{array}{r} 4 \\ 4 \\ 7 \\ 4 \\ 7 \\ \underline{7} \end{array}$$

1. Add the numbers in sequence.
2. Group 3 fours and 3 sevens and then add 12 and 21.
3. Form 3 groups of (4 + 7) or 3 elevens.
4. Add 8, 11, and 14.

These and other ways are used by pupils to find the sum. The variety of responses indicates that many pupils try to discover a pattern for grouping numbers. Although a pupil may have been taught to add the numbers named in a column in sequence, he tries to discover a pattern for grouping that he finds helpful in addition.

The pupil should discover and generalize that the way addends are grouped or arranged does not affect the sum. This generalization is a consequence of the commutative and associative properties of addition. A challenging exercise for the more able learner consists in having him give the sequence of steps and the property involved in grouping addends to have a sum of 10, as shown in the following illustration:

3 + 8 + 7	The addends
(3 + 8) + 7	Associative property
3 + (8 + 7)	Associative property
3 + (7 + 8)	Commutative property
(3 + 7) + 8	Associative property
10 + 8	Renaming 3 + 7

An effective exercise for the more able learner is to find the sum of examples in which proper grouping of the addends will facilitate computation. The pupil selects a pair of numbers having a sum that is a multiple of 10, as 13 + 17. The superior pupil in grade 5 should be able to do most of the work without the use of paper and pencil. The addends in the following examples may be grouped as shown to facilitate computation.

24 + 38 + 76 + 12
17 + 19 + 31 + 23
43 + 81 + 54 + 19 + 57
(24 + 76) + (38 + 12)
(17 + 23) + (19 + 31)
(43 + 57) + (81 + 19) + 54

The number of addends could vary from three to as many as five or six. It should be assumed that the pupil understands the principle for grouping addends in a given way.

SUBTRACTING WITH REGROUPING

Methods of subtraction

In order to perform the subtraction algorism in the example at the right, the number named in the sum must be regrouped. Subtraction in an example of this kind is known as *compound subtraction*. There are three well-known methods used in compound subtraction:

$$\begin{array}{rl} 62 & \text{sum} \\ -\ 17 & \text{addend} \\ \hline 45 & \text{addend} \end{array}$$

1. The decomposition method;
2. The equal-additions method;
3. The additive method.

The decomposition and the equal-additions methods are the two subtraction methods most widely used in this country. The additive method and the method of equal-additions employ different thought patterns, but the basic facts used in subtracting in an example are the same for the two methods.

To subtract in the example at the right by the decomposition method, the thought pattern is as follows: "6 from 13 = 7; 5 from 11 = 6; 1 from 6 = 5." The numbers named by the digits in hundreds' and tens' places of 723 were regrouped.

$$\begin{array}{r} 723 \\ -156 \\ \hline \end{array}$$

Using the same example to subtract by the method of equal additions, one finds that the thought pattern is as follows: "6 from 13 = 7; 6 from 12 = 6; 2 from 7 = 5." The cardinal number named in the hundreds' and tens' places of 156 were increased by 1. The cardinal number named in tens' and ones' places in 723 were increased by 10.

Using the additive method, one finds the thought pattern to be as follows: "6 and 7 are 13, write 7 and add 1 to 5; 6 and 6 are 12, write 6 and add 1 to 1; 2 and 5 are 7, write 5." It may be seen that the additive method uses the same basic combinations in subtraction in the given example as are used in the method of equal additions, but the phraseology of each method is different. In the given example the basic number pairs used by both methods are 13 − 6, 11 − 5, and 6 − 1. There are variations of the method described that combine the additive principle with regrouping of the sum (minuend).

The additive method, sometimes known as the Austrian method of subtraction, was more widely used during the 1920s and 1930s than at present. The proponents of this method assumed that the same *bond* is used in learning a fact in both addition and subtraction. Thus if a pupil establishes the bond that 4 and 8 are 12, it was thought that he should know the answer to the two number pairs

$$\begin{array}{r} 4 \\ +8 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{r} 12 \\ -4 \\ \hline \end{array}$$

The advocates of this method assumed that there was an easy transfer from the addition notation to the subtraction notation. It followed that a saving in time would result in teaching the two operations as a unified process rather than as separate operations.

By the additive method of subtraction the pupil learned addition and subtraction as representing the same operation. Today the pupil learns that addition and subtraction are opposite operations. The one operation undoes the other. Therefore, the additive method of subtraction should not be used in a program that emphasizes meaning and structure. We shall give no further consideration to this method.

Mathematical basis of the two methods

The teacher should use either the decomposition method or the method of equal additions in presenting compound subtraction. Each method illustrates one or more mathematical principles that the pupil must understand if this phase of subtraction is to be meaningful to him. To subtract in the example at the right by the decomposition method, 54 must be regrouped as 4 tens and 14 ones. The pupil should be familiar with regrouping numbers and therefore should be able to understand the procedure in subtracting in the example.

$$\begin{array}{r} 54 \\ -28 \\ \hline \end{array}$$

The method of equal additions takes its name from the procedure used in this form of subtraction. To subtract

by this method in the above example, it is necessary to add 10 ones to 4 ones. It is then possible to subtract 8 ones from 14 ones. Since 10 ones are added to 54, a like amount must be added to 28 to keep from changing the difference between the two numbers in the given example. Adding 1 ten is the equivalent of adding 10 ones, hence add 1 ten to the 2 tens of 28. The two mathematical principles involved in changing numbers in subtraction by the method of equal additions are:

1. Adding the same number to the sum and to the given addend does not change the missing addend.

2. Adding 10 to a number named by a digit in the sum is equivalent to adding 1 to the number named by the digit one place to the left in the given addend.

The class need not learn these two principles in order to understand subtraction by the decomposition method. For this and the following two reasons the pupil should find it easier to understand the decomposition method than the method of equal additions. First, the procedure of the decomposition method is easier to model, using markers, while the equal-additions method is not well adapted to the use of classroom materials.

Second, in the decomposition method the pattern of regrouping is more uniform. In this method regrouping in subtraction is the inverse of regrouping in addition. In example (a), the 13 ones in the sum are regrouped as 1 ten and 3 ones. The addition is then continued. Now use the decomposition method in example (b). The number in the sum (43) is regrouped as 3 tens and 13 ones and then the subtraction completed.

$$\begin{array}{r} 25 \\ +18 \\ \hline 43 \end{array} \qquad \text{b.} \quad \begin{array}{r} 43 \\ -18 \\ \hline 25 \end{array}$$

The regrouping in subtraction is done in the reverse order from that done in addition. Since each operation is the inverse of the other, the regrouping in these operations should be done in the inverse order.

Now use the method of equal additions to subtract in example (b). The sum (43) is not regrouped. Instead, 10 ones are added to 3 ones to make 13 ones. The given sum is changed to 4 tens and 13 ones. Next, 1 ten is added to the given addend (18) to make 2 tens and 8 ones. The subtraction is then performed. The changes made in the sum and in the given addend are not related to the corresponding example (a) in addition. The equal-additions method, therefore, does not help the pupil to discover the inverse relationship between addition and subtraction, while the decomposition method helps to accentuate this relationship. The difference with respect to both ease and depth of understanding in favor of the decomposition method is the chief reason for teaching compound subtraction by this method. This is especially true for introductory work in this topic.

Research dealing with compound subtraction

Probably more studies have been devoted to determining the relative value of the decomposition and equal-additions methods of subtraction than to other topic in arithmetic. A list of such investigations is included in an earlier edition of this text.[1] Most of the results showed that the equal-additions method is superior when meaning and understanding are not factors. An investigation by Brownell and Moser took these

[1] Foster E. Grossnickle and Leo J. Brueckner, *Discovering Meanings in Elementary School Mathematics* (fourth ed.; New York: Holt, Rinehart and Winston, Inc., 1963), pp. 150–151.

factors into account, however, and in this case the decomposition method was preferred.[2] The conclusions of the various studies may be interpreted as follows: *If the pupil is to understand the work in compound subtraction, teach the decomposition method; if the pupil is to learn to subtract mechanically, teach the equal-additions method.*

Steps in teaching compound subtraction

Compound subtraction involves regrouping. The sequence of activities for teaching compound subtraction should parallel those suggested for addition involving regrouping. The topic may be introduced by having the pupil solve the subtraction situation presented by the following problem: A newsboy had 45 papers to deliver. After he delivered 27 papers, how many remained to be delivered?

The teacher wrote the example on the chalkboard as shown and asked how this example differed from other examples that the class solved in subtraction.

$$\begin{array}{r} 45 \\ -27 \\ \hline \end{array}$$

The pupils stated that they did not know how to subtract 7 ones from 5 ones. The teacher could be sure that each pupil understood the objective of the lesson, which consisted in learning how to proceed in subtraction when a digit in the sum named a smaller number than the digit in the corresponding place in the given addend.

The following activities may be included in introducing compound subtraction:

1. Have the class suggest ways to find the answer. Some member of the class may suggest counting forward from 27 to 45 or counting backward from 45 to 27. Another pupil may give the following thought pattern: "27 and 10 are 37; 3 more are 40, and 5 more are 45; $10 + 3 + 5 = 18$."

2. Demonstrate how to find the answer on a number ray (Fig. 9.3).

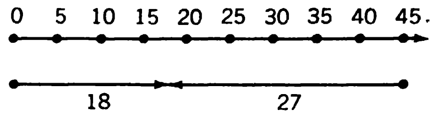

Figure 9.3

3. Have the pupil use his rectangular strips and sequences to find the difference.

4. Use a place-value chart to represent the subtraction situation, as shown in Figure 9.4. Example (A) in the figure shows how to represent 45 as 4 tens and 5 ones; (B) renames 45 as 3 tens and 15 ones; (C) shows how to represent the remainder after removing 2 tens and 7 ones.

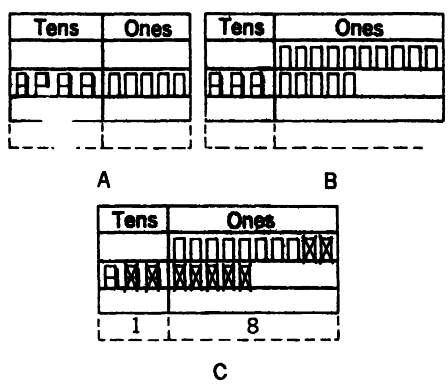

Figure 9.4

5. Write each digit with its place value.

45 = 4 tens 5 ones = 3 tens 15 ones
− 27 = 2 tens 7 ones = 2 tens 7 ones
1 ten 8 ones = 18

[2]W. A. Brownell and H. E. Moser, *Meaningful vs. Mechanical Learning. Study in Grade III Subtraction* (Durham, N.C.: Duke University Press, 1949).

6. Write each numeral in expanded form.

$$\begin{array}{rcccl} 45 & = & 40 + 5 & = & 30 + 15 \\ -27 & = & \underline{20 + 7} & = & \underline{20 + 7} \\ & & & & 10 + 8 = 18 \end{array}$$

7. Write the digit in each regrouped place in the numeral. The 5 ones are changed to 15 ones and the 4 tens are changed to 3 tens. The circled numeral shows that the ones' place is overloaded.

$$\begin{array}{r} {}^{3}\cancel{4}{}^{15}\cancel{5} \\ -27 \\ \hline 18 \end{array}$$

8. The class refers to the textbook and reads the presentation given. The work the pupils did with models and numerals written in expanded form aids their understanding of the verbal presentation of the topic.

9. Write the two number sentences for the given example. They are:

$45 - 27 = 18$ and $45 - 18 = 27$

The development is not complete until the pupil is able to relate the example in subtraction to the corresponding example in addition. This phase of the new work may not take place until one or more lessons following the initial presentation of compound subtraction. The pupil relates the two operations by identifying the different numbers named in each example, as shown.

18	Addend	45	Sum
+ 27	Addend	− 18	Addend
45	Sum	27	Addend

The class should be able to write the four number sentences for the given examples.

The introduction of modern mathematics resulted in the terminology of the numbers used in subtraction. The terms "sum" and "addend" formerly applied only to addition. Different terminology was used for subtraction, as shown in the example that follows. The term "difference" is more widely used in mathematics than "remainder."

45	Minuend
− 18	Subtrahend
27	Remainder, or difference

The terms "minuend" and "subtrahend" have no special significance to the pupil at the lower-grade level. Indeed, they have limited significance in later work in mathematics, hence there is little, if any, justification for introducing these terms in the elementary school.

After one or two lessons with introductory work in compound subtraction, the pupil should know how to subtract in an example in which regrouping is needed in tens' place, as in example (a), or in hundreds' place, as in example (b), or in both places. These examples illustrate the need for regrouping before subtraction can be completed. The pupil should discover that the pattern for regrouping in example (c) applies to regrouping in examples (a) and (b).

a.		b.		c	
	738		5469		43
	− 145		− 2805		− 18

In compound subtraction, the procedure for regrouping is the same regardless of the place in a numeral. When a pupil makes this discovery he understands how to proceed in compound subtraction. The teacher should then challenge the more able pupil to verbalize the generalization pertaining to the need and the procedure to follow in subtraction of this kind.

Supplementary symbolic aids

Some teachers do not approve of the symbolic representation of the example shown at the right. They do not permit their pupils to

$$\begin{array}{r} {}^{3}\cancel{4}{}^{15}\cancel{5} \\ -27 \\ \hline \end{array}$$

regroup the numerals in written symbolic form. In that case the pupil has to perform a mental operation on numbers with unseen numerals in order to do the subtraction. It is much easier for a pupil to understand the regrouping process when the work is written in full, as shown above, than when the example is written in the form shown at the right. The pupil who understands the process and performs the subtraction in this example operates at a higher level of maturity than the one who subtracts in the example with the numeral 45 written in the regrouped form as 3 tens and 15 ones. Since learning is a growth process, it seems reasonable to suggest that a pupil should begin the process at a level at which he is certain to understand the work and progress to a higher and a more mature level of subtraction.

$$\begin{array}{r} 45 \\ -27 \\ \hline \end{array}$$

The example given may be rewritten in three different ways, as follows:

a
$$\begin{array}{rcl} 45 & = & 4 \text{ tens } 5 \text{ ones} \\ -27 & = & \underline{2 \text{ tens } 7 \text{ ones}} \\ & = & 3 \text{ tens } 15 \text{ ones} \\ & = & \underline{2 \text{ tens } \;\; 7 \text{ ones}} \\ & & 1 \text{ ten } \;\; 8 \text{ ones} = 18 \end{array}$$

b
$$\begin{array}{rcccl} 45 & = & 40 + 5 & & 30 + 15 \\ -27 & = & \underline{20 + 7} & - & \underline{20 + \;\; 7} \\ & & & & 10 + \;\; 8 = 18 \end{array}$$

c
$$\begin{array}{r} {}^{3\,15} \\ \cancel{45} \\ -27 \\ \hline 18 \end{array}$$

The teacher may show different forms for regrouping 45. It is important to bear in mind that the pupil does not operate at a mature level in performing the operation of subtraction if it is necessary for him to regroup numbers and write the numerals for these regrouped numbers.

Writing the regrouped numeral in a subtraction example is sometimes known as a *crutch*. Many teachers disapprove of the use of a crutch in computation. Suppose a pupil does not succeed well in subtraction when the number to be regrouped is not named by written numerals, but he does succeed when this number is named by written numerals. Then the procedure is not a crutch. In that case the written work is an essential visual ***symbolic learning aid*** for that pupil. On the other hand, the pupil who, with a feeling of security and confidence, could succeed without the use of this learning aid but persists in writing the work in full uses a crutch. The crutch causes this pupil to operate at a lower level of maturity than that at which he should be working. In this case it would be difficult for him to show growth in dealing with numbers. It follows, then, that what may be a crutch for one pupil may serve as a learning aid for another. There should be no occasion during the carrying out of the class program when the use of a symbolic learning aid is discontinued or prohibited. However, the pupil should be urged to operate without the use of this aid. He should be encouraged to show pride in the fact that he is able to subtract in the same manner as most adults.

Multiple regrouping in subtraction

After a pupil learned how to regroup in any one place in a numeral in the sum in addition, he learned how to regroup in two or more places in a numeral in the sum. That same sequence should apply in compound subtraction. When the pupil learns how to regroup in ones' place in a numeral in subtraction, he should learn to subtract in an example of the type shown at the right. The number named in the sum (752) must be regrouped twice before

$$\begin{array}{r} 752 \\ -387 \\ \hline \end{array}$$

the conventional algorism for subtraction can be completed.

The list of pupil activities for introducing examples involving multiple regrouping should be limited compared to that given earlier for the initial work in compound subtraction. Such activities should be confined to the following:

1. Writing the numerals in expanded notation
2. Writing the digit or digits in each regrouped place in the numeral
3. Reading and interpreting the presentation given in the textbook.

The expanded notation for the numerals in the given example would be as follows:

a

$$\begin{array}{rcl} 752 & = & 700 + 50 + 2 = \\ \underline{-387} & = & \underline{300 + 80 + 7} = \end{array}$$

b

$$\begin{array}{l} 700 + (40 + 10) + 2 = \\ \underline{300 + 80 + 7 =} \end{array}$$

c.

$$\begin{array}{l} 700 + 40 + 12 = \\ \underline{300 + 80 + \ \ 7} = \end{array}$$

d e

$$\begin{array}{rcl} (600 + 100) + 40 + 12 & = & 600 + 140 + 12 \\ \underline{300 \qquad + 80 + \ \ 7} & = & \underline{300 + \ \ 80 + \ \ 7} \\ & & 300 + \ \ 60 + \ \ 5 = 365 \end{array}$$

The numerals in parentheses in steps (b) and (d) illustrate how different names are used for the same number. Steps (c) and (e) illustrate the associative property of addition. The class should be able to give the reason for each step in the solution.

The second activity for introducing multiple regrouping in compound subtraction consists in writing the digit or digits for each place, as follows:

$$\begin{array}{ccccccccccc} H & T & O & & H & T & O & & H & T & O \\ 7 & 5 & 2 & = & 7 & 4 & 12 & = & 6 & 14 & 12 \\ \underline{-3} & \underline{8} & \underline{7} & = & \underline{3} & \underline{8} & \underline{7} & = & \underline{3} & \underline{8} & \underline{7} \\ & & & & & & & & 3 & 6 & 5 \end{array}$$

The letters H, T, and O represent hundreds, tens, and ones, respectively.

Finally, the pupil should read and interpret the presentation given in his textbook.

Comparing regrouping in the two operations

The class should compare multiple regrouping in an example in addition with regrouping in the corresponding example in subtraction. The following two examples show the procedures for comparison.

$$\begin{array}{ccccccccccc} H & T & O & & & & & & & & \\ 2 & 5 & 8 & & & & & & & & \\ \underline{+3} & \underline{9} & \underline{4} & & H & T & O & & H & T & O \\ 5 & 14 & 12 & = & 5 & 15 & 2 & = & 6 & 5 & 2 \end{array}$$

$$\begin{array}{ccccccccccc} H & T & O & & H & T & O & & H & T & O \\ 6 & 5 & 2 & = & 6 & 4 & 12 & = & 5 & 14 & 12 \\ \underline{-3} & \underline{9} & \underline{4} & = & \underline{3} & \underline{9} & \underline{4} & = & \underline{3} & \underline{9} & \underline{4} \\ & & & & & & & & 2 & 5 & \end{array}$$

The illustrations indicate that regrouping in addition is in the inverse order of regrouping in subtraction. In addition, regrouping is done from right to left; in subtraction, from left to right. The teacher should have the class point out the inverse procedures to follow in the two operations.

Zeros in subtraction

Many of the early studies dealing with difficulties that pupils experience in subtraction indicated that zeros were the source of a great number of errors. These investigations were made at a time when little emphasis was given to understanding the work. Many of the difficulties encountered in subtracting resulted from a lack of understanding of the function of zero. The pupil should understand that zero in a numeral, such as in 702, holds the tens' place and represents the number of tens in that place. This knowledge should be supplemented by the procedure to follow in regrouping a number in which zero occupies one or more places in a numeral.

There are three stages of growth in a pupil's procedure in dealing with an example of the type shown below. These stages of operation are as follows:

$$\begin{array}{r} \\ 702 \\ -\,135 \end{array} \qquad \begin{array}{r} 6(10) \\ \cancel{7}\cancel{0}2 \\ -\,135 \end{array} \qquad \begin{array}{r} 6\ \cancel{10}\ (12) \\ \cancel{7}\cancel{0}2 \\ -\,135 \end{array}$$

1. The number named in hundreds' place is regrouped first, then the number of tens is regrouped, and finally the number of ones, as shown. Thus the 7 hundreds are changed to 6 hundreds and 10 tens; then one of the tens is regrouped as 10 ones. The final regrouped number is 6 hundreds, 9 tens, and 12 ones.

2. The numeral 702 is given a grouped value of 70 tens and 2 ones. This number is then regrouped as 69 tens and 12 ones, or 6 hundreds, 9 tens, and 12 ones.

3. The digits of 702 are given a value without regard to the regrouping process. To subtract in the given example, the pupil thinks of the 2 as 12, the 0 as 9, and the 7 as 6.

Pupils who subtract by using each of the three patterns of thinking given above operate at different levels of understanding. The pupil who follows the process of regrouping as described in the first plan should use his kit material to objectify the process.

According to the second plan, the pupil must understand that the number named by the digits in a numeral may have both a grouped and an ungrouped value. If he knows that there are 70 tens in 702, it is easy to understand why 702 can be regrouped as 6 hundreds, 9 tens, and 12 ones.

According to the third plan, the pupil has used a mature pattern of thinking as applied in compound subtraction.

Round numbers

When we state that the distance from coast to coast is 3000 miles, we use *round numbers*. Round numbers are an approximation of the true amount. It is much easier to think intelligently with round numbers than with exact numbers. To illustrate, let us assume that a trip is to be made by car between two places 287 miles apart. This distance can be rounded off as 300 miles. At 50 miles per hour, the driving time would be approximately 6 hours and the gas consumption would be about 20 gallons. If the actual distance of 287 miles were used, it would be difficult to estimate either the driving time or the gas consumption. The illustration shows that a major function of rounded numbers is to increase literacy in dealing with numbers. Buswell found that many high school and college students were unable to reasonably estimate an answer by using round numbers.[3]

The work dealing with round numbers should be introduced not later than grade 3. When the pupil adds or subtracts in examples containing two- or more-place numerals, he should round off the numbers to check on the reasonableness of the sum or difference.

$$\begin{array}{rcr} 47 & \rightarrow & 50 \\ +\,36 & \rightarrow & 40 \\ \hline 83 & & 90 \end{array}$$

In the illustration, the sum expressed in exact numbers is 83 and the sum of the rounded numbers is 90. The indicated sum is sensible, since it is near the approximate sum of 90.

A problem of the following type may be used to introduce the topic of round-

[3]G. T. Buswell, *Patterns of Thinking in Problem Solving* (Berkeley, Calif.: University of California Press, 1956), p. 134.

ing off numbers: The flagpole by the school is 34 feet high. Is the height of the pole nearer to 30 feet or to 40 feet? Use a number ray to have a pupil verify his answer (Fig. 9.5). He identifies 34 on the ray and shows that 34 is nearer to 30 than to 40. Similarly, he gives all the whole numbers between 30 and 40 that are nearer to 30 than to 40 and all that are nearer to 40 than to 30. When we express numbers between 30 and 40 as either 30 or 40, we say that they are rounded off to the nearest 10.

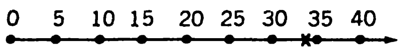

Figure 9.5

In the way described, the pupil identifies in a given decade other whole numbers that are nearer to the smaller or to the larger multiple of 10. He will discover that a number can be rounded off to the smaller multiple of 10 when the digit in ones' place in a numeral is 4 or less; to the larger multiple of 10 when the digit in ones' place is 6 or more. The pupil must then learn how to round off a number to the nearest 10 when the digit in ones' place in the numeral naming that number is 5. The teacher must tell the class that mathematicians agreed to round off a number of this kind to the next higher multiple of 10. Thus, 35 to the nearest 10 is 40.

In order to round off a number to the nearest 10, the pupil must be able to do the following:

1. Identify the multiple of 10 that is the next smaller and larger number than the given number.

2. Decide if the given number is nearer to the smaller or larger multiple of 10 found in 1

3. Know how to deal with a number that is midway between the two multiples of 10.

The ability to perform two of the three items enumerated depends upon the pupil's knowledge of ordinal numbers. He must know that 46 is nearer to 50 than it is to 40 in order to round off 46 correctly. The position of a number in a decade with relation to the midpoint of the decade determines to which multiple of 10 a number should be rounded off for accuracy to the nearest 10. The teacher should have the pupil supply the missing numerals or signs in examples of the following type.

1. Write the numeral for the number midway between
   a 40 and 50 c 150 and 160
   b. 70 and 80 d. 320 and 330
2. Write the missing numeral:
   a 70, 75, __ c 140, 145, __
   b. __, 55, 60 d 670, __, 680
3. Insert the correct sign, >, , or =, to make the number sentence true.
   a. 46 ( ) 45 d 165 ( ) 165
   b 31 ( ) 35 e. 175 ( ) 172
   c. 138 ( ) 135 f 244 ( ) 245

The pattern for rounding off a number to the nearest 10 applies to rounding off a number to the nearest 100 or 1000 or any power of 10. By the time the pupil completes the work in grade 4 he should have a mastery of how to round off whole numbers to a given degree of accuracy. After he understands how to round off numbers and has developed insight into the procedure, the following rules for rounding off may be formulated:

1. Use a 0 to replace each digit dropped in rounding off a number.

2. When the first digit to be dropped is 5 or more, increase by one the next digit to the left.

3. When the first digit to be dropped is less than 5, drop that digit and all digits to the right of that digit.

The number of digits to be retained in a numeral naming a number to be rounded off depends upon the use to

be made of the rounded off number. The usage often presents a difficulty that cannot be dealt with effectively in the elementary school. For purposes at this level, rounding off is intended to ease computation in finding an approximate answer to a number situation. For most purposes, one or at most two significant digits in a numeral name the number to give the degree of precision needed.

Checking addition

The usual procedure for checking addition is to add in the opposite direction. The writers recommend adding downward because the usual pattern for subtraction is upward. Since these operations are opposites, it is well to emphasize this fact in performing these operations. Therefore, the accepted pattern for addition is downward and that for subtraction is upward.

In the example at the right, the pupil would add downward and check the work by adding in the reverse order. After adding upward in the first column, he should write the regrouped digit 3 in tens' place in the sum, as shown. Then he should add the tens, beginning at the bottom of the column. A similar plan should be followed in dealing with each column in an example.

$$\begin{array}{r} {\scriptstyle 2\ 3\ \ } \\ 3\ 4\ 8 \\ 7\ 6\ 9 \\ 5\ 8\ 7 \\ 9\ 0\ 2 \\ \underline{4\ 5\ 6} \\ 3\ 0\ 6\ 2 \end{array}$$

The more able pupils should discover other methods of checking both addition and subtraction. Some other ways to check the example shown above are as follows:

1. Write the sum of each column as shown at the right and find the sum of these subtotals. Thus, the sum of the first column is 32; of the second, 230; and of the third, 2800. The sum of these subtotals is 3062, which is the same total as found for the sum in the example. The zeros used as place holders in 230 and 2800 may be omitted by the pupil who understands what each sum represents.

$$\begin{array}{r} 32 \\ 230 \\ \underline{2800} \\ 3062 \end{array}$$

2. The example may be separated into two parts and then added. The sum of the first three addends may be added to the sum of the other two addends. This procedure is permissible because the way in which addends are grouped does not affect the sum.

Other ways may be discovered for checking addition. The illustrations given are typical of the patterns that the more able pupil may discover for checking addition.

Checking subtraction

The check most widely used in subtraction is to add the remainder or difference to the given addend. If there are no errors in either subtraction or addition, the sum of these two numbers is equal to the given sum. This check is valuable because it demonstrates that addition and subtraction are opposite operations.

The standard way to check the example at the right is to add 546 and 158. The sum should equal 704. Other ways of checking the example are as follows:

$$\begin{array}{r} 704 \\ \underline{-\ 158} \\ 546 \end{array}$$

1. Subtract the difference from the minuend, as 704 − 546 = 158.

2. Subtract 100 from both 704 and 158. Then subtract these differences, as 604 − 58 = 546.

3. Subtract 158 from 200 and subtract 200 from 704. Then add these differences, as 200 − 158 = 42; 704 − 200 = 504; 504 + 42 = 546.

4. Subtract 150 from both 704 and 158. Then subtract 8 from these differences, as 554 − 8 = 546.

In plans (2), (3), and (4), the sum or the given addend, or both, are regrouped so that the operations are performed with different numbers from those in the original example. There are many different ways a pupil may discover to check his work in subtraction. The more capable pupil in mathematics should be encouraged to discover different ways to check an example, just as he should be encouraged to find a variety of ways to solve the example.

If checking is taught effectively so that it is a thinking process, the pupil must have time to perform the operation. Then the number of examples that a pupil can solve in a given period of time will be approximately half the number that he can solve when he does not check the solutions. If a teacher requires a pupil to solve and check as many examples in a class period as he would normally solve without checking the work, the checking operation will be perfunctory and of little value.

DISCOVERING RELATIONSHIPS BETWEEN ADDITION AND SUBTRACTION

Addition and subtraction are inverse, or undoing, operations. There are many cases, therefore, in which a procedure applies to addition but not to subtraction. The teacher should have the class identify as many of these situations as the group discovers. Table 9.1 compares the two operations.

TABLE 9.1

Comparison of Addition and Subtraction of Whole Numbers

Addition	*Subtraction*
1. Addition describes a joining process.	1. Subtraction describes a separating process.
2. To find the sum of $A + B$, start with A and count forward B more.	2. To find the difference of $A - B$, start with A and count backward B less. ($A > B$.)
3. Begin with two addends and end with the sum.	3. Begin with the sum and one addend and end with the other addend.
4. Show addition on a number ray, working from left to right.	4. Show subtraction on a number ray, working from right to left.
5. In column form, perform the operation downward.	5. In column form, perform the operation upward.
6. Regroup, proceeding from right to left.	6. Regroup, proceeding from left to right.
7. Add and then regroup.	7. Regroup and then subtract.
8. If B is added to A, the larger the B, the larger the sum.	8. If B is subtracted from A, the larger the B, the smaller the difference.
9. The sum of any two whole numbers is in the set of whole numbers.	9. The difference of two whole numbers is not always in the set of whole numbers.
10. The order of adding two numbers does not affect their sum.	10. The order of subtracting numbers affects their difference.
11. The grouping of three numbers in addition does not affect their sum.	11. The grouping of three numbers in subtraction affects their differences.
12. There is an identity element for addition (0).	12. There is no identity element for subtraction.

Table 9.1 emphasizes the differences between the two operations; however, it is advisable to identify the points of likeness between addition and subtraction. The two main points of likeness are: (1) both operations deal with numbers and (2) both are *binary* operations. Addition and subtraction deal with numbers and not things. We add and subtract numbers. We use objects and different materials to model a situation that involves joining or separating things.

Addition and subtraction are binary operations, as are all four of the operations. This means that an operation may be performed on only two numbers at one time to form a third number. A pupil may think that he can add three numbers at a time. He must make a grouping of two of the three numbers before he can find their sum. His reaction time to make the grouping may be a fraction of a second, but the numbers must be added in groups of two. It is not possible to find the sum of the numbers named by $2 + 4 + 5$ without first selecting and adding a pair of numbers, as $2 + 4$, and then adding that sum to the next number named, or 5. In the example $2 + 0 + 5$, the pupil may form only one group to be added $(2 + 5)$, because he has discovered that 0 as an addend may be omitted in finding the sum of a set of numbers

Properties of addition

Included in the list of ways in which addition and subtraction differ (Table 9.1) are the properties of addition: (1) *closure;* (2) *commutative;* (3) *associative;* and (4) *identity element.*

A fifth property of addition is a consequence of the commutative and associative properties. The principle derived from these two properties may be stated as follows: *The way numbers are grouped or rearranged does not affect their sum.* Rearranging the addends makes it possible to select the addends in a random sequence in a column to form groups having sums of 10. In the column at the right, it is possible to find the sum by forming the groups $(7 + 3)$ and $(6 + 4)$ and then adding $10 + 10 + 8$.

$$\begin{array}{r} 7 \\ 6 \\ 3 \\ 8 \\ \underline{4} \end{array}$$

Properties of subtraction for whole numbers

There are three apparent properties of subtraction. If a and b are a pair of any numbers, the following number sentences are true:

1. $a - 0 = a$ for all a.
2. $a - a = 0$ for all a.
3. $(a - b) + b = a$ if $a \geq b$.

The third statement is true because addition and subtraction are inverse operations. Subtracting b from a and then adding b to that difference is the equivalent of subtracting 0 from a, which is equal to a.

An important procedure that applies to addition and subtraction involves keeping like places in the same columns. This idea may be explained in terms of the *distributive property.* This property involves both multiplication and addition. It will be treated more fully in Chapter 10. At this stage, however, one may observe that in the addition $43 + 26$, the steps are justified by the distributive property as follows:

$$\begin{array}{rr} 43 & 4 \cdot 10 + 3 \times 1 \\ \underline{+26} & \underline{2 \cdot 10 + 6 \times 1} \\ 69 & 6 \times 10 + 9 \times 1 \end{array}$$

The properties mentioned above are essential in the development of addition and subtraction of natural numbers. An additional property is required for work with directed numbers, as in algebra.

EXERCISES

1. What is an addition situation? a subtraction situation? Illustrate each.
2. Illustrate the different levels of maturity in solving a number sentence that is an equation; that is an inequality.
3. List the minimum number of activities you would use with a class to introduce regrouping in addition.
4. Give your thought pattern as you find the sum of the addends in the column at the right.

$$\begin{array}{r} 3 \\ 3 \\ 4 \\ 3 \\ 4 \\ \underline{4} \end{array}$$

5. Identify some of the advantages and disadvantages of the decomposition and equal-additions methods of subtraction.
6. Chapter 3 lists principles of learning. Identify the principles that are applied in the presentation of regrouping in compound subtraction.
7. Give one illustration in addition and in subtraction of each of the 12 properties or characteristics of these operations as listed in Table 9.1.
8. Identify the properties of addition for the set of whole numbers.
9. Differentiate between checking a sum for accuracy and for reasonableness.
10. In the following equations, decide if n represents an addend or a sum:

a. $n - 15 = 28$	d. $35 + 28 = n$
b. $27 + n = 45$	e. $n + 23 = 82$
c. $47 - 15 = n$	f. $n - 34 = 67$

11. Solve for n in problem 10.
12. In the following number sentences, insert the correct sign, $=$, $>$, or $<$, to make each sentence true:
    a. $4 + 9 \bigcirc 8 + 5$
    b. $37 - 14 \bigcirc 13 + 12$
    c. $51 + 40 \bigcirc 11 + 80$
    d. $27 - 16 \bigcirc 12 + 8$
    e. $64 - 16 \bigcirc 21 + 17$
    f. $n + 6 \bigcirc n + 5$
13. In the following number sentences, find the least whole number that will make each statement true for sentences (a–c); the greatest whole number that will make each statement true for sentences (d–f):

a. $n - 8 > 12$	d. $n + 7 < 16$
b. $a - 12 > 7$	e. $n + 13 < 24$
c. $24 < c - 15$	f. $17 > n + 5$

14. Write the four equations that can be made with the elements of set A; of set B:
    A: $\{32, 48, 80\}$ B: $\{47, 18, 29\}$

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PATTERNS FOR TEACHING THE BASIC FACTS IN MULTIPLICATION AND DIVISION

A child learns rational counting as opposed to rote counting after he understands one-to-one correspondence between the elements of two sets. Once he is able to count rationally, he can enumerate the elements in the sets that he joins and separates. From experience with sets he learns that number is a common property of sets. He also finds that the operations of addition and subtraction may be performed on numbers and that these operations describe the joining and separation of sets. Figure 10.1 depicts the progression from a matching of sets to the mature ways of dealing with numbers.

The column on the left in the figure shows how addition, multiplication, and powers are related. The relationship between addition and multiplication applies when dealing with whole numbers. Raising a number to a power

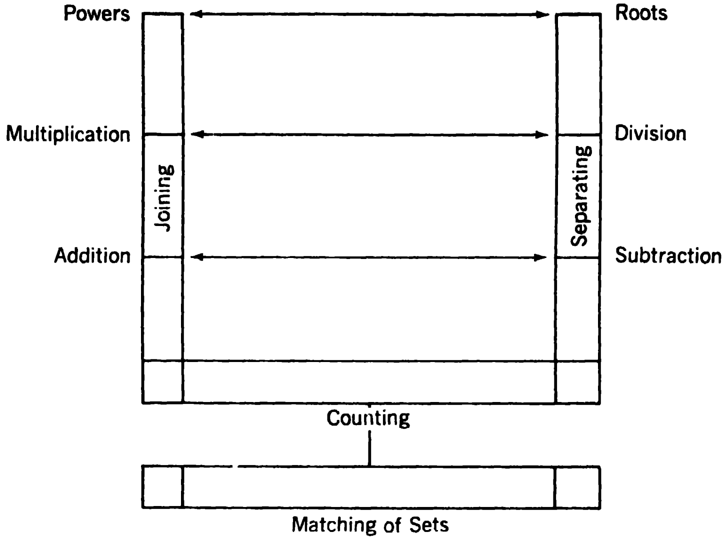


Figure 10.1

is a high-powered way of multiplication, as $2^3 = 8$, which means $2 \times 2 \times 2$. Multiplication by a whole number is a short form of addition when all the addends are equal. The column on the right shows how subtraction, division, and roots are related. Finding a root of a number is a high-powered form of division, as $\sqrt{64} = 8$. To find the square root of a number it is necessary to find two equal factors that have a product equal to the given number. Since $8 \times 8 = 64$, $\sqrt{64} = 8$. Division is a short way of subtraction to find how many equivalent subsets can be formed from a given set.

The horizontal lines in Figure 10.1 indicate another important relationship between the operations. The operations represented by the arrowheads on each line segment are inverses (opposites). The teacher must help the pupil to understand both the horizontal and vertical relationships in the figure. Chapter 9 emphasized the inverse relationship between addition and subtraction. The present chapter will be concerned with the inverse relationship between multiplication and division and with the direct relationship between addition and multiplication as well as between subtraction and division when dealing with whole numbers. The following topics are covered: situations conveyed by multiplication and division; properties of whole numbers for multiplication and division; patterns for teaching the facts in multiplication and division; formation of tables.

SITUATIONS CONVEYED BY MULTIPLICATION AND DIVISION

A multiplication situation

Chapter 9 indicated that there is one problematic situation for addition and two for subtraction. An intuitive response would suggest that there is one problematic situation for multiplication and two for division. This inference is correct. A problematic situation in multiplication consists in finding the result of combining a given number of equivalent subsets of known size into a single set. The following problem

can be used to show a multiplication situation:

Find the cost of 5 stamps at 6 cents a stamp.

The problem can be solved by repeated addition, as indicated by the equation:

$$6 + 6 + 6 + 6 + 6 = \square$$

While it is feasible to find the cost of 5 stamps by repeated addition, it is impractical to find the cost of 200 stamps in this manner. Examples of this type provide the basis for introducing multiplication by a whole number as a short cut for repeated addition. In this light, the equation for the above problem may be written[1]:

$$5 \times 6 = \square \quad \text{or} \quad 5 \times 6 = n$$

Thus, 5 sets of 6 cents is a set of 30 cents. The number 30 may be obtained by repeated addition of 6 or by multiplying 6 by 5. In the number sentence $5 \times 6 = 30$, 5 and 6 are *factors* and 30 is the *product*.

Division situations

Addition and subtraction are inverse (opposite) operations. Addition of 3 can be undone by subtracting 3, and subtraction of 5 can be undone by adding 5. In the number sentence $a + b = c$, a and b are the *addends* and c is the *sum*. In a simple addition situation, two addends are known and the sum is to be determined. In a simple subtraction situation, the sum and one addend are known and the other addend is to be determined.

In a similar manner, multiplication and division are inverse (opposite) operations. Multiplication by 3 may be undone by dividing by 3, and division by 5 may be undone by multiplying by 5. In a simple multiplication problem, two factors are known and the product is to be determined. In a simple division situation, the product and one factor are known and the other factor is to be determined.

The following problems illustrate the two situations that involve division:

1. How many 6-cent stamps can be purchased for 30 cents?

2. Tom bought 6 candy bars (each costing the same) for 30 cents. What is the cost of 1 bar?

Problem 1 may be reworded in set language as:

a. n sets of 6 cents is a set of 30 cents. Equation: $n \times 6 = 30$

b. 30 cents can be broken into n sets of 6 cents. Equation: $\frac{30}{n} = 6$

In early multiplication and division work it is probably more effective to stress the notion of combining sets, as in (a). Pupils should associate the combining of equivalent sets with multiplication.[2] Pupils should also eventually learn to associate the separation of a set into equivalent subsets with division. It is important to understand that both approaches lead to different forms of the same sentence. Either equation may be obtained from the other by the

[1]The open sentence on the left in the equation uses a frame and the one on the right a letter. Both the letter and the frame are technically called *variables*, since they hold a place for a numeral and represent an unspecified number. The frame is most useful in circumstances in which the pupil can write the numeral in the frame on expendable material and on the chalkboard. The teacher should gradually introduce the use of letters even if the text uses only frames. Letters, as *n* or *t*, are easier to write and are standard in later work.

[2]Equivalent sets have the same cardinal number (see p. 45). Two distinct sets of 5 cents cannot be equal because the coins in one set cannot be in another, and equal sets must have the same elements. Care should be taken not to confuse equivalent sets with equal sets.

multiplication-division relationship, as is true with the two equations shown above. The use of set language and open sentences gives pupils an opportunity to recognize more clearly the relation between multiplication and division by making it possible to use the language of multiplication and multiplication-type open sentences, as illustrated in (a), to analyze division problems. Different pupils have different ways of understanding and interpreting problems, and the teacher should allow much freedom in this regard.

Problem 2 may be reworded in set language as:

a. 6 sets of n cents is a set of 30 cents. Equation: $6 \times n = 30$

b. 30 cents may be broken into 6 sets of n cents. Equation: $\frac{30}{6} = n$

In problem 1, the number of equivalent subsets is unknown and the size (cardinal number) of each subset is known (6). The size of the original (final) set is also known (30). If the problem is interpreted as combining n sets of 6 into a set of 30, the set of 30 is the final set. If the problem is interpreted as breaking up a set of 30 into n sets of 6, then the set of 30 is the original set. In problem 2, the number of equivalent subsets is known (6) and the size of each is unknown.

Problem 1 illustrates the *ratio* or *comparison*[3] division situation, while problem 2 illustrates the *partitive*, or *rate*, type of division.

The modern tendency is to stress the common property of the above two problems rather than their difference in terms of ratio or partition. Both problems have a product (30) and one factor (6) given, with the other factor (5) to be determined. Set sentences, as 6 sets of n cents is a set of 30 cents, or n sets of 6 cents is a set of 30 cents, illustrate the difference and are helpful in identifying which number is a factor and which is a product.

Both simple multiplication and division situations can be interpreted as combining equivalent sets into a single set. Table 10.1 illustrates this fact.

The addition-subtraction pattern is illustrated by the following:

$2 + 3 = 5$	$\text{Addend}_1 + \text{addend}_2 = \text{sum}$
$3 + 2 = 5$	$\text{Addend}_2 + \text{addend}_1 = \text{sum}$
$5 - 3 = 2$	$\text{Sum} - \text{addend}_2 = \text{addend}_1$
$5 - 2 = 3$	$\text{Sum} - \text{addend}_1 = \text{addend}_2$

The above relation between addition and subtraction was stressed and its uses indicated in Chapter 8. Experience

[3]Ratio or comparison division is sometimes referred to in the literature as measurement or quotative division.

TABLE 10.1

Simple Multiplication and Division Situations

Number of Equivalent Subsets (factor)	*Number of each Equivalent Subset (factor)*	*Number of Combined Set (product)*	*Operation*
Given	Given	Missing	Multiplication
Missing	Given	Given	Division (Ratio)
Given	Missing	Given	Division (Rate)

with the above pattern should be helpful in learning the following:

$2 \times 3 = 6$ Factor$_1$ × factor$_2$ = product
$3 \times 2 = 6$ Factor$_2$ × factor$_1$ = product
$6 \div 3 = 2$ Product ÷ factor$_2$ = factor$_1$
$6 \div 2 = 3$ Product ÷ factor$_1$ = factor$_2$

The above relation may also be stated in terms of the quotient, divisor, and dividend:

Divisor × quotient = dividend
Quotient × divisor = dividend
Dividend ÷ divisor = quotient
Dividend ÷ quotient = divisor

The above pattern provides the basis for relating multiplication and division facts as well as the means for solving simple equations involving multiplication or division.

PROPERTIES OF WHOLE NUMBERS FOR MULTIPLICATION AND DIVISION

This chapter centers around a small subset of the whole numbers, namely, the set of one-place numbers and their products. The whole numbers do not form a number field (see Chap. 6). The whole numbers do not have inverses for addition or multiplication but do meet all the other conditions for a number field. Table 10.2 summarizes the properties of whole numbers with respect to multiplication and division.

The table does not include the distributive property, since it involves two operations, but is confined to properties defined in terms of a single operation. Multiplication is distributive with respect to addition for whole numbers. For all whole numbers a, b, and c:

1. $a(b + c) = ab + ac$
2. $(b + c)a = ba + ca$

The commutative property of multiplication insures that if equation (1) is true, then equation (2) must also be true. Equation (1) illustrates what is sometimes called the left-hand distributive property, while equation (2) illustrates the right-hand distributive property. There is no need to make this distinction on an elementary level except to the extent that it may clarify the situation with regard to division. It is frequently said that division is distributive with respect to addition.

$(a + b) \div c = a \div c + b \div c$ for all a, b, and c

The above is a direct consequence of the distributivity of multiplication over addition, since

$$(a + b) \div c = (a + b) \times \frac{1}{c}$$

TABLE 10.2

Whole Numbers

Property	*Multiplication*	*Division*
Closure	Yes	No
Associative	Yes	No
Commutative	Yes	No
Identity	Yes	No[a]
Inverse	No	No[b]

[a]The number 1 may be described as a right-hand identity for division, for example, $n \div 1 = n$ for all n, but the number 1 does not meet the condition $n \div 1 = 1 \div n = n$ for all n.

[b]Since the inverse concept is defined in terms of the identity element, if there is no identity then there can be no inverse element.

The following statement, however, *is not true:*

$$c \div (a + b) = c \div a + c \div b$$

For this reason, it is sometimes said that the right-hand distributive property of division over multiplication holds but not the left. Left or right, however, is not a good description in the following cases, which do illustrate the distributive property of division over addition:

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c} \qquad c\overline{)a + b} = c\overline{)a} + c\overline{)b}$$

The lack of commutativity for division results in a more limited form of the distributive property of division with respect to subtraction. Minimum confusion in this regard should be encountered if pupils learn to relate the distributive situation for division to that for multiplication, although this cannot be done until the rational numbers have been introduced.

$$\begin{aligned}\frac{a + b}{c} &= (a + b) \times \frac{1}{c}\\ &= a \times \frac{1}{c} + b \times \frac{1}{c}\\ &= a \div c + b \div c\end{aligned}$$

$$\frac{c}{a + b} = c \times \frac{1}{a + b},$$

which is not equal to $c \times \frac{1}{a} + c \times \frac{1}{b}$

A reference to the distributive property is almost invariably interpreted as the distributive property of multiplication over addition. This is so because any of the other distributive properties, as division over addition or subtraction, are direct consequences of the distributive property of multiplication over addition, as illustrated above.

EXERCISES

1. Give examples to illustrate that multiplication is commutative and associative.
2. From the number field properties for whole numbers, how can one determine that division is not closed for this set?
3. Give an example to prove that whole numbers are not closed for division.
4. Rename each of the following, using the property named in parentheses:
   a. 3 × 5 (commutative property for multiplication)
   b. (4 × 2) × 50 (associative property for multiplication)
   c. 25 × 44 (rename 44 and apply associative property for multiplication)
   d. 12 (identity property for multiplication)
   e. 42 (identity property for addition)

PATTERNS FOR TEACHING THE FACTS IN MULTIPLICATION AND DIVISION

The number of facts

A *number fact for multiplication* is a sentence that indicates the correct product for two one-place numbers (numbers represented by one-digit numerals), for example, $2 \times 3 = 6$ or $4 \times 4 = 16$. Each of 10 one-place numbers may be multiplied by 10 one-place numbers to give 100 possible products of one-place numbers or number facts. Table 10.6 on page 175 presents an example of one of the most convenient ways for writing the 100 multiplication facts.

Of the 100 possible quotients of one-place numbers, 10 must be eliminated

because division by 0 is not possible.[4] Therefore there are 90 division facts. It is not necessary to write a separate table for division facts. When the pupil understands the relation between multiplication and division, he can discover that the table of multiplication facts referred to above also contains all of the division facts.

Pupil materials for multiplication

Each pupil should have strips of paper, preferably construction paper, containing designs to be used as markers. The designs may be circles, squares, or other geometric figures. All the pupils need not have the same kind of marker. One set of markers may contain circles while others may contain squares or other suitable designs. For initial work with the multiplication facts, the class should use different kinds of markers to show a fact. However, the use of different kinds of markers should be confined to class activities during the period of readiness for introducing the facts in multiplication.

The 9 or 10 sets of designs for representing each factor may be arranged on one large sheet or cut into separate strips to correspond to the number to be presented. Figure 10.2 shows a set of circles for modeling the fours in multiplication. To show 3 fours, the pupil would fold his paper containing the designs so that 3 sets of fours would show. If the sheet were cut into strips of fours, the pupil would arrange 3 of these strips to model 3×4. There should be either 9 or 10 strips. Ten strips may be used to model multiplying by 10, as 10×4.

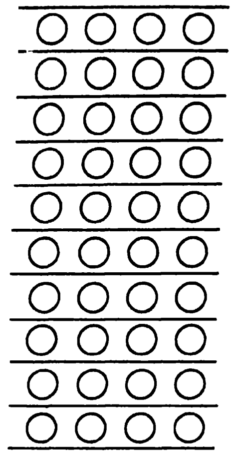
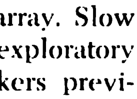


Figure 10.2

The pupil uses markers to model or discover multiplication facts. The textbook may show arrays for the pupil to use to find products. The array shows 3 fours in the horizontal form and 4 threes in the vertical form. Just as a pupil can use his markers to find the answer in a fact, so he can use an array to find that answer. Many pupils can discover the multiplication and division facts from an array. Slow learners may need to use exploratory materials, such as the markers previously described, in order to discover the facts. The teacher should have the class use markers during the initial work in multiplication and their use be discontinued as soon as the pupil is able to discover a fact from an array or from facts that he already knows.



Readiness for multiplication and division facts

It was emphasized in Chapter 9 that the pupil should participate in certain

[4]Many pupils are confused about the difference between $0 \div 3$ and $3 \div 0$. The example $0 \div 3$ may be renamed as $0 \times \frac{1}{3}$, which is equal to 0, since the product of any number and 0 is 0. On the other hand, if it is assumed that $3 \div 0$ has a quotient n, then the relation between multiplication and division demands that $3 = 0 \times n$, which contradicts the fact that 0 times any number is 0.

activities to create readiness for the addition and subtraction facts. The same situation applies to multiplication and division. The pupil should use strips or squares to model different facts. To model the fact 3×4, the pupil should show 3 strips of fours and interpret the representation. Some typical responses are:

Three sets of 4 circles is a set of 12 circles.

Three sets of 4 squares is a set of 12 squares.

The above set sentences may be described by the equation

$$4 + 4 + 4 = \square$$

The basis is then provided for introducing the ideas of multiplication of whole numbers as a short cut for repeated addition or for describing the combining of equivalent sets into a single set. The above equation may be rewritten as

$$3 \cdot 4 = \square$$

Other typical activities for readiness are:

1. Use the number ray to model the situations described above (Fig. 10.3).

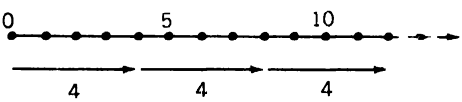


Figure 10.3

2. Draw the number ray shown in Figure 10.4 on the chalkboard and ask the pupils to write a number sentence involving addition; multiplication. After the sentences $5 + 5 + 5 = 15$ and $3 \times 5 = 15$ are obtained, ask for a verbal description or set sentence similar to those given above.

3. After the pupils have used their strips to model a number of facts, draw an array on the chalkboard, as

shown. Then, help the pupils interpret this array as two sets of 4 or as 2 fours. Have them write the equations $4 + 4 = \square$ and $2 \times 4 = \square$. Have pupils draw a number ray and use strips.

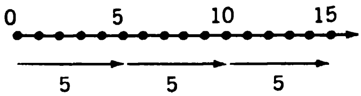


Figure 10.4

4. Write a sentence, for example, $2 + 2 + 2 = \square$, on the chalkboard. Ask the pupils to rewrite it as a sentence involving multiplication. Ask pupils to interpret the multiplication equation with set sentences, for example, 3 sets of 2 apples is a set of n apples. Ask the class to use their strips to make a model for the sentence. Ask a pupil to draw an array on the chalkboard. Ask the pupil to draw a number ray illustrating the situation.

Two major goals in readiness work in multiplication are:

1. To relate the new multiplication symbolism, as 2×3, to the familiar idea of addition, as $3 + 3$

2. To relate both types of symbols shown above to the idea of combining equivalent sets.

Introducing the facts in multiplication

When readiness work has been completed, a systematic approach to learning multiplication number facts must begin. It is common to work with the twos, since they are related to doubles in addition as well as being the simplest set of facts other than those for 1 and 0.

The work with the twos involves no essentially new ideas but continues with the procedures already discussed in connection with multiplication readiness. The ideas and activities are con-

centrated on the twos, with the goal of enabling the pupil to gain mastery of the basic facts involving the twos. As indicated earlier, mastery will be attained most thoroughly and quickly with the widest possible variety of meaningful activities.

Early activity may be along the following lines:

$2 \times 3 = 3 + 3 = 6$ $\quad$ $2 \times 3 = 6$

Two threes are six

• • •
• • •

$2 \times 4 = 4 + 4 = 8$ $\quad$ $2 \times 4 = 8$

Two fours are eight

• • • •
• • • •

$2 \times 5 = 5 + 5 = 10$ $\quad$ $2 \times 5 = 10$

Two fives are ten

• • • • •
• • • • •

Each of the above sentences may be interpreted by a number ray and modeled by specific sets of objects in the classroom and described with sentences, for example, 2 sets of 5 books.

The beginning work with the twos should probably omit the facts 2×0 and 2×1. When enough work of the type shown above is completed, pupils should recognize quickly that $2 \times 0 = 0 + 0$ and that $2 \times 1 = 1 + 1 = 2$.

In initial work with the twos in multiplication, arrays should be used frequently. When the pupil interprets an array correctly as 2 sets of 3, or 2 threes, the array should be rotated as illustrated on the right and interpreted as 3 sets of 2 or 3 twos.

• • •
• • •

• •
• •
• •

The array on the right may be interpreted in two ways without rotation. It may be interpreted as two sets of rows (of 4 in each row) or four sets of columns (with 2 in each column). To avoid confusion, it is probably best in the beginning to interpret such arrays only in terms of rows.

• • • •
• • • •

By rotating arrays in the manner described above, pupils can discover the commutative property for multiplication. This idea can be reinforced with other activities.

1. Activities of the following kind will help pupils recognize the commutative property of multiplication:

$$2 \times 3 = 3 + 3 = 6$$
$$3 \times 2 = 2 + 2 + 2 = 6$$
$$2 \times 5 = 5 + 5 = 10$$
$$5 \times 2 = 2 + 2 + 2 + 2 + 2 = 10$$

2. The number ray is also useful in helping pupils recognize the commutative property of multiplication for whole numbers (Fig. 10.5).

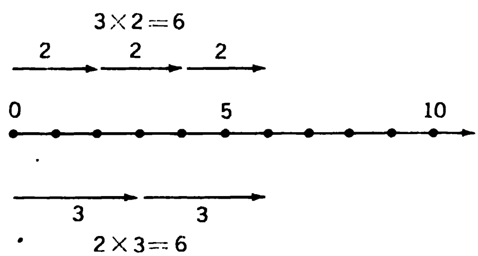


Figure 10.5

3. Pupils should be encouraged to rename 2×3 as 3×2 as well as with any other correct name.

4. Pupils should be helped to recognize that if $3 \times 5 = 15$ is a true sentence, then $5 \times 3 = 15$ is also true.

The commutative property can help pupils make an important discovery about the table of the twos. When the pupil has learned that $2 \times 3 = 6$, the following should enable him to find that $2 \times 4 = 8$.

$$2 \times 3 = 3 \times 2 = 2 + 2 + 2$$
$$2 \times 4 = 4 \times 2 = 2 + 2 + 2 + 2$$

The above work should make it clear that 2×4 is two more than 2×3 and help the pupil understand why each new fact in the table of the twos is two more than the previous fact. The distributive property may also be used for this purpose.

$$\begin{aligned} 2 \times 5 &= 2 \times (4+1) \\ &= 2 \times 4 + 2 \times 1 \\ &= 2 \times 4 + 2 \end{aligned}$$

These same ideas should be used to show that facts in the table of the threes increase by three, and so on. The commutative property is probably more effective for the initial facts, while the distributive property probably has an advantage for later facts. Writing 8×9 in terms of repeated addition is too time consuming and should be used only when necessary.

Use of tables

A table is useful for summarizing facts. Table 10.3 shows only facts of the type $2 \times n$. For facts of the type $n \times 2$, a vertical table is needed. Most tables do not show both the indicated multiplication, as 2×3, and the product, 6. Early work with tables may be more meaningful to the pupils if this practice is followed. Later only the product may be written. It is not necessary to construct a table for the complete set of facts of the type $2 \times n$. The use of smaller tables as shown in (a)–(d) is recommended.

TABLE 10.3

·	0	1	2	3	4	5	6	7	8	9
2	2×0 0	2×1 2	2×2 4	2×3 6	2×4 8	2×5 10	2×6 12	2×7 14	2×8 16	2×9 18

a.

$\times$	2	3	4
2	4	?	?

b.

$\times$	3	7	8
2	?	?	?

c.

$\times$	3	4	?	7	?
2	2×3	?	12	?	18

A vertical table, as shown in (d), stresses the commutative property of multiplication:

d.

$\times$	2
3	3×2
?	8
5	?
6	?
?	18

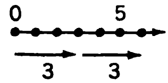
Help the pupils construct a table similar to Table 10.4.

Such a table with several lines may be placed on the chalkboard for a week or longer and used as reference. A more permanent form on oak tag or similar material may also be useful.

It may be helpful to start with a table similar to Table 10.4 with only one entry on each line and then let the pupils supply the missing information, as in Table 10.5.

TABLE 10.4

A Student Aid in Learning the Multiplication and Division Facts

Array	*Set Sentence*	*Abbreviated Set Sentence*	*Addition Equation*	*Multiplication Equation*	*Number Ray*
• • • • • •	Two sets of 3 books is a set of □ books	Two threes are *n*	$3 + 3 = \square$	$2 \times 3 = \square$	

Use objects in the classroom as models for equations and set sentences. Use 3 sets of 2 books on a desk or table. Use 2 sets of 3 children. Use 5 pairs of boots or gloves. Interpret each such example with equations, sentences, arrays, and number rays.

Introduce the vocabulary for multiplication with a sentence, for example, $3 \times 5 = 15$, and identify 3 and 5 as *factors* and 15 as the *product*. Use other sentences to further illustrate this new vocabulary.

Open sentences involving basic facts

Open sentences should be introduced early in the work with multiplication facts. The equation can provide reinforcement activity for learning facts as well as for the concept of factor and product. Introductory equations should have both factors given, as in $2 \times 7 = \square$. When an equation of the type $2 \times \square = 16$ is first given, its "solution" should involve only the concept that when the numeral 8 is placed in the frame the open sentence becomes a true statement. Knowledge of the basic fact $2 \times 8 = 16$ is all that is required to "solve" such an equation. Such work provides a meaningful variety of activity involving basic facts as well as better understanding of the nature of open sentences.

Exercises with open sentences of the following type can be used to provide additional reinforcement of the commutative property for multiplication:

$2 \times 4 = 8$	$5 \times 7 = 35$
$4 \times 2 = \square$	$7 \times 5 = \square$
$2 \times 9 = \square$	$2 \times 7 = \square \times 2$
$9 \times 2 = \square$	$7 \times 3 = 3 \times \square$

An example involving the fives or sevens, as shown above, is not out of place when pupils are working with the twos.

Work with pairs of equations is useful in stressing both the vertical and horizontal relations discussed earlier

TABLE 10.5

Class Table for Learning the Multiplication and Division Facts

Array	*Set Sentence*	*Abbreviated Set Sentence*	*Addition Equation*	*Multiplication Equation*	*Number Ray*
?	?	?	$4 + 4 = \square$	?	?
?	?	Two fives are *n*	?	?	?
?	?	?	?	$2 \times 6 = \square$	?

in this chapter. Some of the ways of illustrating these ideas are as follows:

$3 + 3 = 6$ $2 \times 3 = \square$	$3 \times 4 = 12$ $4 + 4 + 4 = \square$
$12 \div 3 = 4$ $3 \times \square = 12$	$18 \div 2 = 9$ $2 \times \square = 18$
$3 \times 5 = 15$ $15 \div 5 = \square$	$4 \times \square = 20$ $20 \div 5 = \square$
$4 + 4 + 4 = 3 \times \square$ $2 + 2 + 2 = \triangle \times 2$	$5 + \square + \square = \triangle \times \square$ $\square + \square + \square = \triangle \times \square$

Just as the triangle and square above hold a place for a numeral, a circle is frequently used to hold a place for a sign of operation as $+$, $\times$, $-$, or $\div$. In the following, the pupil must replace the circle by the sign of operation that will make the sentence true.

$$2 \bigcirc 4 = 8 \qquad 8 \bigcirc 2 = 16$$
$$12 \bigcirc 4 = 3 \qquad 8 \bigcirc 4 = 2$$

Equations with two frames or variables may also be used, but they are of limited use until the pupil learns a substantial number of facts.

$$\triangle \times \square = 12 \qquad \triangle \times \square = 16 \qquad \square - \triangle = 2$$

The pupil should learn the difference between the equation $\square \times \square = 16$ and $\triangle \times \square = 16$. Where both frames are squares, the same number must be represented; when a square and a triangle are used, however, the numbers may be the same or different. The equation $4 \times 4 = 16$ may be obtained from both $\square \times \square = 16$ and $\triangle \times \square = 16$, but the equation $2 \times 8 = 16$ may be obtained only from the latter.

The vocabulary of factor and product should also be emphasized in the following manner:

Factor $\times$ *factor* $=$ *product*
$3 \times 4 = 12$
$5 \times \square = 30$

Product $\div$ *factor* $=$ *factor*
$12 \div 4 = 3$
$30 \div 5 = \square$

Many lessons in initial work with multiplication can profitably begin with a renaming session. The pupils should be asked to rename 2×5. The earliest answers will probably be $5 + 5$ or 10. The answer 5×2 should also be obtained. In such a session it is important to welcome any correct answer as $2 \times 5 + 0$ or $2 \times 5 \times 1$. Imaginative pupils may deliberately try to give unusual answers. Such answers should be encouraged but a skillful teacher can guide any renaming session in the direction desired. It is desirable that 2×5 be renamed as $2(4 + 1)$ or $2(3 + 2)$ in several such sessions before introducing the use of the distributive property as a useful means for obtaining new facts from old (see p. 163). It is equally important in such renaming sessions for pupils to rename 6 as 2×3 or 3×2. It is also useful to rename $3 \times 7 + 3 \times 13$ as $3(7 + 13)$.

In traditional mathematics at its worst, the multiplication facts were learned by rote repetition, with little or no effort to give the pupil an understanding of fundamental patterns. In the modern approach, every effort is made to obtain variety and understanding by using such different devices as markers, number rays, physical models, arrays, patterns, and equations. It is a mistake to conclude that no drill, maintenance, or reinforcement activity is necessary. While the more able pupils may need little or none of such activity, it is almost certain that some pupils require repetitive work in order to master the facts. The teacher must assess each class in this regard and provide additional activity as needed. Limited use of show cards or other such devices may be useful or necessary, but these should be employed only after the more meaningful activities have been explored. Brief, well-motivated rote-

learning experiences may provide useful additional variety. Efficient computation is impossible without a reasonable mastery of the facts. In a modern approach, mastery of addition and multiplication facts usually insures mastery of subtraction and division facts.

Work with the twos must introduce the operation of division as the inverse operation of multiplication. Every effort should be made to help pupils recognize the similarity between the multiplication-division and the addition-subtraction relationships. The multiplication-division pattern shown on page 158 is of major importance here. When the pupil learns that $2 \times 3 = 6$, he learns to conclude that $3 \times 2 = 6$, $6 \div 3 = 2$, and $6 \div 2 = 3$ are also true statements. Each multiplication fact with unequal factors should lead to a second multiplication fact and to two division facts. A multiplication fact with equal factors, as $3 \times 3 = 9$, leads to only one division fact, $9 \div 3 = 3$.

Notation for facts

There are two accepted ways of writing the facts in multiplication. The facts may be written in vertical form or in equation form, as in (a) and (b), respectively:

a $\begin{array}{r} 6 \\ \times\ 3 \\ \hline 18 \end{array}$ b $3 \times 6 = 18$

There are more ways to represent a basic fact in division than in multiplication. Examples (c–e) are the accepted forms for writing a basic number pair in division:

c $3\overline{)18}$ d $18 \div 3$ e. $\frac{18}{3}$

Arithmetic textbooks seldom use the notation shown in (e) in presenting the basic facts in division. A wider use of this notation should be helpful to the pupil in dealing with rational numbers.

A challenging activity for the more able learner is to rename $18 \div 3$ in a variety of ways, for example:

$(18 \div 3) \times 1$ $1 \times \frac{18}{3}$ $\frac{18}{3} \times 1$

$(9 + 9) \div 3$ $(24 - 6) : 3$

If rational numbers are used, there are many more ways to rename $18 \div 3$, as $\frac{1}{3} \times 18$. Since $\frac{1}{3}$ is not in the set of whole numbers, $\frac{1}{3} \times 18$ does not rename $18 \div 3$ in that set.

Division in relation to subtraction

While it is possible to learn all of the division facts from multiplication facts, a pupil's knowledge of the division operation is not complete until he understands its relation to repeated subtraction. The following activities may be helpful for this purpose:

The answer to $6 \div 2$ may be obtained by using a number ray (Fig. 10.6). Begin at 6 and mark off groups of twos, as shown below the ray. Similarly, the number sentence $6 \div 3 = 2$ shows the representation above the ray. The pupil may rename 6 using multiplication.

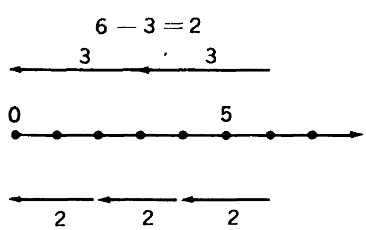


Figure 10.6

Then the number sentences are $6 = 2 \times 3$ and $6 = 3 \times 2$. The answer may also be obtained by subtraction. To find the number of twos there are in 6, subtract, as in (a). The representation

shows that 2 may be subtracted from 6 three times, hence there are 3 twos in 6:

$$\text{a.}\quad \begin{array}{rl} 6 & \\ \underline{-2} & 1 \\ 4 & \\ \underline{-2} & 1 \\ 2 & \\ \underline{-2} & \underline{1} \\ & 3 \end{array} \qquad \text{b.}\quad \begin{array}{rl} 6 & \\ \underline{-3} & 1 \\ 3 & \\ \underline{-3} & \underline{1} \\ & 2 \end{array}$$

These facts may then be written on the chalkboard:

$$6 \div 2 = 3 \qquad 6 \div 3 = 2$$

The teacher tells the class how to read a fact, as "6 divided by 2 equals 3." The pupil uses the equation form for writing the facts until he has discovered the meaning and properties of division. The division sign, $\overline{)}$, is used primarily in examples involving computation, as in $2\overline{)45}$. It is important that quotients be associated with set sentences, for example, n sets of 4 is a set of 12, or 3 sets of n is a set of 12, and the associated equations, $n \times 4 = 12$ and $3 \times n = 12$. Solution of verbal problems, in a modern approach, centers in the ability of the pupil to write a correct number sentence. Readiness for problem solving can be created by giving the class a number sentence and then have the pupil interpret the equation in terms of set sentences and verbal descriptions. Equations (a) and (b) illustrates this procedure.

a. *Equation.* 2×7

Set sentences: 2 sets of 7 cents is 14 cents; 2 sets of 7 books is a set of 14 books; 2 sets of 7 glasses is a set of 14 glasses.

Verbal situations: Find the cost of 2 candy bars at 7 cents a bar. What is the total weight of 2 packages if each package weighs 7 pounds?

b. *Equation* $2 \times \square = 12$

Set sentences: 2 sets of n books is a set of 12 books; n sets of 2 books is a set of 12 books; a set of 12 books may be broken into 2 sets of n books; a set of 12 books may be broken into n sets of 2 books.

Verbal situations: If 2 stamps cost 12 cents, what is the cost of 1 stamp? How many 2-cent stamps can be bought for 12 cents?

The following two set sentences are quite different:

a. 2 sets of n cents is a set of 12 stamps.

b. n sets of 2 stamps is a set of 12 stamps.

However, both sentences are represented by the equation $2 \times \square = 12$. It is true that the second set sentence should be represented by the equation $\square \times 2 = 12$, but because of the commutative property of multiplication, the latter equation may be written as $2 \times \square = 12$.

Combining 2 sets of n cents to get a set of 12 cents and breaking 12 cents into 2 sets of n cents are both represented by the equation $2 \times \square = 12$. Technically, the second set sentence should be represented by the equation $12 = 2 \times \square$, but this equation is the same as $2 \times \square = 12$, since mathematical equations can be read in either direction.

The ability to analyze situations in terms of set sentences requires high verbal ability and will be acquired much more readily by some pupils than others. Many pupils can perform the analysis on a nonverbal basis. Each teacher must learn to judge how much emphasis to put on verbalization for a specific class. Recognition of factor and product is probably less demanding and is a useful activity in helping pupils to write equations (open sentences) in the solution of verbal problems.

When a pupil learns that $2 \times 7 = 14$, he should immediately conclude that $7 \times 2 = 14$. In learning the twos, a pupil learns about other facts. He also learns the concept of division and its relation to multiplication, as well as the problematic situations associated with these two operations. Work with the threes, fours, and fives requires no new concepts but must proceed in a manner that strengthens the concepts introduced with the twos. As new facts are learned,the old ones must be maintained, and the teacher must constantly search for a variety of meaningful activities to help pupils learn more efficiently. All of the activities described in connection with the twos may again be used in connection with learning other facts.

Learning new facts

The following activities are typical of those that may be used in helping pupils to learn new facts:

1. Have pupils discover patterns by supplying additional numbers in sequences similar to the following:

a 1, 2, 3, 4, 5, · · ·
b 2, 4, 6, 8, · · ·
c 3, 6, 9, 12 · · ·
d 2, 5, 8, 11, · · ·
e. 4, 5, 10, 11, 16, 17, · · ·

Ask the pupils to rename numbers in sequence to show common properties. The above sequences may be renamed in this manner as follows:

a $1 \times 1, 1 \times 2, 1 \times 3, \cdots$
b $1 \times 2, 2 \times 2, 3 \times 2, \cdots$
c. $1 \times 3, 2 \times 3, 3 \times 3, \cdots$
d. $1 \times 3 - 1, 2 \times 3 - 1,$
$3 \times 3 - 1, \cdots$
e. $1 \times 3 + 1, 2 \times 3 - 1,$
$3 \times 3 + 1, 4 \times 3 - 1, \cdots$

Sequence (a) should not be renamed as illustrated until the identity element for multiplication has been introduced and discussed. More difficult sequences, such as (d) and (e) should be used sparingly and the teacher should be guided by class reaction to them. Sequences such as (b) and (c) should be used frequently and the pupils should identify them as the twos and threes and eventually as the *multiples* of 2 and the multiples of 3.

2. The use of tables was described earlier in this chapter. As pupils learn new facts, a variety of tables may be constructed that will provide useful variety in pupil activity. The earliest extension of the work on page 163 may be as follows:

×	3	4
2	2×3	8
5	15	5×4

The table need not be restricted to 2×2, as illustrated above. Tables may be used that are 2×3 or 2×4 or 3×3. Work of this kind should eventually lead to construction of the table listing all the facts (Table 10.6).

3. Different types of tables may be used, as shown in (a), (b), and (c).

a

×	2	4
3		△
5	t	n

$3 \times 2 =$ $\quad$ $3 \times 4 = \triangle$
$5 \times 2 = t$ $\quad$ $5 \times 4 = n$

b.

×	2	n	△
3	t	12	15
	10	20	25

$3 \times 2 = t$ $\quad$ $\square \times 2 = 10$
$3 \times n = 12$ $\quad$ $\square \times n = 20$
$3 \times \triangle = 15$ $\quad$ $\square \times \triangle = 25$

c.

×	?	?	?
3	6	12	15
5	10	?	?

Tables (a) and (b) show how the pupil can form number sentences from given data. Table (c) can be used effectively for showing the inverse relationship between multiplication and division.

4. A useful activity involves a set of 3 numbers, as $\{2, 8, 16\}$. The pupil is asked to write the facts associated with the 3 numbers. These are $2 \times 8 = 16$, $8 \times 2 = 16$, $16 \div 2 = 8$, and $16 \div 8 = 2$. As work progresses, the numbers in the set may lead to true statements that are not facts, for example, $4 \times 14 = 56$. This activity parallels that given for addition and subtraction in Chapter 8.

5. The set may include a variable, as in $\{3, 6, n\}$, which leads to the equations $3 \times 6 = n, 6 \times 3 = n, n \div 3 = 6, n \div 6 = 3$. The set $\{4, n, 24\}$ leads to $4 \times n = 24$, $n \times 4 = 24$, $24 \div n = 4$, and $24 \div 4 = n$. This activity provides the basis for solving equations involving larger numbers.

6. Have the pupil write in sequence the numerals for the numbers from 1 to 30 for the threes, from 1 to 40 for the fours, and so on. Then cross off each third numeral for the threes and every fourth numeral for the fours. As the pupil crosses off each numeral, he writes the number pair for that product. If he crosses off 21, the number pair is 3 and 7. He would then write the following four facts derived from this number pair: $3 \times 7 = 21$, $7 \times 3 = 21$, $21 \div 3 = 7$, and $21 \div 7 = 3$.

The identity element

Multiplication involving 0 and 1 can be performed by repeated addition, but recognition of 1 as the identity element for multiplication cannot be achieved until facts other than the twos are known. The earliest desirable point for introducing 1 as the identity element for multiplication is probably after the class is familiar with the fives. The following pattern may be helpful:

$$\begin{array}{ll} 2 \times 1 = 1 + 1 = 2 & 2 \times 1 = 2 \\ 3 \times 1 = 1 + 1 + 1 & \\ \quad = 3 & 3 \times 1 = 3 \\ 4 \times 1 = 1 + 1 + 1 & \\ \quad + 1 = 4 & 4 \times 1 = 4 \\ 5 \times 1 = 1 + 1 + 1 & \\ \quad + 1 + 1 = 5 & 5 \times 1 = 5 \end{array}$$

Activity of this type may be followed by using equations of this type:

$$\square \times 1 = \square \qquad n \times 1 = \square \qquad \square \times 1 = t$$

The similarity between 0 in addition and 1 in multiplication may also be pointed out in the following manner:

$$\begin{array}{ll} 2 \times 1 = 1 \times 2 = 2 & 2 + 0 = 0 + 2 = 2 \\ 3 \times 1 = 1 \times 3 = 3 & 3 + 0 = 0 + 3 = 3 \\ 4 \times 1 = 1 \times 4 = 4 & 4 + 0 = 0 + 4 = 4 \end{array}$$

The above may be summarized by the open sentences $n \times 1 = 1 \times n = n$ and $n + 0 = 0 + n = n$ and by the verbal statements that the product of 1 and any number is that number and the sum of 0 and any number is that number.

One way to strengthen this concept is to encourage pupils to frequently rename 5 as $5 + 0$ or $0 + 5$ and as 5×1 or 1×5. A pupil who can readily rename 5 as 5×1 should readily rename $\frac{1}{2}$ as $\frac{1}{2} \times 1$. The latter concept is important in order that a pupil have a mathematical understanding of the equality $\frac{1}{2} = \frac{2}{4}$ (see p. 233).

Multiplication by zero

Zero is the identity element for addition, as indicated in the previous section, but it also has a special property with regard to multiplication, as illustrated by the following:

$2 \times 0 = 0 + 0 = 0$	$2 \times 0 = 0$
$3 \times 0 = 0 + 0 + 0 = 0$	$3 \times 0 = 0$
$4 \times 0 = 0 + 0 + 0 + 0 = 0$	$4 \times 0 = 0$

The above activity should enable pupils to discover that $n \times 0 = 0$. The commutative property of multiplication should then lead to the conclusion that $0 \times n = 0$. When one factor is 0, the product is 0.

The teacher should understand that the above procedure provides a rational approach for recognizing that multiplication by 0 gives a product of 0. This procedure does not constitute a mathematical proof nor does it apply to multiplication of 0 by a fraction. A mathematical proof that the product of 0 and any number is 0 can be constructed from the field postulates, but such an activity is not appropriate at the elementary level. The fact that a factor of 0 leads to a product of 0 is frequently accepted without proof on the secondary level.

Developing the facts greater than the fives

The activities listed for presenting the facts in multiplication and division include the use of objective and visual materials. As the class develops and understands working with more facts in multiplication, there is less need for using exploratory material. For the facts not known in the table of the sixes, sevens, eights, and nines, an array or a number ray may still be useful. Patterns and number properties should be used more frequently at this level. The distributive property is particularly useful for work with facts greater than the fives.

The teacher should help the pupil to apply this property to find new facts in terms of old. This is especially true for facts in which both factors are greater than 5. In such cases the pupil renames one of the factors and then applies the distributive property. The plan to follow may be illustrated by finding the product of 6×7:

$6 \times 7 = 6 \times (2 + 5)$	
$= (6 \times 2) + (6 \times 5)$	Renaming 7 as 2 + 5
$= 12 + 30$, or 42	Distributive property
$6 \times 7 = 42$ and	Renaming numbers
$7 \times 6 = 42$	

In the same way the pupil can find the product of any basic pair by renaming one of the factors and then applying the distributive property.

Pupils should discover at least two different ways to verify a fact, for example, $4 \times 7 = 28$, by applying the distributive property. The pupil renames one of the factors (7) as $2 + 5$ or $3 + 4$. He then expresses 4×7 as $4 \times (2 + 5)$ or $4 \times (3 + 4)$.

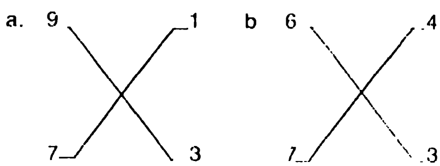
In this way the pupil always has an effective means of discovering the product of a number pair when the factors are between 5 and 10. It should be understood that the use of the distributive property is a long procedure for finding the product. Eventually, the pupil must learn to give a direct response to a basic fact.

St. Andrews cross

Robert Recorde, in the *Grounde of Arte*, published in 1542, described a method of finding the product of a pair of factors between 5 and 10. The plan he introduced was a trick procedure and meaningless to the learner. The products of the number pairs greater than 5×5 could always be found by writing the factors at the ends on the left of a Saint Andrews cross and writing the differences of the factors subtracted from 10 at the other ends. In (a) the factors are 9 and 7 and they are written at the left ends of the cross. The numerals naming the differences of the factors

subtracted from 10 are written at the right ends of the cross. The product of 9×7 can be found as follows:

1. The digit in ones' place in the product of 9×7 will be the product of 1 and 3, or 3.
2. The digit in the tens' place in the product of 9×7 will be the sum of 1 and 3 subtracted from 10, or $10 - (1 + 3) = 6$. Therefore, the product of 9×7 is 63. In the same way, all other products having factors between 5 and 10 can be found. In (b) the digit in tens' place is $10 - (4 + 3)$, or 3. The product of 4 and 3 is 12. Therefore, the product of 6 and 7 is $30 + 12$, or 42.



The procedure described is mechanical and was meaningless to most students who used it. The student who understood algebra could discover why the method worked, but then he would have no need for such a short cut in learning the multiplication facts.

The reader may wonder why this procedure gives the product of the factors between 5 and 10. We may represent the factors as $(10 - a)$ and $(10 - b)$, in which a is the difference between 10 and the first factor and b is the difference between 10 and the second factor. The product of these factors is as follows:

$$\begin{aligned}(10 - a)(10 - b) &= 100 - 10b - 10a + ab \\ &= 10(10 - b - a) + ab \\ &= 10[10 - (b + a)] + ab\end{aligned}$$

The last equation shows that ab is the digit named in ones' place in the product. The sum of $a + b$ subtracted from 10 and then multiplied by 10 names the number in tens' place in the product. The procedure by Recorde was satisfactory when emphasis in learning was not placed on understanding.

Division with a remainder not zero

In a set of 7 elements there are two equivalent subsets of 3 elements with 1 element remaining.

A number ray may be used to show how many threes there are in 7. In Figure 10.7 the points representing the multiples of 3 are circled. Since 7 is one to the right of 6 or 2×3, it follows that $7 = 2 \times 3 + 1$.

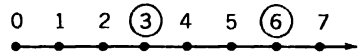


Figure 10.7

In Figure 10.8 the arrows of lengt 3 start at 7 and move to the left. Tw complete arrows of length 3 may b drawn. Since the second arrow ends a 1, it follows that $7 = 2 \times 3 + 1$.

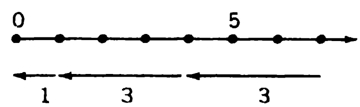


Figure 10.8

Th ivision process may be applied to any o whole numbers (if the divisor is not 0), which means that for any ordered pair of numbers (a, b) there are whole numbers q and r such that $a = qb + r$ where r is less than b.

For (17, 3), $q = 5$ and $r = 2$, so that

$17 = 5 \times 3 + 2$, or $\begin{array}{r} 5\text{ r }2 \\ 3\overline{)17} \end{array}$.

For (3, 17), $q = 0$ and $r = 3$, so that

$3 = 0 \times 17 + 3$, or $\begin{array}{r} 0\text{ r }3 \\ 17\overline{)3} \end{array}$.

If the remainder is not less than the divisor, the division is not complete. It is true that $17 = 4 \times 3 + 5$, but 4 is not considered to be the quotient, since 5 is not less than the divisor 3.

In the ordered pair (a, b), the number b is understood to be the divisor, and therefore r is less than $b (r < b)$, or the greatest remainder is $b - 1$. If the remainder is 0, then a is a multiple of b.

Division of two numbers whose quotient cannot be recognized as the result of a fact can be performed by the division algorism, as shown at the right. When the dividend is less than 10 times the divisor and the divisor is a whole number less than 10 (not 0), the quotient will be a one-place number. The pupil should then be able to give the quotient from his knowledge of the multiplication facts. Often he is unable to recall the factor in an example of the type $a \div b$. In that case he may represent the dividend as the sum of two addends and then apply the right-hand distributive property of division. If the pupil is unable to give the quotient of $52 \div 6$, he may rename 52 as $36 + 16$ or $30 + 22$. One of these addends should be a multiple of 6. The example $52 \div 6$ may now be written as follows:

$$\begin{array}{r} 5 \\ 3\overline{)17} \\ \underline{15} \\ 2 \end{array}$$

$$52 \div 6 = (36 + 16) \div 6 = (36 \div 6) + (16 \div 6)$$

The quotient of $36 \div 6 = 6$. Since 16 is not a multiple of 6 and 12 is the largest multiple of 6 that is less than 16, $16 \div 6 = (12 \div 6)$ with a remainder of 4. Therefore, $52 \div 6 = (36 + 12) \div 6$ with a remainder of 4. The quotient is 8, with a remainder of 4. The same answer can be found by using the division process. The pupil did not recall the largest multiple of 6 that is less than 52, but he knew that $6 \times 6 = 36$. He indicated this fact as shown in the first step. After the subtraction is performed, there is a remainder of 16, which is divided as shown in the second step. The quotient is $6 + 2$, or 8, with a remainder of 4. The same answer can be found by repeated subtraction of 6 from 52. The short forms in (a) and (b) are the standard procedures for finding the number of sixes in 52:

$$\begin{array}{rr} 6\overline{)52} & 6 \\ \underline{36} & \\ 16 & 2 \\ \underline{12} & \underline{} \\ 4 & 8 \end{array}$$

a. $52 \div 6 = 8 \text{ r } 4$

b. $\begin{array}{r} 8 \text{ r } 4 \\ 6\overline{)52} \\ \underline{48} \\ 4 \end{array}$

The pupil uses these forms after he understands the meaning of the process. An understanding of the process implies the following:

1. The answer can be found by repeated subtraction.
2. The product of the quotient and divisor is the largest multiple of the divisor contained in the dividend.
3. The remainder is always at least one less than the divisor.
4. Expressing the dividend as the sum of two addends and then dividing each number by the divisor illustrates the right-hand distributive property of division.

The fourth item is the one least understood by most pupils and is sometimes difficult to apply. Often the pupil does not know how to rename a number as the sum of two addends so that at least one of these addends is a multiple of the divisor.

The class should have little difficulty in finding the quotients for nonmultiples of 2 less than 20, since the remainder will always be 1. The dividends are the odd numbers from 1 through 19. It is primarily when dealing with the divisors 6 through 9 that the class experiences difficulty in finding the quotient named by a one-place numeral of a number that is not a multiple of the divisor.

Division of the nonmultiples of the divisor

We shall first be concerned only with the quotients named by a one-place numeral for the numbers that are not multiples of the divisor. The largest multiple of the divisor that is less than the number divided is then the product of a basic number pair. Finding the quotient in examples of this kind is therefore closely related to finding a missing factor of an open-number sentence that represents a basic fact in division.

The plan for introducing the division process for finding the quotient of nonmultiples of the divisor follows about the same pattern as that used to introduce the division facts. The pupil uses objective materials, such as counters, markers, or rectangular strips, and also makes visual representations of a number pair. He may use both types of aids until he is able to discover the relationship between the multiples and the nonmultiples of a given factor.

Many teachers have the pupil write the numbers in sequence up to 10 times the divisor. The pupil then circles the multiples of the divisor, as shown for the threes.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29

The teacher identifies the multiples of the divisor from the nonmultiples. The pupil expresses the nonmultiples in a number sentence using both multiplication and division. Thus, the different ways to write number sentences for the number pair $13 \div 3$ are as follows:

$$13 = 3 \times 4 + 1 \qquad 13 \div 3 = 4 \text{ r } 1$$

The two number sentences show the relationship between multiplication and division. In a similar manner, the pupil writes number sentences for any nonmultiple of 3. He may also use the division algorism, as shown at the right.

$$\begin{array}{r} 4 \text{ r } 1 \\ 3\overline{)13} \\ \underline{12} \\ 1 \end{array}$$

A number ray is an effective visual aid to enable a pupil to find the quotients of numbers that are not multiples of the divisor. In Figure 10.9, the points representing multiples of 3 are circled. Since 13 is one unit to the right of 12 or 3×4, it follows that $13 = 3 \times 4 + 1$. The nonmultiples are indicated by the points not circled. If one starts at 13 and draws arrows to the left, as in the rays in Figure 10.9, it is clear that $13 = 4 \times 3 + 1$.

The pupil uses a number ray in the same way that he uses any other learning aid. He uses this visual aid to find the quotient of a nonmultiple of the divisor until he discovers the answer by using only numbers. His knowledge of the facts in multiplication and division then enables him to write the quotients of both multiples and nonmultiples of the divisor.

Expressing the remainder in division

When the division process is performed on a number that is not a multiple of the divisor, a remainder greater than 0 will result. The way to ex-

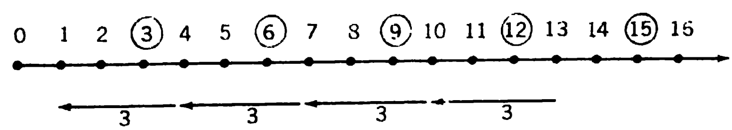


Figure 10.9

press that remainder depends upon the set of numbers in which the operation is performed and upon the division situation. In the set of whole numbers, the quotient may be expressed only as a whole number with a remainder. The set of rational numbers is closed with respect to division (except for 0). The property of closure makes it possible to express the quotient with any number in that set. Thus, the quotient of $8 \div 3$ may be expressed as $2\frac{2}{3}$ as shown:

$$\frac{8}{3} = \frac{6+2}{3}$$

$$= \frac{6}{3} + \frac{2}{3} = 2 + \frac{2}{3}, \text{ or } 2\frac{2}{3}$$

There are two division situations that demand different representations of the remainder in division. One calls for the remainder to be expressed as part of the quotient. The quotient will then be expressed as a fractional numeral. This situation may be illustrated by the following problem.

A piece of wire 9 feet long is cut into 4 equal pieces. What is the length of each piece? The length of each piece is $2\frac{1}{4}$ feet. The answer 2 with the remainder 1 is not sensible here. After the introduction of rational numbers, it is possible to divide 9 by 4 and express the quotient as $2\frac{1}{4}$. In the set of whole numbers, this problem cannot be solved. The need for solving a problem of this kind gave impetus to expansion of the number system to include rational numbers.

The other division situation calls for the remainder to be expressed as a remainder and not as part of the quotient. This condition prevails in the set of whole numbers. The following problem illustrates the second division situation:

How many groups of 4 children can be formed with 9 children?

The quotient is 2 with a remainder of 1. A quotient of $2\frac{1}{4}$ is meaningless in this situation.

The way in which the quotient is expressed in division situations involving nonmultiples of the divisor depends upon the situation. The pupil must be able to interpret the answer. From the standpoint of the structure of the number system, the division process can be applied to any two whole numbers (divisor not 0). The quotient of $a \div b$ is q, with a remainder r ($r < b$). In case a is a multiple of b, r is 0.

FORMATION OF TABLES

Table for the facts in multiplication and division

The pupil should construct a table that includes certain sets of facts as these facts are introduced, for example, the set of facts for the twos, threes, and fours. Such tables are described on page 163. When the teacher has introduced all the facts in multiplication, each pupil should make a composite table of these facts. A composite shows the orderly arrangement of a set of factors and of the products. The class should make a composite table of the facts for display on the classroom bulletin board.

Table 10.6 is a composite table for all the facts in multiplication and division. The pupil must understand that division by 0 is not permitted. Each number named in black type is a product. The numerals in color that are written at the beginning of each row and at the top of each column name the factors of the product, which is written in both a column and a row.

There are two unequal factors for each product except the products named along the diagonal from the upper left corner of the table to the lower

TABLE 10.6

Composite Table of Multiplication and Division Facts

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

right corner. These products are the squares of the numbers from 0 through 9. The square of a number is the product of that number and itself. Each pair of unequal nonzero factors forms a set of four related facts, two in multiplication and two in division (see p. 158).

Interpreting a composite table

After the class has constructed the table of multiplication facts, the teacher should encourage the pupils to discover some of its distinguishing features. The class should find all or some of the following characteristics of the table:

1. Each succeeding number named in a row increases by the same amount as the number named in color in the column at the left.
2. Each succeeding number named in a column from the top down increases by the same amount as the number named at the top of the column.
3. The number named in each row may be found by adding the number named in the left margin to the preceding number. A similar method may be used to obtain the numbers named in a column.
4. If a number named in the table is divided by the factor at the beginning of a row, except 0, the quotient will be the factor at the top of that column.
5. If a number named in the table is divided by the factor at the top of a column, except 0, the quotient will be the factor at the left of the row.
6. Each number named in the tables on the diagonal from the upper-left corner has two equal factors. (These

numbers are the squares of the numbers from 0 through 9.)

7. When one of the factors is 0, the product is 0.

8. If a number is multiplied by 1, the product is that number.

9. Each number in the table is a multiple of the number at the top of its column and at the left of its row.

10. The commutative property applies to the number pairs, as $3 \times 5 = 5 \times 3$.

11. The product of two even numbers is even; the product of two odd numbers is odd.

12. The product of an even number and an odd number is even.

13. In multiplication the factors are given but the product is missing; in division the product and one factor are given but one factor is missing.

Most pupils will not discover all the relationships enumerated. The teacher should encourage the pupils to identify as many distinguishing features of the table as possible.

Making a table from known facts

Each pupil should know how to find the answer to a basic number pair in multiplication, as 4×7, by use of objective materials such as disks or strips of geometric designs, by making drawings, and by addition. If he knows a division fact he should be able to derive the corresponding multiplication fact. All pupils should possess these minimum learning facts.

In every good learning situation there must be provision for more than minimum learnings. One of the writers showed how opportunities for such learning may be provided in the presentation of the multiplication facts.[5]

[5]Foster E. Grossnickle, "Discovering the Multiplication Facts," *The Arithmetic Teacher*, October 1959, 6:195–198.

He proposed three stages in the development of these facts as follows:

1. The minimum plan as described
2. Construction of a table from known facts
3. Discovery of a pattern for a set of tables.

The second and third stages or steps in the presentation are not part of a minimum program. These activities are offered to challenge the more able pupil to discover patterns and relationships among numbers.

The class should make a table, such as the table of the sixes. All pupils should participate in making the table, but the pupils who develop the greatest insight into number should profit most from the activity. Certain facts involving the sixes are given. From the known facts the class derives other facts. At the introduction of each new fact or element of a table the teacher challenges the pupils to offer as many ways as possible of finding a new fact. As each new fact or element in a table is verified, this fact may be used in discovering the remaining facts of the table.

The procedure to follow to make a table of the sixes from known facts may be illustrated by assuming that the following facts are known:

$$1 \times 6 = 6$$
$$2 \times 6 = 12$$

$$10 \times 6 = 60$$

The next element in making the table is to derive the fact $3 \times 6 = 18$. The class offers as many ways of proving this fact as possible by using only the facts assumed to be known and also basic knowledge of the multiplication process. The class should be able to discover that $3 \times 6 = 18$ by giving the following answers:

"Since 1 six is 6 and 2 sixes are 12, add 6 and 12 to find three sixes." This procedure shows the distributive law.

"Add 3 sixes."

"Add 6 threes."

The next step in the construction of the table consists in finding $4 \times 6 = 24$. The following means may be given to find the product of 4 and 6:

"Add 1 six to the product of 3 sixes."

"Since 2 sixes are 12, 4 sixes will be twice as much, or 24."

"Add either 6 fours or 4 sixes."

"Since 10 sixes are 60, 5 sixes will be half as much, or 30. Then 4 sixes will be 1 six less, or 24."

The answers stated above are typical of some of the answers pupils give to express the relationships among the multiplication facts given in the table. The teacher should encourage activities of this kind because the pupil is challenged to exhibit thinking in dealing with number. Learning takes place when thinking is involved. Making a table by using the technique suggested offers some of the best learning experiences the teacher can provide to challenge the class, especially the superior pupils.

Sets of tables

The pupil who develops insight into number synthesizes small groups into larger groups and discovers relationships between the smaller and larger groups. The superior pupil in grades 4 to 6 should be able to discover that the tables may be grouped into *sets.* There are some unifying ideas or factors common to each table within a set. The twos, fours, and eights may be classified in the set of twos; the threes, sixes, and nines in the set of threes. The two remaining tables of fives and sevens cannot be grouped with the other sets of tables.

The tables in the set of threes include the threes and nines, and for purposes of classification, also the sixes. The sixes can partially be classified in the set of the twos and partially in the set of the threes. The more able pupil at about the level of grade 4 should write the facts in the three tables, as shown, and then find the sum of the numbers named by the digits in each product:

$3 \times 1 = 3$	3	$9 \times 1 = 9$
$3 \times 2 = 6$	6	$9 \times 2 = 18$
$3 \times 3 = 9$	9	$9 \times 3 = 27$
$3 \times 4 = 12$	3	$9 \times 4 = 36$
$3 \times 5 = 15$	6	$9 \times 5 = 45$
$3 \times 6 = 18$	9	$9 \times 6 = 54$
$3 \times 7 = 21$	3	$9 \times 7 = 63$
$3 \times 8 = 24$	6	$9 \times 8 = 72$
$3 \times 9 = 27$	9	$9 \times 9 = 81$

$6 \times 1 = 6$	6
$6 \times 2 = 12$	3
$6 \times 3 = 18$	9
$6 \times 4 = 24$	6
$6 \times 5 = 30$	3
$6 \times 6 = 36$	9
$6 \times 7 = 42$	6
$6 \times 6 = 48$ (12)	3
$6 \times 9 = 54$	9

These sums are given in the column at the right of a table. The numbers named by the numerals between two horizontal bars are in the same decade. Some of the characteristics of the products in each subset of the set of threes are as follows:

1. The sequence of the sum of the digits in the products of the threes is 3-6-9 and then the sequence repeats. A new decade appears at each point of repetition.

2. The sum of the numbers named by the digits in each product is 9 in the table of the nines.

3. Another numeral for 9 is $10 - 1$. Each product can be found by multiplying $(10 - 1)$ by a one-place number and applying the distributive property. The application of this property makes it

possible to tell why the ones' digit in each successive product decreases by 1 and the tens' digit increases by 1.

4. All 10 digits are used in writing the products in the threes and nines. This is not true for the sixes or for any of the other tables except the sevens.

5. Alternate products of the threes and the nines are even, but all of the products of the sixes are even. This is also true of the fours and eights, because 4, 6, and 8 have factors of 2.

6. The sequence of the sum of the digits of the products in the table of the sixes is 6-3-9 and then the sequence repeats. If a digit naming a smaller number in the sum follows a digit naming a larger number, the products are in different decades. In the reverse situation, the products are in the same decade. Finding the sum of the digits in a product is the equivalent of casting out nines in that product (see p. 219).

The pattern given for the set of the threes can be used to find the characteristics of the set of the twos. The pupil should be able to discover many of the relationships among the products.

The products in the table of the fives end in either 0 or 5. This is true because 5 is half the number base. There is no apparent pattern for the sequence of the digits in the products of the sevens because 7 is *prime* (it has no factor except 1) with respect to the number base and to all the remaining digits.

Discovering number patterns from a table

The teacher should provide opportunities for the pupil to discover number patterns. The sequence of the multiplication facts in a table forms a pattern all pupils should discover. Changing the sequence of the facts or of the factors often enables a pupil to discover one or more patterns that characterize the table. Sawyer and also Jackson illustrated different patterns in multiplication and stressed the importance of pupil discovery of a pattern.[6] The facts in (a) and (b) illustrate a pattern that a pupil should discover.

$6 \times 6 = 36$	b.	$6 \times 5 = 30$
$7 \times 5 = 35$		$7 \times 4 = 28$
$8 \times 4 = 32$		$8 \times 3 = 24$

11 11 $11 \times 0 = 0$

The teacher supplies a few facts of a table. The pupil is supposed to discover the pattern and then supply the missing numerals. In the examples the pupil should be able to discover the pattern for writing the missing numerals from the sequence either of the factors or of the products. In (a) the difference of the products is the series of odd numbers; in (b) the difference is the series of even numbers.

The relationship shown by the factors in (a) is utilized in algebra to find the product of two factors of the type $(n + a)(n - a)$. In (a), n is equal to 6. The difference of the products of the factors 6×6 and 7×5 is 1. Therefore, $(n + 1)(n - 1) = n^2 - 1$, and the product of $(n + a)(n - a)$ will be $n^2 - a^2$, in which a is the difference between each factor and their average.

The first equations in (a) and (b) should show the pattern for the formation of another set of multiplication facts. The teacher should use the illustrations in (a) and (b) to aid the pupils in giving similar sets of examples.

[6] W. W. Sawyer, "*Why Arithmetic Is Not the End*," *The Arithmetic Teacher*, March 1959, 6:91–96; H. C. Jackson, "Tables and Structure," *The Arithmetic Teacher*, February 1960, pp. 71–76.

Testing understanding of the operations

After the pupil has meaningful experiences with the facts pertaining to the four operations, he should be able to deal with the basic number pairs in these operations. A table of the type shown (Table 10.7) is an effective means of testing a pupil's understanding of the operations. The entries in the first example are complete. Some entries in the remaining examples are vacant. The pupil supplies the correct answer for each blank.

Dramatizing the operations

Dramatization offers an effective means of demonstrating the meaning of an operation. The teacher should ask a group of approximately eight pupils to pantomime a process and have the remaining pupils in the class describe the operation represented in the demonstration.

Five different demonstrations are needed to represent the four operations. The pupils who are to interpret the nature of the pantomime should understand that no operation will be repeated. A demonstration to represent multiplication could also represent addition, but it is possible to have a representation of addition that does not apply to multiplication. The group giving the pantomime must be sure that it is possible for the audience to differentiate between addition and multiplication and between subtraction and division.

Suppose eight pupils participate in a pantomime. The following procedures may be used to dramatize the operations. These pupils should meet in a separate place, as in the hallway outside of the classroom, to decide upon the type of representation to be given.

Addition The eight pupils to participate in the dramatization enter the hallway near the classroom and form three unequal groups, such as groups of 2, 1, and 5. The first group walks to the front of the classroom followed by the 1 pupil and then followed by the group of 5 pupils. The pupils then form one group of eight in a straight line. After a few seconds in this position the group disbands.

Subtraction The eight pupils line up in front of the classroom. Then one group goes into the hallway. The num-

TABLE 10.7

Testing the Pupil's Understanding of the Operations

Example	*Number Pair*	*Answer to a Number Pair*	*Operation*	*Equation or Mathematical Sentence*
1	3, 5	8	Addition	$3+5=8$
2	4, 1	4	?	?
3	?, ?	?	?	$12 \div 2 = 6$
4	5, ?		?	?
5	?, 6	8	Subtraction	?
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

ber in the group leaving the line should not be the same as the number in the group remaining.

Multiplication The eight pupils assemble in the hallway and march into the classroom in equal groups, such as groups of twos. The pupils form a straight line as they arrange themselves before the class.

Division Two representations are needed to show the two usages of division:

1. *Ratio.* Have the eight pupils arrange themselves in a straight line at the front of the classroom. They then leave the line in groups of twos, each group selecting a different position in the classroom. There should be four groups, each containing two pupils.

2. *Rate.* Have the eight pupils form a straight line at the front of the classroom as arranged in the previous demonstration. Now they arrange themselves into two equal groups. The first pupil in the line goes to a certain place in the classroom. The second pupil goes to a different place from the position taken by the first pupil. Then, in order, the six remaining pupils separate to form two equal groups.

Neither the order of the presentation nor the demonstration of a process need be the same as described above. The plan outlined is effective for determining how well the class understands the meaning of the basic operations.

EXERCISES

1. Indicate how you would have a pupil discover the meaning of reversibility as applied to multiplication.
2. What is a multiplication situation? a division situation? Illustrate each.
3. What properties of multiplication are common to addition.
4. Show how finding the product of the example at the right illustrates the distributive property. $\begin{array}{r} 32 \\ \underline{\times\ 3} \end{array}$
5. What is meant by the statement that division illustrates a limited usage of the distributive property?
6. Show why it is not possible to divide by 0.
7. Try to formulate a problem in which it is necessary to multiply a number by 0, for example, $0 \times 4 = 0$.
8. Give four ways a pupil can discover the product of 3×8, or verify the product if he knows the threes through the fives.
9. Use the following sets. Make the set of related facts with the elements of each set.

   A: $\{4, 4, 16\}$

   B: $\{56, 7, 8\}$
10. Decide if each of the following problems represents a ratio or a rate situation:
    a. How many yards are there in 15 feet?
    b. How many quarts are there in 6 pints?
    c. At 15 miles per gallon, how many gallons of gasoline are needed to travel 60 miles?
    d. A car used 4 gallons of gasoline on a 60-mile trip. What was the average mileage per gallon?
    e. If 40 tulip bulbs are planted in 5 equal rows, how many bulbs in a row?
11. What is meant by the statement that the *division process* may be used to divide any two whole numbers instead of using division.

12. Some pupils used the algorism shown in (a) for finding the quotient of the example $53 \div 8$. Other pupils used the one shown in (b). Evaluate these procedures.

    a. $53 \div 8 = 6\ r\ 5$

    b. $$\begin{array}{r|l} 8\overline{)53} & \\ \underline{40} & 5 \\ 13 & \\ \underline{8} & \underline{1} \\ 5 & 6 \end{array}$$

13. A teacher wrote on the chalkboard the following number sentences and asked the class to fill in the table. One of the pupils asked why the numbers behave as they do. Give a satisfactory answer to this question.

    a. $1 \times 9 + 1 = 10$
    b. $2 \times 9 + 2 = 20$
    c. $10 \times 9 + \triangle = \square$

14. Write the set of tables for the twos, fours, and eights. Identify at least three distinguishing features of the products or of the sequence of the digits in the products.

15. Make a list of ways to show how multiplication and division are opposites.

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MULTIPLICATION AND DIVISION OF WHOLE NUMBERS

Chapter 10 was concerned with the properties of multiplication and division. The present chapter deals with the following topics: multiplying and dividing by a one-digit number with no regrouping, multiplying by a one-digit number with regrouping, dividing by a one-digit number with regrouping; multiplying by a two-digit number, dividing by a two-digit number; relationships between multiplication and division.

The topics listed involve the algorisms for multiplication and division. The following material should not only teach the pupil how to perform the algorisms but should also enable him to discover how the algorisms apply the properties of the operations.

MULTIPLYING AND DIVIDING BY A ONE-DIGIT NUMBER WITH NO REGROUPING

Multiplying by 10

Example (a) illustrates multiplication by a one-digit number without regrouping in the product. Example (b) shows the corresponding example in division:

a. $\begin{array}{r} 23 \\ \times\ 3 \\ \hline \end{array}$ b. $3\overline{)69}$

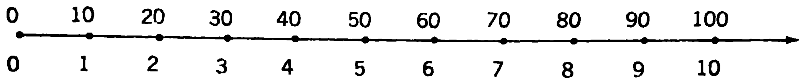

Figure 11.1

Before multiplying in examples such as (a), the pupil should multiply a one-digit number by 10. If he does not know these products, he should be able to find them by counting by tens. A number ray scaled as shown in Figure 11.1 is an effective instructional aid to enable a pupil to discover the pattern for finding the products when 10 is a factor. The use of a ray and a table of the kind given at the right should enable most pupils at the third-grade level to discover the pattern for finding the product when 10 is a factor. Since multiplication is commutative, $3 \times 10 = 10 \times 3$.

$0 \times 10 = 0$
$1 \times 10 = 10$
$2 \times 10 = 20$

In a similar manner, the teacher should have the pupil find the corresponding products when 100 is a factor. The pupil can count by hundreds to find a product, use a number ray, or apply the pattern from part of a table, as:

$1 \times 100 = 100$
$2 \times 100 = 200$

Multiplying by 100 is an extension of the procedure for multiplying by 10, as illustrated by the following:

8×100	The factors
$8 \times (10 \times 10)$	Renaming 100
$(8 \times 10) \times 10$	Associative property
80×10	Renaming 8×10
800	Renaming 80×10

After using a few illustrations that involve multiplying by 10 and 100, the pupil should discover that annexing a 0 to a numeral multiplies the whole number by 10; annexing two 0's multiplies the whole number by 100.

Multiplying by any one-digit number with no regrouping

If a pupil can multiply by 10, he can easily understand how to find the product of an example of the type 3×20. Name the 20 as 2×10. Then 3×20 may be written as $3 \times (2 \times 10)$. Since multiplication is associative, $3 \times (2 \times 10) = (3 \times 2) \times 10$. Therefore the product of $3 \times 20 = 6 \times 10$, or 60. The different steps described may be tabulated as follows:

3×20	The factors
$3 \times (2 \times 10)$	Renaming 20
$(3 \times 2) \times 10$	Associative property
6×10	Renaming 3×2
60	Renaming 6×10

After a pupil discovers how to find the product of a multiple of 10 and any one-digit number, he can readily discover how to multiply when the larger factor is not a multiple of 10. The product of 2×34 may be found by writing 34 in expanded notation, as $30 + 4$. The sequence of steps in the computation is as follows:

$2 \times 34 = 2 \times (30 + 4)$	Renaming 34
$= (2 \times 30) + (2 \times 4)$	Distributive property
$= 60 + 8$, or 68	Renaming numbers

Example (a) shows another procedure for multiplying 34 by 2. Each digit of 34 is given its total value, and the partial products are written as shown. The procedure in (a) emphasizes place value in a numeral. Example (b) represents the standard or conventional algorism for multiplication. Frequently the pupil learns how to compute as in (b) without understanding either structure or place value:

a. $\begin{array}{r} 34 \\ \times\ 2 \\ \hline 8 \\ +60 \\ \hline 68 \end{array}$ b $\begin{array}{r} 34 \\ \times\ 2 \\ \hline 68 \end{array}$

Dividing by a one-digit number with no regrouping

Since division is the inverse operation of multiplication, the pupil who understands the application of the distributive property of multiplication in the example $2 \times 34 = 68$ should understand that he applies the right-hand distributive property of division to each addend in the corresponding example in division. The corresponding example in division for the example $2 \times 34 = 68$ may be written as in (c). The expanded notation in (c) emphasizes place value and also suggests an application of the distributive property of division with respect to addition:

c $2\overline{)68} = 2\overline{)60 + 8}$, quotient $30 + 4 = 34$ d $2\overline{)68}$, quotient 34

The teacher should also instruct the pupil to use the following notation, which applies the distributive property of division over addition:

$$\frac{68}{2} = \frac{60 + 8}{2} = \frac{60}{2} + \frac{8}{2} = 30 + 4, \text{ or } 34$$

Example (d) shows the conventional algorism for division in examples of this type. This form should not be used until the pupil understands place value. He should be familiar with the equation form for finding the product of the quotient when no regrouping is required. Thus, the pupil would express the example 3×23 as shown in (e) and the example $3\overline{)69}$ as shown in either (f) or (g).

e $3 \times 23 = 3 \times (20 + 3) = 60 + 9$, or 69

f $3\overline{)69} = 3\overline{)60 + 9}$, quotient $20 + 3 = 23$

g $3\overline{)69} = \frac{60 + 9}{3} = 20 + 3$, or 23

MULTIPLYING BY A ONE-DIGIT NUMBER WITH REGROUPING

About the same types of experiences may be used to introduce regrouping in the product as are used to introduce multiplication without regrouping. To find the product in (a), it is necessary to regroup the *partial product* in ones' place before multiplying the tens. The following sequence of activities should enable the pupil to understand regrouping in the example 3×24. The teacher should proceed as follows:

a. $\begin{array}{r} 24 \\ \times\ 3 \\ \hline 72 \end{array}$

1. Express 24 in expanded notation and then write the example as $3 \times (20 + 4)$. Apply the distributive property and solve as follows:

$3 \times (20 + 4) = (3 \times 20) + (3 \times 4)$	Distributive propei
$= 60 + 12$	
$= 60 + (10 + 2)$	Renaming 12
$= 60 + 10 + 2$	
$= (60 + 10) + 2$,	Associative proper
or 72	

The sequence of steps in the last line may be shortened to $60 + 12 = 72$.

The pupil should verify the product by finding the sum of $24 + 24 + 24$. The sum in ones' place is overloaded; therefore the numeral 12, which expresses this sum, must be regrouped as 1 ten and 2 ones.

2. Use the vertical form for expressing 24 in expanded notation, as shown in (b).

3. Write each product as shown in (c). The 2 in 24 is given its total value of 20.

b $\begin{array}{rcl} 24 & = & 20 + 4 \\ \times\ 3 & = & \times\ 3 \\ \hline & & 60 + 12 = 72 \end{array}$ c. $\begin{array}{rl} 24 & \\ \times\ 3 & \\ \hline 12 & (3 \times 4) \\ 60 & (3 \times 20) \\ \hline 72 & \end{array}$

4. Introduce the algorisms shown in (d) and (e). In (d) the pupil writes the number of tens found from regrouping the product of the ones. Most pupils do not need this visual aid in multiplication. Example (e) is the standard algorism used for an example of this type.

$$\text{d.}\quad \begin{array}{r} \scriptstyle 1 \\ 24 \\ \times\ 3 \\ \hline 72 \end{array} \qquad \text{e}\quad \begin{array}{r} 24 \\ \times\ 3 \\ \hline 72 \end{array}$$

ADDING BY ENDINGS IN MULTIPLICATION

It has already been demonstrated that adding by endings is used in column addition and multiplication. Example (a) shows how to find the product of 8 and 476 when each partial product is written. Example (b) shows the product of these factors when the partial products are not written in full. Example (b) involves adding by endings but (a) does not:

$$\text{a}\quad \begin{array}{r} 476 \\ \times\ 8 \\ \hline 48 \\ 560 \\ 3200 \\ \hline 3808 \end{array} \qquad \text{b}\quad \begin{array}{r} 476 \\ \times\ 8 \\ \hline 3808 \end{array}$$

In (b) it is necessary to regroup the first partial product (48) as 4 tens and 8 ones. The 4 tens are added to 56 tens (8×7 tens) to make 60 tens, or 6 hundreds and 0 tens. Finally, 6 hundreds are added to 32 hundreds (8×4 hundreds). Adding 4 to 56 and 6 to 32 illustrates adding by endings. The numerals naming the numbers involved in adding by endings in multiplication are unseen. Computing with unseen numerals often creates considerable difficulty for some pupils. If the factor 476 were expressed in expanded notation, adding by endings would not be involved. Therefore multiplication as shown in (a) or in the example $8 \times (400 + 70 + 6)$ should be less difficult for the beginner than multiplication in the standard algorism, as given in (b). The solution in (b) is much shorter than that in the other two forms. The pupil who multiplies as in (b) demonstrates a higher level of maturity than the one who proceeds as in the other two forms.

Pupils frequently find that adding by endings in multiplication is difficult when one of the factors is from 6 through 9. The maximum number to be added to a partial product in regrouping is always one less than the multiplier. If the multiplier is 7, the maximum number to be added to a partial product is 6.

The number of examples involving adding by endings for each two-digit product is one less than the greatest one-place factor of that product. If the product is 36, the one-place factors are 4, 6, and 9. The greatest factor is 9, hence the maximum number in regrouping a product to be added to 36 is 8. The possible examples of this kind are as follows:

a	b	c
36 + 1	36 + 4	36 + 7
36 + 2	36 + 5	36 + 8
36 + 3	36 + 6	

The three examples in column (a) have sums in the same decade as the product; the examples in columns (b) and (c) have sums in the next decade.

The class should practice with examples that are given orally. The teacher should dictate a two-place product of a basic number pair and a one-place number. The one-place number should be at least one less than the greatest (one-digit) factor of the dictated product. The pupil should write the sum of these numbers. An exercise of this kind is usually effective in helping the class to deal with numbers named by unseen numerals, as used in regrouping in multiplication.

Enriching work with one-place factors

An effective program for teaching the basic operations in arithmetic has two characteristics. First, there is provision for a minimum acceptable achievement for all pupils. Second, there is provision for the superior pupils to gain mastery in the operations and to develop a level of insight into the meaning of the operations, which most of the class will not attain. These pupils should discover ways other than the conventional one for dealing with multiplication of a two-place factor by a one-place factor. For example:

1. To multiply by a *composite number* (a number having other factors besides itself and 1), multiply by the factors of that number. Thus to multiply by 6, first multiply by one of the factors, as 3, and then multiply that product by the other factor, or 2.

2. To multiply by 9, rename 9 as 10 − 1 and then apply the distributive property of multiplication with respect to subtraction. Thus, $9 \times 67 = (10 - 1) \times 67$, or $670 - 67$. The pupil may discover a short cut for performing the multiplication when a factor is 10 − 1. If the other factor is 67, the work may be performed as shown.

$$\begin{array}{r} 670 \\ -\ 67 \\ \hline 603 \end{array}$$

The pupil annexes a 0 to 67 and then subtracts 67 from that product. The superior pupil should perform the computation without paper and pencil and should understand the sequence of steps in the solution.

3. Use expanded notation for expressing the larger factor and then apply the distributive property of multiplication. All computations should be made without writing the partial products. This type of exercise stresses "mental arithmetic." To find the product of 7×48, express 48 as either $40 + 8$ or $50 - 2$, multiply each number by 7, and find the sum or difference of the products without writing any of the numerals except the numeral naming the answer, or 336. The larger factor should not contain more than three digits if the computation is to be done with unseen numerals.

DIVIDING BY A ONE-DIGIT NUMBER WITH REGROUPING

The solution of the example $3\overline{)72}$ involves regrouping. To solve examples of this type by using the conventional algorism, the pupil must be able to: (1) select the largest multiple of the divisor in each partial dividend; and (2) find the difference in (1) and express that difference as part of the number named by the digit one place to the right. In the given example, the pupil must select the correct multiple of 3 in 7 tens, which is 6 tens. The difference of 1 ten between 7 tens and 6 tens must be changed to 10 ones to be added to 2 ones to make a total of 12 ones. According to Bruner, "regrouping represents one of the two most difficult procedures a pupil must encounter with the operations."[1] The pupil must rely on his knowledge of the multiplication facts in order to find the largest multiple of the divisor contained in a partial dividend. Page 173 explained how to teach the pupil a way of finding a missing answer in an example of the type $27 \div 4 = \square \text{ r } \triangle$. If the pupil solves an example of the type $6\overline{)522}$ by short division, most of the work is done with numbers that are not named by numerals. This task

[1] Jerome Bruner, *The Process of Education* (Cambridge, Mass.: Harvard University Press, 1963), p. 41.

is too difficult for almost all pupils at the grade level at which the topic is introduced.

One of the principles of learning given on page 31 pertains to growth in maturity in dealing with concepts and operations. This principle should be applied as well to division involving regrouping. Several decades ago many schools presented only the adult pattern for dividing, as in the example $6\overline{)522}$. This pattern could be designated *short division,* as in (a), or *long division,* as in (b). The teacher should not use the algorism shown in (a) to introduce division by a one-place divisor when regrouping is involved in the solution:

$$\text{a}\quad \begin{array}{r} 87 \\ 6\overline{)522} \end{array} \qquad \text{b.}\quad \begin{array}{r} 87 \\ 6\overline{)522} \\ \underline{48} \\ 42 \\ \underline{42} \end{array}$$

All experimental evidence indicates that the scores made by pupils using the long form were significantly higher than those made by the pupils using the short form.[2] Although the use of form (b) will result in greater accuracy than the use of form (a), the long form is not the one preferred for introducing division involving regrouping.

Teaching division involving regrouping

The pupil should know that division is the undoing of multiplication. If he can find the product of $3 \times (20 + 4)$, he should be able to find the quotient in the example $72 \div 3$. A set of opposite problems may be used to introduce division of this type. One is as follows

[2]See Foster E. Grossnickle and Leo J. Brueckner, *Discovering Meanings in Elementary School Mathematics*, 4th ed. (New York: Holt, Rinehart and Winston, Inc., 1963) p. 190.

Each of 3 sections of a class contains 24 pupils. How many pupils are there in a class?

The pupil writes 24 in expanded notation and makes the following computation:

$$3 \times (20 + 4) = 60 + 12 = 72$$

The problem using the inverse operation is as follows:

A class containing 72 pupils is to be divided into 3 equal groups. How many pupils will there be in each group?

The pupil writes 72 in expanded form as $70 + 2$. Since 70 is not a multiple of 3, he renames 70 as $60 + 10$. Now $3\overline{)70 + 2}$ may be expressed as $3\overline{)60 + 12}$. He then completes the solution as shown in either (a) or (b).

$$\text{a}\quad 3\overline{)72} = 3\overline{)70 + 2} = \begin{array}{r} 20 + 4\text{, or } 24 \\ 3\overline{)60 + 12} \end{array}$$

$$\text{b}\quad \frac{72}{3} = \frac{70 + 2}{3} = \frac{60 + 12}{3} = \frac{60}{3} + \frac{12}{3}$$
$$= 20 + 4\text{, or } 24$$

Check: $3 \times 24 = 3 \times (20 + 4)$
$= 60 + 12\text{, or } 72$

The slow learners in the class should use their squares and rectangular strips to demonstrate the solution (see Fig. 11.2). Each pupil would form three groups by placing a strip in each group and then repeat the procedure. There would be one strip of tens remaining, which he would exchange for 10 ones to make in all 12 ones. Next he would distribute the ones into three equal groups. Each group would contain 2 strips of tens and 4 ones to represent 24. Therefore the quotient of $72 \div 3$ is 24.

Finally, the teacher would show the slow learner how to write the steps in the solution as given above. The use of objective materials supplemented with proper questioning pertaining to the procedure followed in separating the strips should enable the pupil to dis-

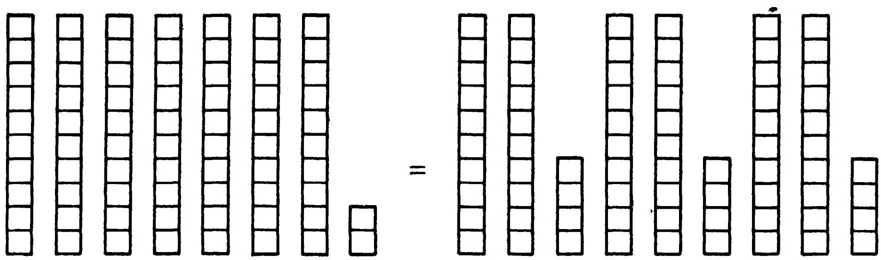

Figure 11.2

cover how and why the number divided must be regrouped to make the number of tens a multiple of the divisor.

The next step in learning to divide when regrouping is involved is to introduce the standard notation for division. The expanded notation is effective for showing the relationship between multiplication and division when the numerals in multiplication are written in that form. However, the expanded notation is not the conventional one for writing the numerals in an example in multiplication. Example (a) shows the most familiar notation for writing an example in multiplication. Example (b) represents the most familiar notation for writing an example in division:

```
 78      b   4)98
· 6
---
```

Therefore the pupil should learn to divide by using the accepted notation for writing the numerals when this operation is applied.

The class used the example 72 ÷ 3 to find the quotient when the dividend was written in expanded notation. That same example should be used to introduce the new notation, as illustrated by the work in (c):

```
c.  3)72   20
      60
      --
      12    4
      12   --
      --   24
```

The pupil knows that the quotient must be greater than 10 because $3 \times 10 = 30$ and $3 \times 20 = 60$ and 72 is greater than 60. From his knowledge of multiplication, he knows that the largest multiple of 3 and 10 in 72 is 60. He then writes 20 at the right of the dividend and completes the example as shown. The quotient is the sum of 20 and 4, or 24.

The procedure in (c) may represent a low level of maturity in dealing with division. If the pupil does not know the largest multiple of the divisor, he can find the correct quotient by repeated subtraction, as shown in (d):

```
d   3)72   10
      30
      --
      42   10
      30
      --
      12    4
      12   --
      --   24
```

We shall designate this method the *subtractive method.* The subtractive method may be used to introduce division involving regrouping for the following two reasons.

1. It emphasizes division as a shortened form of subtraction.
2. It is not necessary to select the largest multiple of the divisor in the dividend in order to find the quotient.

Van Engen and Gibb experimented with the subtractive method of division and compared it with the conventional

or standard method of division.[3] Their study showed that the difference in achievement made by the two groups was not significant.

The subtractive method may be modified slightly by writing the quotient above the dividend, as shown in (e):

```
e      24
        4
       20
     3)72
       60
       12
       12
```

The quotient is the sum of 20 and 4, or 24. This procedure may be called the *pyramid method* of division. If the pupil is unable to select the largest multiple of the divisor in the dividend when dividing by this method, he can find the quotient by repeated subtraction, as in the subtractive method. Therefore, the two methods are similar. The pupil using the pyramid method shows slightly more maturity than when he uses the subtractive method because of emphasis on place value. Each quotient digit must be correctly placed with respect to the digits of the dividend. The subtractive method does not give consideration to this feature. Since the two procedures are similar, the method to use in introducing regrouping in division is mostly a matter of choice.

Because the pyramid method gives more emphasis to place value, we shall consider this method in greater detail. The mature way of dealing with this method consists in selecting the correct multiple of the divisor for each partial dividend. In order to make this selection, the pupil must know the following:

1. The number of places in the quotient
2. How to estimate each quotient digit.

The pupil can find the number of places in the quotient by multiplying the divisor by a power of 10. In the example $4\overline{)275}$, the quotient will be a two-digit number, as shown by the following:

$$10 \times 4 = 40 \qquad 40 < 275$$
$$100 \times 4 = 400 \qquad 400 > 275$$

The number sentences on the right show that the quotient must be greater than 10 but less than 100. *The first step in division involving regrouping should always be to determine the number of digits in the quotient by multiplying the divisor by a power of 10.*

The second essential in division by the pyramid method is to find each digit in the quotient. The pupil may use either of two procedures. First, he may write the factors in the quotient from his knowledge of the corresponding example in multiplication. In the example $4\overline{)275}$, the pupil determines that the quotient must be a two-place number. He also knows that $4 \times 60 = 240$ and $4 \times 70 = 280$, hence he writes the factor 60 in the quotient.

Second, the pupil determines the factor of the quotient by using only the cardinal value of a digit in the dividend. In example (f), the pupil identifies the 7 in 72 as 7 tens and divides 7 tens into 3 groups of 2 tens, or 20, and writes 20 in the quotient. In (g) he identifies each digit in the dividend with its place-value name. There are 2 hundreds that cannot be divided as hundreds, hence there are 26 tens to be separated into 4 groups of 6 tens each. He then writes 6 tens, or 60, in the quotient. The remainder of the solutions in (f) and (g) are the same as illustrated in (e):

[3]Henry Van Engen and E. Glenadine Gibb, *General Mental Functions Associated with Division* (Cedar Falls, Iowa: Iowa State Teachers College, 1956), p. 181.

```
       20            60
f.  3)72    g.  4)275
       60           240
```

The second method may be the easier one for the pupil because he deals with smaller numbers. This procedure is based on the pupil's knowledge of the multiplication or division facts. He can use this pattern for finding a digit in the quotient when the divisor is a two-or more-place numeral.

Refining the pyramid method

After the pupil understands the procedure in (h), he may refine it by using a short cut, as illustrated in (i). He writes 4 in the quotient in tens' place and not 40, but he writes the product of 3 and 40 as the first partial dividend. He then completes the solution by writing 8 in ones' place in the quotient:

```
     48     i     4
      8        3)144
     40          120
  3)144           24
    120
     24
     24
```

Example (j) shows the next refinement in the division process. The procedure illustrated is usually known as the conventional algorism for division when regrouping is involved and the divisor is a one-digit number:

```
      48
j  3)144
     12
      24
      24
```

When the pupil progresses through successively higher levels of maturity in using the division process, he demonstrates growth in dealing with this operation. Examples (k–n) illustrate different levels of maturity in dealing with division by a one-place divisor.

```
                                46
                                 6
                                40
k. 6)276 |        l.  6)276
     120 | 20              240
     156 |                  36
     120 | 20               36
      36 |
      36 |  6
           46

     46                   46
n  6)276          n.  6)276
    240                24
     36                 36
     36                 36
```

The solution in (n) represents the conventional adult procedure for dividing by a one-place number. Slow learners may require a much longer time to progress from the level of operating successfully at (k) to the adult level of performance at (n) than other members of the class.

The solution given in (o) represents a higher degree of maturity than that in (n). In (o) the pupil writes only the quotient without writing any of the work involved in the solution. A limited number of pupils can divide effectively by using the short form in (o), and for this group the short procedure is effective. The teacher should not attempt to have the whole class master the procedure shown in (o):

```
       46
o  6)276
```

Dividends not a multiple of the divisor

In all the illustrations given so far, the dividend is a multiple of the divisor. The same algorism for division applies whether or not the dividend is a multiple of the divisor. In the illustration at the right, the quotient is 23, with a remainder of 2. The pupil should write a number sentence to show the relationship among divisor, divi-

```
     23 r 2
  4)94
    8
    14
    12
     2
```

dend, and quotient. The number sentence or equation for the given example is $94 = (4 \times 23) + 2$. If D = dividend, d = divisor, q = quotient, and r = remainder, then $D = (d \times q) + r$ if $r < d$.

The best check for division is to show that the number sentence for a given example is a true statement. The number sentence $94 = (4 \times 23) + 2$ is a true statement if 94 and $(4 \times 23) + 2$ are different names for the same number. The pupil shows that the equation is true by writing 23 in expanded form and then proceeds as follows:

$$4 \times 23 + 2 = 4 \times (20 + 3) + 2$$
$$4 \times (20 + 3) + 2 = (80 + 12) + 2 = 92 + 2$$
$$94 = 94$$

MULTIPLYING BY A TWO-DIGIT NUMBER

There are three different types of examples to be considered in multiplying by a two-digit number:

1. Both factors are multiples of 10
2. Only one factor is a multiple of 10
3. Neither factor is a multiple of 10.

Both factors multiples of 10

The teacher should introduce multiplication by a two-digit number with factors that are multiples of 10, for example, 20×30. The pupil should write each numeral as a factor of 10, as $20 = 2 \times 10$ and $30 = 3 \times 10$. The example may then be written as follows:

$$\begin{aligned} 20 \times 30 &= (2 \times 10) \times (3 \times 10) \\ &= (2 \times 3) \times (10 \times 10) \\ &= 6 \times 100, \text{ or } 600 \end{aligned}$$

The pupil learned that the way factors are arranged does not affect the product. This generalization is a consequence of the commutative and associative properties of multiplication.

One factor a multiple of 10

In the second type of example involving two-digit factors, only one of the factors is a multiple of 10, for example, 20×43. The pupil writes the factor 43 in expanded notation. The multiplication may then be performed as follows:

$20 \times 43 = 20 \times (40 + 3)$	Renaming 43
$= (20 \times 40)$ $+ (20 \times 3)$	Distributive property
$= 800 + 60$, or 860	Renaming $800 + 60$

Example (a) represents the conventional procedure for multiplying when the factors are written in the vertical form.

a
$$\begin{array}{r} 34 \\ \times\ 20 \\ \hline 680 \end{array}$$

Neither factor a multiple of 10

In the third type of example involving multiplication of two-digit factors, neither factor is a multiple of 10, for example, 12×34. The pupil writes one of the factors in expanded notation, as $12 = 10 + 2$. The example 12×34 may then be expressed as $(10 + 2) \times 34$, which is equivalent to $(10 \times 34) + (2 \times 34)$. He may find it easier to complete the computation when the numerals are written in vertical form, as in (b), than when they are written in horizontal form:

b
$$\begin{array}{r} 34 \\ \times\ 10 \\ \hline 340 \end{array} + \begin{array}{r} 34 \\ \times\ 2 \\ \hline 68 \end{array} = 408$$

The procedure shown in both vertical and horizontal forms illustrates the distributive property of multiplication over addition.

Example (c) shows the conventional procedure for multiplying 12 and 34. It is a combination of the two multiplications given in (b). The pupil should use the algorism shown in (c) to multiply

two-digit factors that are not multiples of 10. He may refine the work by not writing the terminal 0 in the second partial product, as in (d). The pupil does not use the short cut shown in (d) until he has discovered the function of the terminal 0 as a place holder in the second partial product:

```
c       34                 d.    34
      × 12                     × 12
        68 (2 × 34)              68
       340 (10 × 34)             34
       408                      408
```

Both factors of the example 12×34 may be written in expanded notation as $(10 + 2) \times (30 + 4)$. Many pupils at the fourth-grade level find it difficult to apply the distributive property in examples of this type. For that reason, the procedure shown in (e) should be used for enrichment purposes for the more able pupils.

e $(10 + 2) \times (30 + 4) = 10 \times (30 + 4)$
$\qquad + 2 \times (30 + 4)$
$= (10 \times 30) + (10 \times 4)$
$\qquad + (2 \times 30) + (2 \times 4)$
$= 300 + 40 + 60 + 8$
$= 408$

The pupil can verify the partial products given in (e) by using the vertical notation, as in (f). He can readily understand the procedure in (f) but not that in (e):

```
f       34
      × 12
         8 (2 × 4)
        60 (2 × 30)
        40 (10 × 4)
       300 (10 × 30)
       408
```

The sequence of steps for presenting multiplication by two-digit factors that are not multiples of 10 is as follows:

1. Have the class give the positional value of each digit in both factors.
2. Multiply, using the distributive property of multiplication, as shown in (b).
3. Have the class tell the sequence of steps, as in (c).
4. Have the class interchange the factors and multiply, as in (g). The factors may be interchanged because of the commutative property of multiplication.

```
g.      12
      × 34
        48
       360
       408
```

Checking multiplication

There are two acceptable ways of checking multiplication by a two-place number. One method is to go over the work; the other is to interchange the factors and then multiply. Both are satisfactory. It was demonstrated earlier that the pupil should discover different procedures for checking addition and subtraction. The same is true for multiplication and division. Two other procedures for checking multiplication by a two-place factor include (a) an application of the distributive property and (2) casting out nines.

A check for the example 23×35 consists in giving each digit its total value in a factor. Thus the example may be written as $(20 + 3) \times (30 + 5)$ to illustrate the distributive property, as $20 \times (30 + 5) + 3 \times (30 + 5)$. The pupil performs the indicated operations. The final answer provides a check of the product found in the conventional algorism.

Casting out nines

Casting out nines provides a useful check for multiplication. This form of checking is based on the *excess of nines* in the factors. The excess of nines in a number is the remainder resulting from dividing that number by 9. Thus, the excess of nines in 43 is 7. The excess of

nines in a number may be found by adding the digits in that numeral. The sum of the digits in 43 is 7. The sum of the digits in 45 is 9, which is a multiple of 9; therefore the excess of nines in 45 is 0. (See page 219 for a description of how to use the excess of nines to check multiplication and for an explanation of why the check works.)

The example below checks by casting out nines. The product of the excesses of nines in the factors is 35. The excess of nines in 35 is 8, which is the same as the excess of nines in the product of the two factors 32 and 547. Very probably the product, 17,504, is correct.

```
                    Excess
                    of 9's
    547   →           7
  × 32    →         × 5
  ------            ---
   1094              35
  1641                |
  ------              |
  17504               |
    ↓                 ↓
   [8]      =        [8]
```

The reader who is not familiar with this method of checking multiplication should check the products in the following examples.

```
a      76        b      409
     × 35             ×  26
     ----             ------
     2660             10,634

c     356        d      749
     × 47             ×  54
    ------            ------
    16,632            40,446
```

One of the products given is not correct. A check by casting out nines will detect this incorrect product.

Example (e) checks by casting out nines. However, the second partial product is misplaced. The 8 of this product should be in hundreds' place and not in tens' place, therefore the product of the example is not correct. The number 534 was multiplied by 23 and not by 203:

```
                        Excess
                        of 9's
e.     534    →           3
     × 203    →         × 5
     -----              ---
      1602               15
     1068                 |
     -----                |
     12282                |
       ↓                  ↓
      [6]       =        [6]
```

The difference between these numbers is 180 (203 − 23), which is a multiple of 9. If the order of the digits in a numeral is changed, the excess of nines will be the same. Thus, the excess of nines in 213, 312, or 123 is the same. Therefore if the product of two factors is 312 and the pupil writes 321, the example will check by casting out nines. This check will not reveal an error resulting from the misplacement of digits in a numeral, such as occurs when reversing the order of certain digits. Many left-handed pupils have a tendency to reverse digits in a numeral, as 736 for 763. Casting out nines will not reveal an error of this kind. If an answer does not check by casting out nines, the example is incorrect. If an answer checks by casting out nines, the example could contain an error resulting from misplacement of one or more digits in the solution. However errors of this kind are infrequent.

A three-place factor

The pupil who understands how to multiply by a two-place factor should readily understand how to deal with a three-place factor, such as 235. The numeral 235 may be expressed in expanded form as 200 + 30 + 5. To multiply in the examples at the right, multiply 413 by the total value of each digit. The product of 235 and 413 is the sum of the three partial products. Then combine the three examples, as in (d).

```
   413
 × 235
```

```
a      413      b      413
     ×   5           ×  30
      2065           12390

c.     413      d      413
     × 200           × 235
     82600            2065
                     12390
                     82600
                     97055
```

An example of the kind given at the right shows why the 0 of the multiplier should be treated as a place holder.

```
  423
× 204
```

The number 204 is equal to 200 + 4. The pupil multiplies by 4 ones and then by 2 hundreds. The solutions given in (a) and (c) are acceptable. Solution (b) is not recommended.

```
a      423     b      423     c      423
     × 204          × 204          × 204
      1692           1692           1692
     84600          8460            846
     86292          86292          86292
```

In (a) the zeros in the second partial product hold the empty places. In (c) the zeros are omitted. The pupil using this form should understand the work and have discovered that the first digit of a partial product is always written in the same column as the corresponding digit of the multiplier. This pupil uses a mature method and can operate efficiently without the use of the zeros to hold the empty places.

In (b) the pupil uses the terminal zero as a guide in the placement of the partial product. This procedure is meaningless and may result in serious error if the multiplier contains zeros in alternate positions, as in the numeral 20,304.

Rounded numbers in multiplication

The use of rounded factors provides a good check on whether a product is sensible. The pupil should round off each factor so that it contains only one *significant digit*. Each of the numerals 34, 476, and 705 contains only one significant digit when rounded off, as 30, 500, and 700, respectively.

The product in (a) is sensible because 2774 is near 2800. The answer in (b) is not sensible because the product must be greater than 210,000:

```
a      73        70
     × 38      × 40
      584      2800
     219
     2774

b     736          700
    × 304        × 300
     2944      210,000
    2208
    25024
```

When the factors are two-digit numbers that are not multiples of 10, the teacher should have the pupil round off each factor to the two nearest multiples of 10. The given factor will be between these two multiples. The two multiples of 10 that are nearest to 34 are 30 and 40.

The product of 34 and 76 must be between the products of 30 × 70 and 40 × 80. The product of the smaller multiples of 10 is the lower limit of the product of the two given factors. The product of the larger multiples of 10 is the upper limit of the given factors. In (c) the product of the factors is 2584, which is between 2100 and 3200, therefore the answer is sensible. The class does not check all products of two-place factors by this procedure. The teacher uses this method chiefly as a means of enriching the pupil's understanding of numbers.

```
                |80
c.     76  ↗    |70      40 × 80 = 3200
     × 34  ↘    |40      30 × 70 = 2100
      304       |30
     228
     2584
```

DIVIDING BY A TWO-DIGIT NUMBER

Grade placement of the topic

Division by a two-digit number is one of the most difficult topics in computational arithmetic. There is a great range of difficulty in division of this kind when the standard algorism is used, as in examples (a) and (b). In (a) the estimated quotient is the true quotient. It is not necessary to regroup in either multiplication or subtraction. In (b) the reverse is true. The estimated quotient is not the true quotient, and regrouping is needed in both multiplication and subtraction. Because of the wide range in difficulty, great care should be exercised in gradation of examples if the pupil is to use the algorism shown. The procedures in (a) and (b) represent adult usage of division by a two-place divisor:

```
        21             56 r 14
a   31)651    b    16)910
       62              80
       31              110
       31               96
                        14
```

The pupil should not begin division by a two-place divisor by using an adult procedure.

For ease in learning to divide by a two-place divisor, two points must be considered if only the standard algorism is presented. First, the topic must be spaced through two or more grades, and second, the examples must be selected carefully with respect to difficulty. If the pupil begins to divide by a two-place divisor by using the subtractive method or the pyramid method and then progresses to the adult pattern, the gradation of examples according to difficulty is not as important as when only the adult procedure is used. The pupil using the standard adult pattern in (c) estimates 9, but the true quotient is 7. He uses the subtractive method in (d). Now a variety of estimations are possible, so that he has little difficulty in finding the quotient:

```
        7 r 12
c   24)180        d   24)180    5
       168               120
        12                60
                          48    2
                          12    7 r 12
```

solution in (d) is easier than that in (c) because in (d) it is not necessary to estimate the true quotient digit, while in (c) this must be done. In a program using only the adult procedure, an example of the type 24)180 would be deferred until the pupil had mastered the process when the estimated quotient is the true quotient. The work in (d) shows that the pupil can solve the example when he understands the meaning of division as repeated subtraction.

Although the problems of grade placement of topics and of graduation of examples are not as vital to success in learning division in a modern program as in a conventional program, each factor deserves some consideration. The work in grade 4 involving division by a two-digit divisor should be confined to divisors that are multiples of 10, as in the example 20)80. Division by any two-place divisor can be completed in grade 5 by the pyramid method. Many pupils may not achieve mastery of the adult pattern for division by a two-place divisor until grade 6. At least the topic should be spaced through grades 4 and 5. By the time the pupil completes grade 5 he should know how to divide any number by a two-place divisor by the pyramid method. In grade 6 he should be able to refine this method so that it will be the same as the standard method. According to this plan, the topic will be spaced through grades 4, 5, and 6.

Methods of estimation

There are many methods of estimation of the quotient by a two-digit divisor. These methods may be grouped under the *one-rule method* or the *two-rule method*. According to the one-rule method, the divisor is rounded off downward to the nearest 10. The divisors in a decade, such as 21–29, would be rounded off to 20 for purposes of estimation.

If the two-rule method is used, the divisor may be rounded off both downward and upward. For the divisors 20–24, the divisor is rounded off as 20; for divisors 25–29, the divisor is rounded off upward as 30. This plan follows the usual pattern for rounding off whole numbers. In many arithmetic textbooks, for estimation purposes divisors ending in 5 are rounded off downward, and only divisors ending in 6, 7, 8, and 9 are rounded off upward.

The tens' digit of a two-place divisor is the *guide figure*. The guide figure may be given either its *cardinal value* or its *total value*. By the one-rule method, the guide figure for each of the divisors 31–39 may be given its cardinal value of 3 or its total value of 30.

There are many arguments for the relative merits of each procedure. Experimental evidence indicates that there is no significant difference in the results obtained with the two methods, and adult competence can be achieved using either method.[4]

It is possible to use a one-rule method and round off the divisor upward instead of downward. By the upward method for the divisors 21–29, the divisor is considered to be 30. Hartung demonstrated that this method will produce greater accuracy in estimation than when the divisor is rounded off downward.[5] Since it is rarely used, no further consideration will be given here to the upward method.

The modern mathematics program does not emphasize the method of estimation to the extent that the traditional program did. This is true because the modern program gives greater consideration to the inverse relationship between multiplication and division than the traditional program. The value of a given method of estimation of the quotient in a modern program has not been determined by experimentation.

Examples

There are three situations in division when the divisor is a two-digit number:

1. When the divisor is a multiple of 10
2. When the divisor is not a multiple of 10, but the estimated quotient is the true quotient
3. When the estimated quotient is not the true quotient.

The first situation is the easiest, and the teacher would use a divisor that is a multiple of 10 to introduce the topic to the class. The pupil should discover the relationship between the equations

$$\text{Factor} \times \text{factor} = \text{product}$$
$$\text{Product} \div \text{factor} = \text{factor}$$

to enable him to deal with examples in this classification.

The teacher is confronted with the problem of what materials to employ and to what extent they should be used. The use of objective materials should be limited in dealing with a two-place

[4]A detailed appraisal of the two methods is given in Foster E. Grossnickle and Leo J. Brueckner, *Discovering Meanings in Arithmetic* (New York: Holt, Rinehart and Winston, Inc., 1959), pp. 220–224.

[5]M. L. Hartung, "Estimating the Quotient in Division," *The Arithmetic Teacher*, April 1957, 4:100–111.

divisor. Such materials are valuable in helping the pupil to discover the meaning of division, and they should be employed for this purpose with a one-digit divisor. Their use is not effective in enabling the pupil to estimate the quotient.

Divisor a multiple of 10

The teacher can introduce division by a multiple of 10 by using the following problem:

How many bundles of 20 tickets each can be made with 60 tickets?

Most of the class can give the answer to this problem without writing the numerals. The teacher should use this easy problem to enable the pupil to discover a pattern for finding the quotient.

First, determine the number of places there will be in the quotient by multiplying the divisor by a power of 10:

$$1 \cdot 20 = 20 \qquad 20 < 60$$
$$10 \cdot 20 = 200 \qquad 200 > 60$$

The number sentences on the right show that the quotient is greater than 1 but less than 10, hence the quotient is a one-place number.

Second, have the pupil discover the relationship between multiplication and division:

$$2 \cdot 20 = 40 \qquad 20\overline{)40} = 2\overline{)4}$$
$$3 \cdot 20 = 60 \qquad 20\overline{)60} = 2\overline{)6}$$

The teacher may employ a number ray to have the pupil find the number of twenties in 60 (Fig. 11.3) or use addition or subtraction to find that number. Regardless of the means used, the pupil should discover that he can always find the answer by either of the following procedures:

1. Make a table using the divisor as one factor and the digits in order beginning with 1 as the second factor. Continue the table until a product is the given dividend or until the dividend is between two consecutive products.

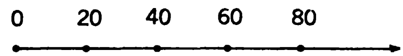

Figure 11.3

2. Divide the number of tens of the dividend by the number of tens in the divisor for a one-place quotient.

Both procedures are correct, but the use of a table is time consuming and inefficient. The pupil should be able to apply his knowledge of the multiplication or division facts to estimate the quotient. The second plan uses this knowledge. The point may be raised that the pupil will not understand the mathematical basis of this procedure. This criticism may be true for teachers who assume that a pupil must understand every phase of any computational procedure. All pupils may not understand that both dividend and divisor are multiplied by $\frac{1}{10}$ or divided by 10. The teacher can give a few illustrations of the following type to enable the class to see that the quotient remains the same when divisor and dividend are divided by 10.

a	$40 \div 10 = 4$	c	$60 \div 20 = 3$
	$4 \div 1 = 4$		$6 \div 2 = 3$
b.	$80 \div 20 = 4$	d	$90 \div 30 = 3$
	$8 \div 2 = 4$		$9 \div 3 = 3$

Examples (a–d) illustrate the use of the identity element of 1. The example $20\overline{)60}$ may be expressed as

$$\frac{60}{20} = \frac{6 \cdot 10}{2 \cdot 10}$$

If we multiply each set of factors by $\frac{1}{10}$, the example becomes

$$\frac{6 \cdot 1}{2 \cdot 1} = \frac{6}{2} \text{ or } 2\overline{)6}$$

Even if all pupils do not discover the application of the identity element for multiplication in the illustrations, the pupil should divide the number of tens in the dividend by the number of tens in the divisor to find a one-place quo-

tient. The pupil should understand the meaning of division, the sequence of steps in the solution, and how to determine the number of places in the quotient. The procedure for estimating a quotient digit does not affect the fundamental meaning pertaining to division. Therefore it is relatively immaterial to meaningful learning whether or not the pupil understands this time-saving procedure for estimating the quotient.

The next situation involves a three-place dividend, as $20\overline{)120}$. The pupil follows the pattern described. First, he discovers that the quotient is a one-place numeral by multiplying 20 by a power of 10. Second, he thinks, "$12 \div 2 = 6$."

The third situation includes those examples in which the dividend is not a multiple of the divisor, such as $20\overline{)42}$, $20\overline{)70}$, $20\overline{)123}$, and $20\overline{)137}$. The pupil follows the same pattern for estimation of the quotient in the examples of this type as he used when the dividend was a multiple of the divisor:

$$\begin{array}{r} 2\text{ r }14 \\ 20\overline{)54} \\ \underline{40} \\ 14 \end{array} \qquad 54 = (2 \times 20) + 14$$

The illustration shows the pattern for the written work. It is important to have the pupil write the corresponding number sentence in multiplication, as shown.

Divisor not a multiple of 10—estimated quotient true quotient

The pupil should apply the same pattern for dealing with divisor of the type 21–29 when the estimated quotient is the true quotient as he applies when the divisor is a multiple of 10. For the initial treatment of divisors that are not multiples of 10, the examples should be limited to one-place quotients. In the example $21\overline{)63}$, the thought pattern for estimation would be as follows: "Very probably the number of twenty-ones in 63 is the same as the number of twos in 6."

There is a great range of difficulty in examples in division in which the estimated quotient is the true quotient and the divisor is not a multiple of 10:

a.	$21\overline{)63}$	Two-place dividend a multiple of divisor
b	$21\overline{)126}$	Three-place dividend a multiple of divisor
c.	$21\overline{)65}$	Remainder in quotient
d	$21\overline{)75}$	Guide number not a factor of number of tens
e	$24\overline{)72}$	Regrouping in multiplication
f	$23\overline{)74}$	Regrouping in subtraction
g	$34\overline{)142}$	Combination of e and f
h.	$34\overline{)325}$	Apparent quotient seems to be 10 or more

In examples of the last type, the estimated quotient seems to be 10 or more. In this case the pupil always multiplies the divisor mentally by 10 and compares this product with the number to be divided. Since the dividend is not 10 times the divisor, he estimates the quotient to be 9. Of course, 9 is not the true quotient in all estimations of this kind, as shown by the example $24\overline{)213}$. Examples of the latter type represent a very difficult phase of division.

In all of these examples the estimated quotient is the true quotient. The pupil may not always estimate correctly, hence he should have a means of finding the quotient. He should also be able to use the subtraction method, as shown, or the pyramid method:

$$\begin{array}{rl} 24\overline{)155} & 5 \\ \underline{120} & \\ 35 & 1 \\ \underline{24} & \underline{} \\ 11 & 6\text{ r }11 \end{array}$$

The two methods are similar and are effective when the pupil is unable to estimate the true quotient.

Two-or-more-place quotients

After the pupil becomes skillful in finding a one-place quotient, he should be able to find a quotient with two or more digits. Regardless of the number of digits in the quotient, the procedure is the same. The steps are as follows:

1. Determine the number of digits in the quotient by multiplying the divisor by a power of 10.
2. Estimate each quotient digit.
3. Give each digit in the quotient its total value.
4. Write the number sentence for each example.

In (a) the pupil determines the number of places in the quotient as follows:

$10 \times 21 = 210 \qquad 210 < 672$

$100 \times 21 = 2100 \qquad 2100 > 672$

```
a     3
      2
     30
21)672
   630
    42
    42
```

Check: $672 = 21 \times 32$

The two number sentences containing the inequalities show that the quotient is between 10 and 100, hence it is a two-digit number.

The pupil estimates the quotient by rounding off 21 as 20 and then thinks $6 \div 2 = 3$. Since the quotient is a two-place number the 3 is 3 tens, or 30, which he writes. He uses the pyramid method. Example (a) gives the complete solution.

The final step consists in writing the number sentence. The teacher may have the pupil express the quotient in expanded notation. The number sentence would then be as follows:

$672 = 21 \times (30 + 2)$	Expanded notation for 32
$= (21 \times 30) + (21 \times 2)$	Distributive property
$= 630 + 42$, or 672	Renaming numbers
$672 = 672$	

The illustration shows that the number sentence is true, and therefore the solution is correct.

Estimated quotient not true quotient

Examples (a–c) show that the estimated quotient is not always the true quotient.

```
a 24)1776    80      c 26)1776    50
     1920                 1300
                           476    10
                           260
b 24)1776    70            216     7
     1680                  182
       96     4             34     1
       96                   26
             74              8    68 r 8
```

$1776 = 24 \times 74$

The first step in division is to determine the number of places in the quotient numeral. Since 1776 is between 240 and 2400, the quotient is a two-digit number. In (a), the pupil rounded off 24 as 20 and estimated the first quotient digit as 8, which is 8 tens, or 80. Since the product of 80 and 24 is greater than the dividend, the estimated quotient is too large. When the divisor is rounded off downward and the estimation is not correct, the estimated number is always too large.

In (b), the pupil rewrote the example and used 7 tens as the trial quotient. The quotient is 70 + 4, or 74. The number sentence for the example is $1776 = 24 \times 74$.

In (c), the divisor 26 may be rounded off either downward or upward. Because of ease of solution, the 26 should be rounded off upward as 30. The first estimated digit is then 5, which represents

5 tens, or 50. The remainder is greater than the divisor, hence another estimation is made. The quotient digit is 1 ten, or 10. Since the remainder of 216 is less than 10 times the divisor, the next quotient number will be a one-digit number. The estimated quotient is 7 ($3\overline{)21}$), but the remainder 34 is greater than the divisor, so 34 is treated as a new partial dividend. The quotient is the sum of 50, 10, 7, and 1, or 68, with a remainder of 8. The number sentence for the example is $1776 = (26 \times 68) + 8$.

In (c), the first and third estimates were incorrect, but it was not necessary to erase the product of divisor and quotient because each remainder is considered to be a new partial dividend. If the divisor is rounded off upward and the estimated quotient is not the true quotient, the estimated quotient is always too small. The correction can then be made by including an extra estimate, as in (c). The quotient may be written at the right of the dividend, as in (b) and (c) (the subtractive method), or above the dividend as in the pyramid method.

Since the pupil should always multiply the divisor by a power of 10 to determine the number of places in the quotient, he knows the product that is 10 times the divisor. Half this product is 5 times the divisor. He should write the product of 5 times the divisor as a guideline or a supplementary aid in estimation. Examples (d) and (e) show how to use such guidelines.

d	$10 \cdot 24 = 240$	
	$5 \cdot 24 = 120$	
	$24\overline{)1776}$	60
	1440	
	336	10
	240	
	96	4
	96	
		74

e	$10 \cdot 26 = 260$	
	$5 \cdot 26 = 130$	
	$26\overline{)1776}$	60
	1560	
	216	7
	182	
	34	1
	26	
	8	68

In (d), the pupil estimates 8. Since $5 \times 24 = 120$, very probably the pupil can see that 8 would be too large, so he tries 6. He can then be certain that 10 is the next estimated quotient. In (e), the pupil estimates 5 tens as the first quotient digit. Since $5 \times 26 = 130$, he sees that the quotient would be at least 6 tens, which is the true quotient. The other two estimations also give the true quotient.

The illustrations show that the following sequence of steps should be followed in finding the quotient by a two-place divisor:

1. Multiply the divisor by a power of 10 to determine the number of places in the quotient.
2. Write 5 times the divisor as a guideline in estimation.
3. For purposes of estimation, round off the divisor in the same way that any whole number is rounded off. Thus, round off 21–24 as 20; round off 25–29 as 30.
4. If the remainder is larger than the divisor, treat this remainder as a new partial dividend.
5. Record the quotient to the right of the dividend or above it, preferably above it. This procedure illustrates the pyramid method of division.
6. Write a number sentence to illustrate the example.
7. The best check for division is to show that the number sentence in (6) is true.

Refining the method of division

The pattern for refining the introductory method of division by a one-place divisor applies for a two-place divisor. The pupil begins with the subtractive method and changes very quickly to the pyramid method. He refines the pyramid method which he uses in grades 4 and 5 for a two-place divisor. At the grade 6 level he uses the conventional algorism for division. Ex-

amples (f–i) illustrate different levels of maturity in dealing with a two-place divisor.

```
f.  10 × 36 = 360          g.  10 × 36 = 360
     5 × 36 = 180               5 × 36 = 180
                                        54
                                         4
                                        50
   36)1944   50                 36)1944
      1800                         1800
       144    4                     144
       144                          144
             54

   1944 ÷ 36 = 54              1944 ÷ 36 = 54

h.  10 × 36 = 360          i.
     5 × 36 = 180
        54                           54
   36)1944                      36)1944
      1800                         180
       144                          144
       144                          144

   1944 ÷ 36 = 54              1944 ÷ 36 = 54
```

Examples (f) and (g) illustrate the subtractive and pyramid methods of subtraction, respectively. In (h), the pupil estimates 5 and writes 5 in tens' place in the quotient. He multiplies the divisor by 50 and not 5. In (i), he gives each quotient digit its cardinal value, but he writes each digit in its correct place. At the adult level, as in (i), the pupil mentally computes the product of divisor and the estimated digit to be sure that the quotient digit is correct. It is then of little significance whether he uses a one- or a two-rule procedure to estimate a quotient digit.

Checking division

One of the best ways to check division is to show that the number sentence for an example is true. The number sentence for the example given is $895 = (34 \times 26) + 11$. The pupil verifies the statement by showing that $(34 \times 26) + 11$ and 895 are different numerals for the same number. He can write one of the factors in expanded form, as follows, to show that the statement is true:

```
 26 r 11
34)895
```

$$\begin{aligned} 895 &= (34 \times 26) + 11 \\ &= 34 \times (20 + 6) + 11 \\ &= 680 + 204 + 11 \\ 895 &= 884 + 11 = 895 \end{aligned}$$

The pupil must understand the function of a check, why it is used, and the relationship between the inverse operations involved. Checking that is done in a mechanical, meaningless manner should be eliminated from the mathematics program.

Casting out nines in division

It has already been recommended that the more able pupils check multiplication by casting out nines. This check also should be used by the same group for division. Casting out nines in multiplication applies to division, provided the final remainder is 0. Then the divisor is a factor of the dividend. If the divisor is not a factor of the dividend, as in the example at the right, subtract the final remainder from the dividend. The difference between these two numbers is a multiple of the divisor. In the given example, subtract 28 from 912. The difference is 884, which is a multiple of 34. Then the example, $34 \times 26 = 884$, is checked by casting out nines as any other example in multiplication is checked.

```
 26 r 28
34)912
```

Enrichment in multiplication

An effective exercise for enriching the multiplication program consists in having the pupils rank examples according to the size of the product in examples of the following type:

a.	57	57	57	57	57
	× 6	× 12	× 22		

II	$\begin{array}{r}36\\ \times 15\\ \hline\end{array}$	$\begin{array}{r}71\\ \times 15\\ \hline\end{array}$	$\begin{array}{r}14\\ \times 15\\ \hline\end{array}$	$\begin{array}{r}83\\ \times 15\\ \hline\end{array}$	$\begin{array}{r}27\\ \times 15\\ \hline\end{array}$

The teacher has the pupil try to rank the examples by inspection before multiplying. If he cannot discover the pattern to enable him to rank the examples, he can multiply and then rank them. The teacher should challenge the very superior pupil by having him write a generalization that applies to each set or to both sets. A concise generalization pertaining to the two sets is as follows: If one of the two factors in different examples is the same but the second factor is different, the smaller (or larger) the unequal factor, the smaller (or larger) will be the product.

The work with whole numbers should not be limited to finding the value of a variable in an equation. The pupil should be able to solve for the variable in an inequality of the type $3 \times 12 > 3 \times n$. Any whole number less than 12 will make the sentence true. In order to limit the number of answers that will make the sentence true, the pupil should supply the largest or smallest whole number that the variable may represent. Thus, the greatest whole number that may replace n in the mathematical sentence $3 \times 12 > 3 \times n$ is 11. Similarly, the smallest whole number that may replace n in the mathematical sentence $4 \times 15 < 4 \times n$ is 16. If the variable in a mathematical sentence is an element of the larger member of the inequality, the pupil should find the smallest number that may replace that variable. In the reverse situation, the pupil should find the largest number that may replace the variable.

Enriching work in division

The plan described for enriching the mathematics program in multiplication can be followed in division. The pupil ranks examples of the type that follow according to the value of the missing numerals. He then checks his rankings by solving the examples. In case he cannot discover from inspection the pattern to use to rank the examples, he solves the examples and then ranks them. The following sets of examples show how divisor, quotient, and dividend may vary.

I	$\begin{array}{r}8\\ ?\overline{)144}\end{array}$	$\begin{array}{r}48\\ ?\overline{)144}\end{array}$	$\begin{array}{r}2\\ ?\overline{)144}\end{array}$	$\begin{array}{r}16\\ ?\overline{)144}\end{array}$
II.	$\begin{array}{r}24\\ ?\overline{)576}\end{array}$	$\begin{array}{r}24\\ ?\overline{)144}\end{array}$	$\begin{array}{r}24\\ ?\overline{)312}\end{array}$	$\begin{array}{r}24\\ ?\overline{)720}\end{array}$

In (I), the dividend is constant. The divisor and quotient vary. If the product of two factors is a constant, an increase in one factor causes a corresponding decrease in the other factor. If the divisor is multiplied or divided by a number, the quotient is divided or multiplied by that number. The superior pupil in grade 6 should be able to express verbally the relationship between the factors as illustrated in (I).

In (II), one factor is constant, but the other factor and the product vary. As the given factor increases or decreases, there is a corresponding increase or decrease in the dividend. The examples in (II) above correspond to the examples in (I) and (II) in multiplication on page 201 and at the top of this page.

RELATIONSHIPS BETWEEN MULTIPLICATION AND DIVISION

A pupil's depth of understanding of multiplication and division depends upon his ability to identify points of difference between these operations. The teacher should have the class point out these differences in a pair of examples. Table 11.1 lists the differences between multiplication and division in the set of whole numbers.

TABLE 11.1

Comparison of Multiplication and Division

Multiplication	*Division*
1. Multiplication is a shortened form of addition of equal addends.	1. Division is a shortened form of repeated subtractions.
2. Usually we operate from right to left on a numeral.	2. Usually we operate from left to right on a numeral.
3. Two factors are given to find the product.	3. The product and one of two factors are given to find the second factor.
4. Zero times any number is zero.	4. Zero may not be used as a divisor.
5. One is the only number multiplied by itself that has a product of 1.	5. Every number (except 0) divided by itself has a quotient of 1.
6. If a counting number *a* is multiplied by a counting *b*, increasing *b* increases the product.	6. If a counting number *a* is divided by a counting number *b*, increasing *b* decreases the quotient.
7. Multiplication is commutative.	7. Division is not commutative.
8. Multiplication is associative.	8. Division is not associative.
9. Both left and right hand distributive properties apply to multiplication.	9. Only the right-hand distributive property applies to division.
10. The identity element for multiplication is 1.	10. There is no identity element for division.
11. The set of whole numbers is closed with respect to multiplication.	11. The set of whole numbers is not closed with respect to division.

EXERCISES

1. List at least four different ways of finding the product in the example 6×58.
2. Show how the distributive property may be applied in the example $4\overline{)72}$.
3. Evaluate the use of short and long division for a one-place divisor.
4. Illustrate four different levels of maturity in solving the example $7\overline{)5104}$ (see p. 190).
5. Express the factors 34×57 in expanded notation and multiply by applying the distributive property.
6. Check the example $26 \times 38 = 988$ by casting out nines.
7. The example in problem 6 will check by casting out nines if the digits in the product are rearranged, as 898 or 889. Why is this true?
8. Use rounded numbers to give the upper and lower limits of the product of 36×53.
9. Some teachers do not present any method of estimating the quotient. The pupil multiplies the divisor by the digits, beginning with 1. Evaluate this plan.
10. Find the quotient in the example $38\overline{)2485}$. Check the solution by casting out nines.
11. Refer to sets (A) and (B). Give all the examples that use the members of each set.

    A: $\{12, 15, 180\}$ B: $\{912, 15, 57\}$
12. Enumerate some of the procedures a teacher should use to help a slow learner succeed in division with a two-place divisor as well as ways to challenge the quick learners.

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PRIMES, COMPOSITES, AND INTEGERS

Chapters 7–11 considered the set of whole numbers, $\{0, 1, 2, 3, 4, 5, 6, \cdots\}$. In the logical development of the number system, the whole or cardinal numbers are followed by the integers, $\{\cdots, -3, -2, -1, 0, +1, +2, +3, \cdots\}$.

This chapter deals with the set of integers and some important subsets of the integers. The discussion will include the following topics: multiples of a number, prime numbers; composite numbers and common multiples, signed numbers, modular arithmetic.

MULTIPLES OF A NUMBER

The ability to recognize important characteristics of a number is useful to the pupil in his study of mathematics. Identification of the *multiples* of a number demonstrates recognition of an important relationship. The number 35 is a multiple of both 5 and 7 because $35 = 5 \times 7$. If $z = xy$, then the integer z is a multiple of both x and y.[1] The number zero is a multiple of every whole number. Since 45 is a multiple of 9, it follows that 9 is a factor of 45. If z is a multiple of x, then x is a factor of z (divisor of z) provided z is not equal to 0. The three stages in identifying the

[1]Every integer is a multiple of itself and the opposite of itself and the numbers -1 and $+1$. Thus -15 is a multiple of 5 and -5, 3 and -3, $+1$ and -1, $+15$ and -15.

distinctive characteristics of number at the elementary level are:

1. Recognition of patterns
2. Formulation of a rule
3. Informal mathematical proof.

The following set of graded activities illustrates the manner in which many pupils might learn about multiples of 3 in the stages indicated.

a. The sequence 3-6-9-12 is written on the chalkboard. Pupils are asked to supply additional numbers in the same pattern. Most pupils in grade 1 can perform this type of activity without knowing multiplication facts if they are given a reasonable amount of practice.

b. The pupil should become familiar with the name of this sequence by hearing the teacher refer to it by its name. When the pupil knows the multiplication facts, he should be able to write the sequence when it is called for by name.

c. As work progresses, the pupils should be able to formulate rules or descriptions of the properties of the sequence. One such property is that each member of the sequence is divisible by 3. A less obvious rule is that the sum of the digits of a numeral representing a multiple of 3 is also a multiple of 3 (or is divisible by 3).

d. An informal proof can be constructed for the fact that the sum of the digits in a numeral representing a multiple of 3 is divisible by 3. The first step in developing this proof involves the recognition of multiples of 3 in less obvious forms such as:

$$4 \times 6 \qquad 5 \times 9 \qquad 7 \times 66 \qquad 4 \times 99$$

Alert pupils will observe that a product is a multiple of 3 if at least one factor is a multiple of 3.

The following are also multiples of 3:

$$6 + 9 \qquad 2 \times 9 + 24 \qquad 4 \times 99 + 3 \times 9$$

The distributive property should help the pupil to recognize that the sum of two multiples of 3 is also a multiple of 3.

The final stage of the "proof" involves the renaming of numbers, such as 432 and 487, in the following manner:

$$\begin{aligned} 432 &= 4 \times 100 + 3 \times 10 + 2 \\ &= 4(99 + 1) + 3(9 + 1) + 2 \\ &= (4 \times 99 + 3 \times 9) + (4 + 3 + 2) \\ 487 &= 4 \times 100 + 8 \times 10 + 7 \\ &= 4(99 + 1) + 8(9 + 1) + 7 \\ &= (4 \times 99 + 8 \times 9) + (4 + 8 + 7) \end{aligned}$$

These and similar examples should aid the learner in recognizing that if N is a whole number, it can be renamed in the following manner: $N =$ a multiple of 3 + the sum of the digits.

It then becomes a consequence of the distributive property that if the sum of the digits is divisible by 3, the number N is divisible by 3. It is a valuable and important skill in mathematics to be able to recognize consequences of mathematical properties.

A formal proof would use the same fundamental ideas given above but would involve a degree of generality and attention to detail not appropriate at this level.

It is important to recognize that patterns can often be used as effective guides before verbal rules are given. Some of the typical patterns that can be used effectively at the early elementary level are:

0, 1, 2, 3, 4, 5, · · ·	Whole numbers
1, 3, 5, 7, 9, · · ·	Odd numbers
0, 2, 4, 6, 8, 10, · · ·	Even numbers
0, 5, 10, 15, 20, · · ·	Multiples of 5 as well as multiples of other whole numbers

RULES OF DIVISIBILITY

The following rules for divisibility may also be discovered and "proved" by many elementary school pupils:

1. A whole number is divisible by 2

(or is a multiple of 2) if its numeral ends in 0, 2, 4, 6, or 8. It is acceptable classroom language for all but the most fastidious to say that a number is divisible by 2 if it ends in 0, 2, 4, 6, or 8. Over precise language may be more confusing than helpful (see p. 75).

The "proof" first requires that the learner recognize that a number is divisible by 2 if its numeral ends in 0. Such a whole number has a factor of 10 and 10 has a factor of 2. It is then a consequence of the distributive property that numbers with numerals ending in 0, 2, 4, 6, 8 are divisible by 2 because

$$22 = 20 + 2 \qquad 34 = 30 + 4$$
$$56 = 50 + 6 \qquad 138 = 130 + 8$$

2. A whole number greater than 100 is divisible by 4 if the number represented by the last two digits in its numeral is divisible by 4. For example, 12,348 is divisible by 4 because 48 is divisible by 4. On the other hand, 3455 is not divisible by 4 because 55 is not divisible by 4. The proof follows that for divisibility by 2, numbers such as 12,348 are renamed as 12,300 + 48. A number represented by a numeral ending in two zeros has a factor of 100 and is therefore divisible by 4. The distributive property requires that a number of the form 12,300 + n be divisible by 4 if n is divisible by 4.

A whole number greater than 1000 is divisible by 8 if the number represented by the last 3 digits in its numeral is divisible by 8. For example, the number 125,888 is divisible by 8 because 888 is divisible by 8. The "proof" is again similar to that for divisibility by 2 and 4. The number 125,888 may be rename as 125,000 + 888. The number 125,000 (and any other whole number represented by a numeral ending in three zeros) has a factor of 1000 and is therefore divisible by 8. As before, the distributive property then requires that every number of the form 125,000 + n be divisible by 8 if n is divisible by 8.

A whole number represented by a numeral ending in 0 or 5 is divisible by 5. A whole number represented by a numeral ending in 0 has a factor of 10 and is therefore divisible by 5. A number such as 325 may be renamed as 320 + 5 and therefore must be divisible by 5 because of the distributive property.

A whole number is divisible by 9 if it is represented by a numeral that has the sum of its digits divisible by 9. The "proof" is similar to that for divisibility by 3.

The number 486 may be renamed as follows:

$$\begin{aligned} 486 &= 4 \times 100 + 8 \times 10 + 6 \\ &= 4 \times (99 + 1) + 8 \times (9 + 1) + 6 \\ &= \underbrace{(4 \times 99 + 8 \times 9)}_{\text{multiple of 9}} + \underbrace{(4 + 8 + 6)}_{\text{sum of digits}} \end{aligned}$$

In a similar manner, any whole number may be renamed as the sum of a multiple of 9 and sum of the digits. The distributive property then insures that if the sum of the digits is divisible by 9 the number must be divisible by 9.

PRIME NUMBERS

Every natural number greater than 1 is either a *prime number* or a *composite* number. The number 1 is called a *unit* and is neither prime nor composite. A prime number has exactly two unequal natural number factors, the number itself and 1. Every prime number is divisible only by itself and 1. The number 2 is the first prime number and is the only even prime number. Every even number greater than 2 must be a *composite number.* A composite number is a nonprime natural number greater than 2. A composite number is

divisible by a natural number other than itself and 1. The number 4 is the first composite number because it is divisible by 2 (other than itself and 1). The number 9 is the first odd composite number.

Pupils may ask why the number 1 is not a prime number. The number 1 is not a prime number by definition. Such an answer will not usually satisfy many pupils. A more satisfactory answer may be given by showing that the number 6 will not factor into prime factors in only one way (uniquely) if the number 1 is a prime number.

$6 = 2 \times 3 \qquad 6 = 2 \times 3 \times 1 \qquad 6 = 2 \times 3 \times 1 \times 1$

If the number 1 is not a prime number, the number 6 factors into prime factors uniquely except for the order in which the factors are written.

The *fundamental theorem of arithmetic* states that every composite number factors uniquely into a product of prime factors except for the order in which they are written. It is sometimes agreed to write the factors in the order of their size, as follows:

$$6 = 2 \times 3$$
$$12 = 2 \times 2 \times 3, \text{ or } 2^2 \times 3$$
$$300 = 2 \times 2 \times 3 \times 5 \times 5, \text{ or } 2^2 \times 3 \times 5^2$$

When this convention is followed, prime factorizations can be performed in only one way.

The following activities may be useful in introducing prime numbers and in helping pupils discover some of the important properties of these numbers.

1. The sequence-pattern approach may be useful in introducing prime numbers. Have the pupil practice with some familiar sequences in which he supplies additional numbers in the same pattern (see p. 206). Now write the following sequence on the chalkboard: 2-3-5-7-11-13. Ask the pupils to give additional numbers. Some hints may be given, for example, that there is no fixed rule to get from one number to the next but that all the numbers in the set have a common property. Several brief sessions of this nature will provide excellent readiness for the introduction of prime numbers.

2. The renaming technique is also useful in introducing prime numbers. After a brief session of renaming numbers, restrict the renaming to the use of multiplication. This restriction still allows the renaming of 3 as $6 \times \frac{1}{2}$. Finally, the renaming should be restricted to the use of multiplication and the set of whole numbers greater than 1. With this restriction, 15 can be renamed as 3×5 or 5×3. After the class has renamed several composite numbers in this fashion, introduce a prime number. A prime number cannot be renamed under these restrictions.

3. One of the most well-known activities for determining the set of prime numbers is the sieve of Eratosthenes. Construct a table similar to Table 12.1.

TABLE 12.1

Determining the Set of Prime Numbers

1	2	3	~~4~~	5	~~6~~
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

The number 1 is eliminated by definition. Circle the numeral 2 (the first prime) and cross out every second numeral following 2. This procedure eliminates all multiples of 2 (even numbers) greater than 2. Now move to the numeral 3 (the next numeral that is not crossed out). Cross out every third numeral following 3. This eliminates all

multiples of 3 that are not even numbers. The multiples of 3 that are even were eliminated in the first step. Now proceed to 5 (the next numeral not crossed out). Cross out every fifth numeral following 5. This eliminates all multiples of 5 that are not even (except 5). Now move to 7 (the next numeral not crossed out. Cross out every seventh numeral after 7. By proceeding in this manner, all composite numbers are eliminated and only primes are left.

There is an advantage to using six columns, as in Table 12.1. All the primes then fall in the first or fifth column. Numbers named in the first column are one more than a multiple of 6 and numbers named in the fifth column are one less than a multiple of 6. It is not difficult to prove that all prime numbers are one more or less than a multiple of 6 (except for 2 or 3). All whole numbers greater than 5 can be expressed in the form $6n$, $6n + 1$, $6n + 2$, $6n + 3$, $6n + 4$, or $6n + 5$ All of these except $6n + 1$ and $6n + 5$ have a factor of 2 or 3 and cannot be prime. Thus all primes must be in the form $6n + 1$ (one more than a multiple of 6) or $6n + 5$ (one less than a multiple of 6).

4. A well-known conjecture in mathematics (the Goldbach conjecture) is that all even composite numbers may be written as the sum of two prime numbers. This theorem has never been proved or disproved and is therefore still an open question mathematically. The evidence collected over many years indicates that the guess is probably correct, but a statement about an infinite set cannot be proved by a finite number of examples.

It is a worthwhile activity to have pupils rename even numbers as the sum of prime numbers. Some numbers can be renamed as the sum of two primes in more than one way.

$$8 = 3 + 5$$
$$10 = 3 + 7 = 5 + 5$$
$$16 = 3 + 13 = 5 + 11$$

Since every even number greater than 6 can be written as the sum of two even composite numbers, such numbers can be written as the sum of four primes if the conjecture is correct.

$$8 = 4 + 4 = (2 + 2) + (2 + 2)$$
$$26 - 12 + 14 = (5 + 7) + (3 + 11)$$

5. Help more able pupils to recognize that there can never be more than four primes in any *decade*.[2] A number greater than 9 with a numeral ending in 0, 2, 4, 5, 6, or 8 cannot be a prime, since it has a factor of 2 or 5. Have the pupils look for sequences of consecutive composite numbers, as 24, 25, 26, 27, and 28. Also have pupils look for *twin primes,* as 11 and 13 or 17 and 19. There is one set of triple primes, 3, 5, and 7, and there is only one set of consecutive primes, 2 and 3.

COMPOSITE NUMBERS AND COMMON MULTIPLES

The set of natural numbers can be partitioned into the set of prime numbers, the set of composite numbers, and the set containing the number 1. The set of composite numbers is infinite, since the set of even numbers greater than 2 is infinite. Euclid proved in a famous mathematical theorem that the set of primes is also infinite.

Separating composite numbers into prime factors is an important mathematical activity. The tree diagram provides a graphic illustration of the process of decomposing composite numbers into products of prime factors. Figure

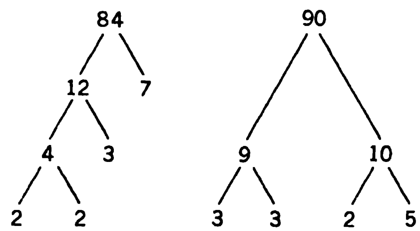
[2]A decade is a set of 10 consecutive whole numbers beginning with a number (represented by a numeral) ending in 0. {70, 72, · · ·, 79} and {140, 142, · · ·, 149} are examples of decades.

12.1 illustrates two ways in which the number 12 may be written as the product of prime factors.

$12 = 3 \times 2 \times 2$ $\qquad$ $12 = 2 \times 2 \times 3$

Figure 12.1

When using the tree diagram, be certain that the pupil uses the diagram to rename the original number. Figure 12.2 illustrates how 84 and 90 may be renamed as the product of prime factors with the help of a tree diagram.



$84 = 2 \times 2 \times 3 \times 7$ $\qquad$ $90 = 3 \times 3 \times 2 \times 5$

Figure 12.2

A more efficient method of renaming a number as the product of prime factors is repeated division by prime factors, as illustrated below:

$$2\overset{42}{\overline{)84}} \qquad 2\overset{21}{\overline{)42}} \qquad 3\overset{7}{\overline{)21}} \qquad 84 = 2 \times 2 \times 3 \times 7$$

When the pupil is familiar with the short method of division, the divisions can be recorded in a more compact arrangement, as shown at the right. This method is probably most efficient when each division is performed by the smallest prime number possible.

$$\begin{array}{r} 2\underline{)84} \\ 2\underline{)42} \\ 3\underline{)21} \\ 7 \end{array}$$

The number 48 is a multiple of both 6 and 8 (by definition). The number 48 is called a *common multiple* of 6 and 8 because it is divisible by 6 and 8. A common multiple of two or more numbers is divisible by each of the numbers. The number 48 is not the *lowest common multiple* of 6 and 8 because a smaller number, 24, is also a common multiple of 6 and 8. The lowest common multiple of a set of numbers is the smallest whole number divisible by each number in the set. The *lowest common denominator* of a set of denominators is the lowest common multiple of the denominators in the set. The following activities are useful in helping pupils understand the concept of the lowest common multiple:

1. Have the class write the multiples of 6. Ask if 6 is a multiple of 8; if 12 is a multiple of 8; if 18 is a multiple of 8; if 24 is a multiple of 8. In this manner the class should discover that 24 is the smallest multiple of 6 that is also a multiple of 8. Now have the class work with multiples of 8 in the same manner and discover that 24 is the smallest multiple of 8 that is also a multiple of 6. Repeat this activity with other pairs of numbers, some without common factors.

2. Use the numbers 6, 10, and 15. Have the class write the set of multiples for each number as follows:

$\{6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, \cdots\}$
$\{10, 20, 30, 40, 50, 60, 70, \cdots\}$
$\{15, 30, 45, 60, 75, \cdots\}$

The number 60 can then be recognized as the lowest common multiple. The class should then discover that it is not necessary to write all three sets of multiples. Repetition of this activity with other sets of numbers should enable the pupils to discover that the lowest common multiple can be obtained most quickly by examining the multiples of the largest number in the

set and choosing the smallest such multiple divisible by the other numbers. To determine the lowest common multiple of 8, 12, and 18, examine the multiples of 18 and discover that 72 is the smallest such multiple divisible by 8 and 12. Therefore, 72 is the lowest common multiple of 8, 12, and 18. This method for determining the lowest common multiple is efficient for most sets of numbers normally encountered by the elementary school pupil but is not efficient for larger numbers. When larger numbers are involved, the most efficient method probably is to use prime factors, as outlined in (4) below.

3. Write the following pairs of numbers on the chalkboard. Also write the product of each pair, as shown.

6, 8	48
5, 7	35
4, 5	20
9, 15	135
8, 9	72

This activity should help the pupil to recognize that the product of two numbers is always a multiple of the numbers (by definition) but is not always the lowest common multiple. The class should then try to predict when the product of two numbers is the lowest common multiple of the numbers. Construction of a table similar to Table 12.2 may be helpful in this regard.

The class should then discover that the product of two numbers is the lowest common multiple of the numbers only if the two numbers have no common natural number factor other than 1. An alert pupil may notice that if the product of the two numbers is divided by the highest common factor of the two numbers, the quotient obtained is the lowest common multiple.

4. Have the class find the lowest common multiple of 10 and 15 by examining multiples of 15. Now have the pupil rename 10 and 15 as products of prime factors. Next have him tell how the correct answer of 30 can be obtained from 2×5 and 3×5. In this case, the correct answer is obtained by multiplying the three different prime factors, 2, 3, and 5. This activity should be repeated with other pairs of numbers in which no prime factor occurs more than once. Now choose the pair 15 and 20 and determine that the lowest common multiple is 60. Rename these two numbers as products of prime factors, $2 \times 2 \times 5$ and 3×5. Have the pupil tell how the lowest common multiple can be determined from these prime factorizations. The answer is $2 \times 2 \times 3 \times 5$. The prime factor 2 is used as a factor twice because 2 occurs twice as a factor in 20. Now examine 20 and 30 and let the pupil discover that the lowest common

TABLE 12.2
Finding the Lowest Common Multiple

Number Pair	*Numbers Renamed as Products of Primes*	*Product*	*LCM*
6, 8	2×3, $2 \times 2 \times 2$	48	24
5, 7	5, 7	35	35
4, 5	2×2, 5	20	20
9, 15	3×3, 3×5	135	45
8, 9	$2 \times 2 \times 2$, 3×3	72	72
18, 60	$2 \times 3 \times 3$, $2 \times 2 \times 3 \times 5$	1080	180

multiple is still 60. The prime factor 2 is again used twice because it occurs no more than twice in either factor. Similar activities should be performed with sets of two and three numbers. The goal is to have the pupil discover that the lowest common multiple of a set of numbers is the product of the different prime factors of the numbers in the set. Each factor is used the greatest number of times that it occurs in any one number.

5. It may provide some variety and appeal to some pupils to examine the multiples of different sets of numbers by means of a set diagram. Figure 12.3 illustrates how the multiples of 6 and 8 are related. The set intersection of the set of multiples of 6 and the set of multiples of 8 is the set of common multiples of 6 and 8. Only multiples less than 50 are shown in the diagram.

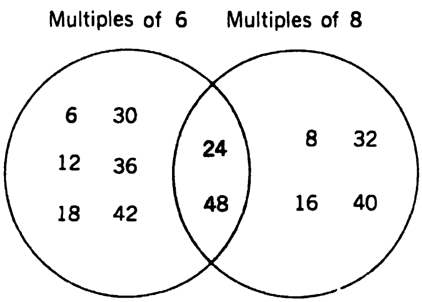


Figure 12.3

Figure 12.4 illustrates how the multiples of 4, 6, and 9 are related. The intersection of these three sets is the set of common multiples of 4, 6, and 9. Only multiples less than 40 are shown in the diagram.

6. Find the common multiples by using the concept of set intersection. Ask the pupils to write the set of multiples of 8 (less than 80) and the set of multiples of 12 (less than 80). Now ask them to write the intersection of these two sets. Help them discover that this intersection is the set of common multiples and that the smallest number in the set is the lowest common multiple.

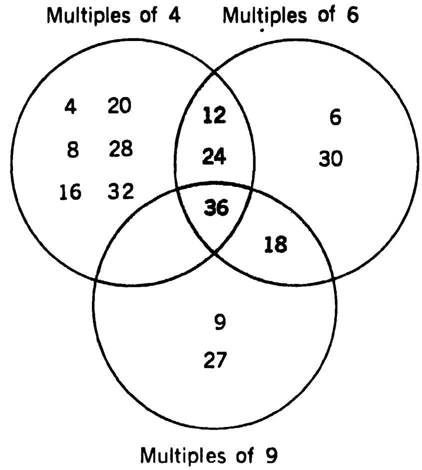


Figure 12.4

The use of prime factors in determining the lowest common multiple is fundamental in arithmetic and algebra and should not be neglected.

The following generalizations are important:

1. Every number has itself and 1 as a divisor.

2. Every prime number has only itself and 1 as a natural number divisor.

3. The product of two prime numbers is always the lowest common multiple of the numbers.

4. The product of two numbers is the lowest common multiple of the two numbers only if the numbers have no common natural number factor (other than 1).

5. A product is divisible by a number n if at least one of its factors is divisible by n.

6. A composite number has at least one factor different than itself and 1.

8. The lowest common multiple of a set of numbers is the product of the different prime factors of the numbers. If a prime factor occurs more than once, use this factor the greatest number of times it occurs in any one number.

9. The set of common multiples of x and y is the intersection of the set of multiples of x and the set of multiples of y.

SIGNED NUMBERS

The number line is one of the best means to enable a pupil to recognize whether a new symbol represents a new number or is another name for a familiar number. When the symbol $\frac{1}{2}$ is assigned to a point that previously had no name, the symbol is a name for a new number. When the symbol -1 is assigned to a point to the left of 0 that previously had no name, the symbol names a new number. When the symbol $+2$ is assigned to the point that also has the name 2, then $+2$ is a new name for a familiar number. Such numbers as -2 or $+3$ are called *signed* (or directed) *numbers*. Signed numbers and 0 constitute the set of integers.

The pupil is familiar with signed numbers from expressing temperature readings on a thermometer. He is familiar with such statements as "below zero," "above or below sea level," and similar phrases. He may not be familiar with the notation for signed numbers. In grades 5 and 6 the pupil should learn how to interpret signed numbers on a number line. The teacher now uses a *number line* and not a *number ray* to introduce signed numbers.

The pupil is familiar with the number ray for representing 0 and numbers greater than 0. Now the number line is used for representing numbers less than 0 (on the left) as well as numbers greater than 0 (on the right).

One plan of action is to place a number line on the chalkboard in which points on both sides of the point for 0 are named with unsigned numerals, as in Figure 12.5. The class should tell why the first point to the left of 0 cannot be named as indicated. Two different points cannot be named by the same number. Hence different names must be given to points to the left of 0 than to those to the right of zero. If the class is asked for suggestions, some interesting answers may result. The teacher then tells the class how to label the point as $^{-}1$ and to read the number as "negative one" and not as "minus one." The term "minus" designates the operation of subtraction. The term "negative" indicates the sign of direction opposite to positive (not plus). In the same way, the symbol $^{+}5$ should be read as "positive five" and not "plus five." The term "plus" is used to designate the operation of addition, while the term "positive" is used to indicate the sign of direction opposite to negative.

5 4 3 2 1 0 1 2 3 4 5

Figure 12.5

The teacher then has each pupil draw a number line and name the first five points on either side of the 0 point, as in Figure 12.6. The number positive 5 is frequently written as $^{+}5$ in early work with signed numbers. In a similar manner, the number negative 6 is written as $^{-}6$. Raising the $+$ and $-$ signs is useful in helping the pupil distinguish between the use of the signs for indicating addition and subtraction, as in $4 + 5$ and $4 - 2$, and their use as a sign of direction, as in $^{+}5$ and $^{-}3$.

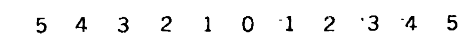


Figure 12.6

After the pupil understands how to name and read signed numbers, he performs the operations of addition and subtraction with the aid of the number

line in a manner similar to that for performing addition and subtraction with whole numbers. The number line provides a visual aid for introducing addition and subtraction of signed numbers just as sets of concrete objects provide visual aids in early work with whole numbers.

The teacher should review addition and subtraction of whole numbers on the number line. Addition of $^+3$ and $^+2$ can be performed as indicated in Figure 12.7. The pupil identifies $^+3$ on the number line and uses this point as the initial point for an arrow of length 2 pointing to the right. The tip of the arrow shows that the sum is $^+5$.

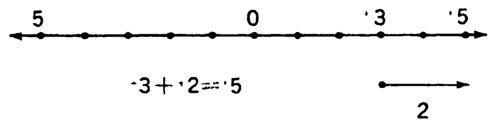


Figure 12.7

To add $^+4$ to $^-2$, place the initial point of an arrow of length 4 at $^-2$ and point the arrow to the right. The point of the arrow indicates that $^-2 + {}^+4 = {}^+2$ (see Fig. 12.8).

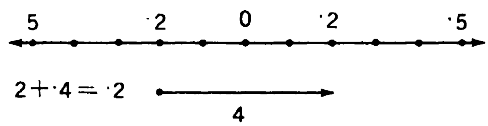


Figure 12.8

Add $^+2$ to $^-3$ by placing the initial point of an arrow of length 2 at $^-3$ and pointing the arrow to the right. The tip of the arrow indicates that the sum is $^-1$ (see Fig. 12.9).

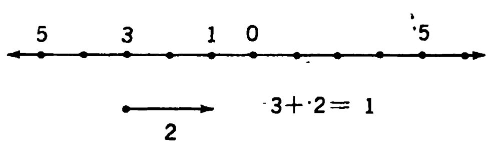


Figure 12.9

In working with whole numbers, arrows point right for addition and left for subtraction. By adding only positive numbers in the previous examples, the arrows always point to the right. In introducing subtraction examples, it is important to subtract only positive numbers, so that arrows always point left in order that the transfer from whole numbers to signed numbers is accomplished with a minimum of confusion.

Subtract $^+4$ from $^+2$ by placing the initial point of an arrow of length 4 at $^+2$ and pointing the arrow to the left. The tip of the arrow indicates that the difference is $^-2$ (see Fig. 12.10).

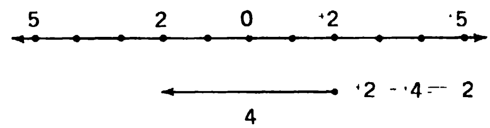


Figure 12.10

Subtract $^+2$ from $^-3$ by placing the initial point of an arrow of length 2 at $^-3$ and pointing the arrow to the left. The tip of the arrow indicates that the difference is $^-5$.

In early work with signed numbers, every numeral should be written with its sign of direction indicated. After the brief introductory period, $^+4$ is usually written as 4. Somewhat later, possibly not until junior high school, $^-4$ will be written as -4.

To add a positive number, as $^+2$, on the number line, move 2 units to the right. To add $^-2$, move 2 units to the left. To add $^-5$ to $^+3$, place the initial point of the arrow of length 5 at $^+3$ and point the arrow to the left. The tip of the arrow indicates that the sum is $^-2$ (see Fig. 12.11).

Have the class indicate which way to point the arrow for subtracting $^-4$ if the arrow is pointed to the left for subtracting $^+4$. To subtract $^-4$ from $^+2$, place the

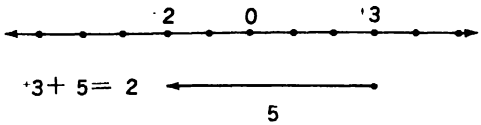


Figure 12.11

initial point of an arrow of length 4 at $^{+}2$ and point the arrow to the right. The tip of the arrow indicates that the difference is $^{+}6$ (see Fig. 12.12).

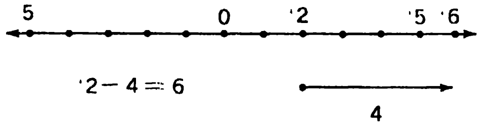


Figure 12.12

The class may accept the above subtraction problem more readily if the addition subtraction-pattern is reviewed at this time, as indicated below (see p. 119).

$$\begin{array}{ll} 2 + 5 = 7 & {}^{-}6 + {}^{+}4 = {}^{-}2 \\ 5 + 2 = 7 & {}^{+}4 + {}^{-}6 = {}^{-}2 \\ 7 - 5 = 2 & {}^{-}2 - {}^{-}6 = {}^{+}4 \\ 7 - 2 = 5 & {}^{-}2 - {}^{+}4 = {}^{-}6 \end{array}$$

The above example shows that the addition-subtraction pattern applies to signed numbers (integers) as well as to whole numbers. The pupil should learn to apply this pattern to sentences involving signed numbers, as follows:

$$\begin{array}{ll} \square + {}^{-}2 = 6 & 3 - \triangle = 5 \\ {}^{-}2 + \square = 6 & 3 - 5 = \triangle \\ 6 - \square = {}^{-}2 & \triangle + 5 = 3 \\ 6 - {}^{-}2 = \square & 5 + \triangle = 3 \end{array}$$

When a pupil recognizes that the equation $6 - {}^{-}2 = \square$ can be rewritten as $\square + {}^{-}2 = 6$, he can use the number line to determine that if one addend is $^{-}2$ and the sum is 6, then the other addend must be 8 (see Fig. 12.13).

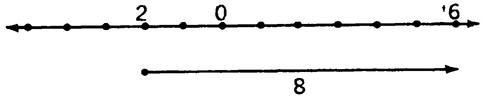


Figure 12.13

A second number line may be used in place of the arrows shown in previous illustrations. The 0 point on the second number line is used as the initial point of the arrow. Positive numbers will be used in place of an arrow pointing to the right and negative numbers in place of an arrow going to the left. To add $^{+}5$ to $^{-}3$, place the 0 of the lower number line in Figure 12.14 at $^{-}3$ of the upper number line and read the sum of $^{+}2$ on the upper line above $^{+}5$ on the lower.

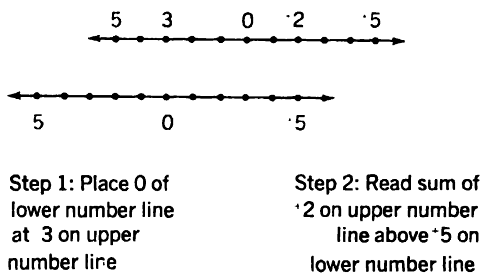


Figure 12.14

It may be a useful activity to ask the pupil to interpret a situation on the number line of Figure 12.15 with an equation. The pupil should discover that both $^{+}2 + {}^{+}3 = {}^{+}5$ and $^{+}2 - {}^{-}3 = {}^{+}5$ apply. Such activities may help the pupil understand that subtracting $^{-}3$

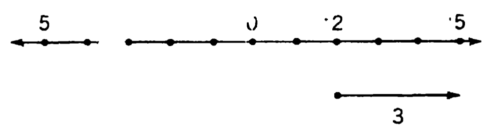


Figure 12.15

gives the same answer as adding $^{+}3$, a basic principle in dealing with signed numbers. The following patterns are related to this situation:

$$\begin{array}{ll} 1 + {}^{-}1 = 0 & \frac{1}{2} \times 2 = 1 \\ 2 + {}^{-}2 = 0 & \frac{1}{3} \times 3 = 1 \\ 3 + {}^{-}3 = 0 & \frac{1}{4} \times 4 = 1 \end{array}$$

Both of the above patterns show pairs of inverse numbers. The pairs on the left are *additive inverses* because their

sum is 0 (the identity element for addition). The pairs on the right are *multiplicative inverses* because their product is 1 (the identity element for multiplication). The point to be made here is shown by the similarity in the following statements:

To divide by a number (as $\frac{1}{2}$), multiply by its multiplicative inverse (2).

To subtract a number (as $^-2$), add its additive inverse (2).

These two statements help to illustrate the similarity in structure between subtraction and division.

Elementary pupils usually add and subtract only by using the number line, but the teacher should be alert for opportunities to lay a sound foundation for the understanding of the concept of pairs of inverse numbers, as:

$2 + 6 =$ [illegible]	[illegible] $\times 6 =$ [illegible]
$2 + (2 + 4) =$ [illegible]	[illegible] $\times (2 \times 3) =$ [illegible]
$(2 + 2) + 4 =$ [illegible]	$(\frac{1}{2} \times 2) \times 3 =$ [illegible]
$0 + 4 =$ [illegible]	$1 \times 3 =$ [illegible]
$4 =$ [illegible]	$3 =$ [illegible]

MODULAR ARITHMETIC

Modular, or clock, arithmetic may be introduced by taking a segment of the number line illustrated in Figure 12.16 and bending it to make the circular number line shown in Figure 12.17.

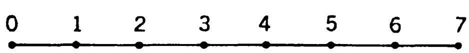


Figure 12.16

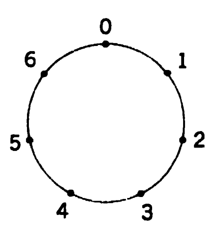


Figure 12.17

It is important to recognize that both 0 and 7 are now associated with the same point and therefore represent the same number. In beginning work of this kind, it is probably best to use 0 as illustrated in Figure 12.17.

The curved number line is now used to perform basic operations in a manner similar to that used with a straight number line. Figure 12.18 illustrates how these basic operations can be performed.

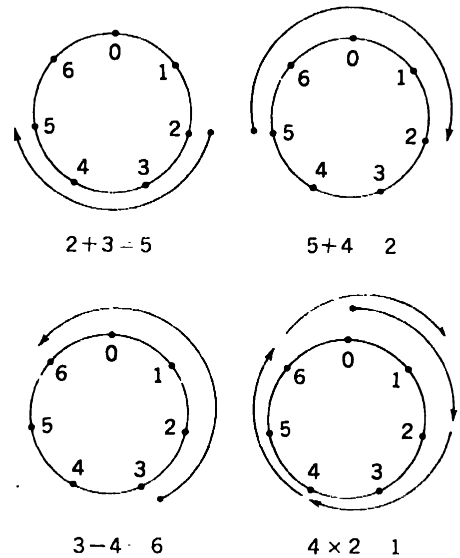


Figure 12.18

Tables of basic facts can be constructed as illustrated in Table 12.3.

The set with seven numbers {0, 1, 2, 3, 4, 5, 6} is usually referred to as the set of *integers mod* 7, often abbreviated *mod* 7.

The two tables of Table 12.3 list the facts for addition and multiplication and are used as the tables of facts in ordinary arithmetic.

Enter the addition table at 3 on the left and continue across on this line to the column under 5 and find that $3 + 5 = 1$.

TABLE 12.3

Addition Facts

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Multiplication Facts

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Enter at 5 on the left and continue across to the column under 3 and find that $5 + 3 = 1$. In this manner it can be shown that addition is commutative in this system.

Enter at 4 on the left and continue across on this line until an entry of 1 is reached. Read 4 at the top of this column and conclude that $1 - 4 = 4$ or $4 + 4 = 1$.

Enter at 5 on the left and move across until an entry of 0 is reached. Read 2 at the top of the column and recognize that $5 + 2 = 0$ or that 2 and 5 form a pair of additive inverses.

The system is closed with respect to addition and multiplication because the only entries occurring in both tables are members of the set $\{0, 1, 2, 3, 4, 5, 6\}$. Therefore, when addition or multiplication is performed on any two members of the set, the result is always a member of the set.

In a similar manner, enter the multiplication table at 4 on the left and move across to the column under 2 and determine that $4 \times 2 = 1$. This also indicates that 4 and 2 form a pair of multiplicative inverses.

Enter the multiplication table at 2 on the left and move across to the column under 5 and determine that $2 \times 5 = 3$. Now enter the table at 3 on the left and move across to the column under 6 and read 4, which shows that $(2 \times 5) \times 6 = 4$. In a similar manner, show that $2 \times (5 \times 6) = 4$.

Enter at 5 on the left and move across until an entry of 1 is reached. Read 3 at the top of this column to show that 5 and 3 are multiplicative inverses.

If the circular number line has nine points, a set of nine members called mod 9 results. The circle may be used with n points if n is a whole number, giving the set of integers mod n. The term "modular arithmetic" is used to refer to one or more of these finite sets with one or more operations. The study of modular arithmetic is helpful because the finite sets have many properties in common with the sets of numbers used in arithmetic. These properties can be proved by testing all possibilities because the sets are finite. It is usually sufficient, however, to try enough cases to convince the class that the properties do apply and in this manner to provide a review of these properties. Important properties that can be verified are:

Closure for addition and multiplication,

The associative property for addition and multiplication,

The commutative property for addition and multiplication,

The distributive property of multiplication over addition,

The existence of an identity element for both addition and multiplication (0 for addition and 1 for multiplication),

The fact that each number in a modular system has an inverse for addition.

The fact that each number in a system with a prime modulus (as in mod 7 or mod 11) has an inverse for multiplication (except for 0).

The above summary indicates that a modular system with a prime modulus forms a field (see p. 80). A study of modular arithmetic should help reinforce important concepts that apply in ordinary arithmetic. One example involves the recognition that 0 and 7 represent the same point and therefore the same number in mod 7. A consequence of this is that 9 and 2 must also name the same number: $9 = 7 + 2 = 0 + 2 = 2$ (in mod 7). This procedure can be applied to any integer:

$$37 = 5 \times 7 + 2 = 5 \times 0 + 2 = 0 + 2 = 2 \pmod 7$$

By proceeding in the above manner, the pupil should be able to discover the rule for replacing any integer by its equal number in the basic set $\{0, 1, 2, 3, 4, 5, 6\}$. This rule is to divide the integer by 7 and to use only the remainder. The rule explains why some people refer to modular arithmetic as remainder arithmetic. By this rule, $75 = 5 \pmod 7$, $36 = 1 \pmod 7$, and so on.

The inverse property for both addition and multiplication can be reinforced. The following equations indicate the existance of inverses for addition and multiplication (except for 0 in multiplication) in mod 7:

$0 + 0 = 0$	$1 \times 1 = 1$
$1 + 6 = 0$	$2 \times 4 = 1$
$2 + 5 = 0$	$3 \times 5 = 1$
$3 + 4 = 0$	$6 \times 6 = 1$

The addition-subtraction pattern holds in modular arithmetic as in ordinary arithmetic. The following illustrate corresponding patterns in modular arithmetic and ordinary arithmetic for addition and subtraction as well as for multiplication and division:

$4 + 5 = 9$	$4 + 5 = 2$	(mod 7)
$5 + 4 = 9$	$5 + 4 = 2$	(mod 7)
$9 - 4 = 5$	$2 - 4 = 5$	(mod 7)
$9 - 5 = 4$	$2 - 5 = 4$	(mod 7)
$3 \times 4 = 12$	$3 \times 4 = 5$	(mod 7)
$4 \times 3 = 12$	$4 \times 3 = 5$	(mod 7)
$12 \div 3 = 4$	$5 \div 3 = 4$	(mod 7)
$12 \div 4 = 3$	$5 \div 4 = 3$	(mod 7)

By comparing the equations $2 - 4 = 5$ and $2 + 3 = 5$ (both in mod 7), it can be recognized that subtracting 4 gives the same result as adding 3. The number 3 is the additive inverse of 4 in mod 7 and 4 is the additive inverse of 3 because $4 + 3 = 3 + 4 = 0$. Hence, modular arithmetic follows the same pattern as ordinary arithmetic where subtraction of $^-3$ is the same as adding 3. The number $^-3$ is the inverse of 3 because $^-3 + 3 = 0$ (see p. 215).

The similarity between the operations of addition and subtraction and those of multiplication and division can be shown by examining the two equations $5 \div 3 = 4$ and $5 \times 5 = 4$. This pair of equations demonstrates that division by 3 in mod 7 is the same as multiplying by 5. The number 5 is the *multiplicative inverse* of 3 because $3 \times 5 = 1$ in mod 7.

The calendar

The calendar is probably the simplest application of the integers mod 7. By

assigning numbers to the days of the week, as illustrated, the following problems may be solved.

Sunday	0
Monday	1
Tuesday	2
Wednesday	3
Thursday	4
Friday	5
Saturday	6

Problem: What day occurs 10 days after a Wednesday?

Solution: The number for Wednesday in the table above is 3. Add 10 to 3 and obtain a sum of 13, which is equal to 6 in mod 7. Therefore, 10 days later than a Wednesday is a Saturday.

Problem: What day is 365 days after Saturday?

Solution: The number for Saturday is 6. Add 6 to 365 and obtain a sum of 371, which is equal to 0 in mod 7. Therefore, 365 days later than a Saturday is a Sunday. Since $365 = 1$ in mod 7, there will always be an advance of one day for a given date in consecutive years when no leap year is involved. Thus, if March 7 falls on Tuesday in a given year, it will fall on Wednesday the next year if no leap year is involved.

Casting out nines

The check of casting out nines is an application of mod 9. The numerals 9 and 0 represent the same number (0) in mod 9 in the same way that 0 and 7 represent the same number in mod 7. Hence, $10 = 9 + 1 = 0 + 1 = 1$ in mod 9. Also, $10^2 = 10 \times 10 = 1 \times 1 = 1$ in mod 9. Therefore, in mod 9, $10^n = 1$. A consequence of the fact that $10^n = 1$ in mod 9 is as follows:

$$\begin{aligned} 321 &= 3 \times 10^2 + 2 \times 10 + 1 \\ &= 3 \times 1 + 2 \times 1 + 1 = 3 + 2 + 1 \quad \text{(mod 9)} \\ 34 &= 3 \times 10 + 4 \\ &= 3 \times 1 + 4 = 3 + 4 \quad \text{(mod 9)} \end{aligned}$$

The given examples illustrate that to rename a number in mod 9 in terms of the basic set {0, 1, 2, 3, 4, 5, 6, 7, 8}, it is sufficient to add the digits in the numeral. Hence, in mod 9, $321 = 3 + 2 + 1$, or 6. This may be verified by dividing 321 by 9 and using the remainder. In a similar manner, $483 = 4 + 8 + 3 = 15$ in mod 9, but $15 = 5 + 1$, or 6. Therefore, $483 = 6 \pmod 9$. In the case of 45, where the sum of 4 and 5 is 9, the numeral 0 is used rather than 9. Here is a check by casting out nines:

$$\begin{array}{r} 321 \\ \times\ 34 \\ \hline 1284 \\ 963 \\ \hline 10914 \end{array} - (6) \qquad \begin{array}{r} 6 \\ \times\ 7 \\ \hline 42 \end{array} - (6)$$

In mod 9, 321 and 6 name the same number, as do 34 and 7. Therefore, the product of 321×34 and 6×7 must be the same in mod 9. The fact that the product of two numbers is always independent of the numerals used to represent the numbers is the basis for the check by casting out nines.

It must be recognized that the check by casting out nines is a check and not a proof. An incorrect answer that differs from the correct answer by a multiple of 9 will check, for example:

$$\begin{array}{r} 134 \\ \times\ 201 \\ \hline 134 \\ 268 \\ \hline 2814 \end{array} - (6) \qquad \begin{array}{r} 8 \\ \times\ 3 \\ \hline 24 \end{array} - (6)$$

Casting out nines indicates that the above answer is correct even though it is wrong because the difference between 26,934 (the correct answer) and 2814 (incorrect answer) is 24,120, a multiple of 9.

In summary, the properties of closure, associativity, and commutativity apply to both addition and multiplication of integers. Zero is the identity ele-

TABLE 12.4

Comparison of Integers and Whole Numbers

Properties	*Integers*	*Whole Numbers*
Closure for addition	Yes	Yes
Closure for multiplication	Yes	Yes
Closure for subtraction	Yes	No
Closure for division	No	No
Associative for addition and multiplication	Yes	Yes
Commutative for addition and multiplication	Yes	Yes
Identity for addition	Yes (0)	Yes (0)
Identity for multiplication	Yes (1)	Yes (1)
Additive inverse	Yes	No
Multiplicative inverse	No	No
Distributive (multiplication over addition)	Yes	Yes

ment for addition and 1 is the identity element for multiplication. The distributive property holds for multiplication over addition. Every integer has an inverse number for addition (additive inverse). No integer has an inverse for multiplication (multiplicative inverse) except the number 1. Table 12.4 compares the set of integers and the set of whole numbers:

The reader should recognize that the existence of inverse elements for an operation implies closure for the inverse (opposite) operation.

EXERCISES

1. Write the set of multiples of 7 that are less than 50.
2. Write the intersection of the set of multiples of 10 less than 61 and the set of multiples of 12 less than 61. Describe this set verbally.
3. Why is it desirable not to include 1 in the set of prime numbers?
4. Find two sets of five consecutive composite numbers greater than 30 and less than 80.
5. Write the set of prime numbers greater than 30 and less than 50.
6. Find the lowest common multiple of each of the following sets of numbers by examining multiples of one of the numbers in the set:

   A: $\{6, 8, 9\}$ C: $\{9, 10, 12, 15\}$
   B: $\{10, 15, 18\}$ D: $\{12, 16, 18, 24\}$
7. Determine the lowest common multiple of the sets in problem 6 by first finding the prime factors of the numbers in each set.
8. Make addition and multiplication tables for the integers mod 3 and determine the pairs of inverse numbers for addition and multiplication.
9. Use the information from problem 8 to change the subtraction $1 - 2$ into an addition problem. In place of subtracting 2, add the inverse of 2 for addition.

10. Use the information from problem 8 to change the division $1 \div 2$ into a multiplication problem. In place of dividing by 2, multiply by the inverse of 2 with respect to multiplication.
11. Use a circular number line (clock) to verify the results in problems 9 and 10.
12. Use a number line to perform the following operations:

    a. $3 + 7$ c. $5 - {}^{+}3$
    b. ${}^{-}3 - {}^{-}3$ d. $4 + 5$

13. Use a double number line (slide rule) to perform the operation indicated in problem 12.
14. Use the addition and multiplication tables given for mod 5 below to complete the following:
    a. 3×4
    b. $3 + 4$
    c. $2 \times (3 \times 4)$
    d. $(2 \times 3) \times 4$
    e. $2 + (3 + 4)$
    f. $(2 + 3) + 4$
    g. Find the additive inverse of 3 in mod 5.
    h. Find the multiplicative inverse of 2 in mod 5.
    i. In Table 12.5 is mod 5 closed for addition? multiplication? division?

TABLE 12.5

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

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Swain, R. L., and E. D. Nichols, *Understanding Arithmetic*. New York: Holt, Rinehart and Winston, Inc., 1965. Chapter 10.

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ADDITION AND SUBTRACTION OF RATIONAL NUMBERS

Neither a carpenter using a ruler graduated in inches nor a merchant using scales marked in pounds would find his measuring instrument satisfactory. For greater accuracy the carpenter would want the inch divided into halves, quarters, or eighths, and the merchant would want the pound similarly divided. Such needs gave rise to a new type of number that may be expressed as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and the like.

Chapter 11 indicated that the set of whole numbers is not closed with respect to division. It is not possible to divide 3 by 4 and obtain a quotient that is a whole number. The number system therefore had to be expanded to meet two problems; namely, to increase accuracy in measurement and to make it possible to divide any two whole numbers. The inclusion of fractional numbers in the number system lent increased accuracy to measurement and made it possible to have closure in division except when the divisor is 0.

This chapter deals with fractional numbers and includes the following topics: rational numbers; instructional

TABLE 13.1

Properties of Rational Numbers

Properties	*Addition*	*Multiplication*
Closure	Yes	Yes
Associative	Yes	Yes
Commutative	Yes	Yes
Identity element	Yes (0)	Yes (1)
Existence of inverse element for each member of set	Yes	Yes (except for 0)
Distributive (multiplication over addition)		Yes

aids; operations with rational numbers; concepts conveyed by fractional numbers; addition of fractional numbers; subtraction of fractional numbers.

RATIONAL NUMBERS

A *rational number* is the quotient of two integers in which the divisor is not 0. In this chapter rational numbers are referred to as fractional numbers. The numerals that represent rational numbers are designated fractional numerals. The rational number $\frac{1}{4}$ can be obtained by many quotients, for example, $3 \div 4$, $6 \div 8$, $9 \div 12$, and so on. A rational number may be indicated by a single quotient. The rational numeral $\frac{6}{8}$ indicates one of the infinite number of quotients ($6 \div 8$) that determines the rational number $\frac{3}{4}$. The rational numeral $\frac{3}{4}$ has a *numerator* of 3 and a *denominator* of 4, while the rational number $\frac{3}{4}$ does not have a numerator or a denominator. In traditional arithmetic the word "fraction" sometimes referred to a fractional number and at other times to a fractional numeral. It was up to the reader to determine which usage was intended by the context of the sentence. Sentences similar to the following occurred frequently in traditional arithmetic:

1. Add the first fraction to the second.
2. Multiply the numerator and denominator of the fraction by 2.

In the first sentence, the word "fraction" refers to a fractional number. Only numbers (not numerals) are added. In the second sentence, "fraction" refers to a fractional numeral. Only numerals (not numbers) have a numerator and denominator. Some educators believe that confusion resulted from this dual use of the word "fraction," and there is still no uniform interpretation of the word. Some programs use the word "fraction" to represent a numeral, while others use it to represent a number.

Properties of fractional numbers

The system of fractional (rational) numbers with the operations of addition and multiplication form a number field (see p. 80). This fact is summarized in Table 13.1.

Whole numbers do not have *inverses* with respect to addition and multiplication.[1] Each rational number, however, has an inverse in these operations except for 0 in multiplication. For every fractional number, such as $\frac{a}{b}$, there is

[1]Except for 0 and 1. $0 + 0 = 0$, therefore 0 is its own additive inverse. $1 \times 1 = 1$; therefore 1 is its own multiplicative inverse.

another number, for example, $\frac{b}{a}$ if a and $b \neq 0$. The product of these two numbers is equal to 1. Each number is the multiplicative inverse of the other. Two numbers that have a product of 1 are also called *reciprocals*. The inverse or reciprocal is a key concept in division of rational numbers, as demonstrated in Chapter 14.

INSTRUCTIONAL AIDS

Both pupil and teacher should have supplementary materials to use in dealing with fractions. Each pupil should have a *fractions kit* consisting of a set of disks and fractional cutouts for modeling fractional numbers. The cutouts should be made from disks about 5 inches in diameter. The fractional cutouts should correspond to the fractional cutouts the teacher uses for class demonstrations. (See Appendix for directions for making the cutouts.) A 6-inch manila envelope is convenient for storing the cutouts.

Instructional aids for the teacher should include a *flannel board*, a *number ray*, and a set of *fraction charts*. The teacher's kit should include a set of disks and cutouts about 10 inches in diameter to be shown on the flannel board. The disks should be covered with flannel or some other material that has a good nap to enable them to adhere to the flannel board. The use of different colors on each side of a cutout, as red and green, increases the clarity of a demonstration of fractional numbers on a flannel board.

The pupils' set of disks should include at least four whole disks and two disks each cut into halves, thirds, fourths, sixths, and perhaps eighths, as shown in Figure 13.1. The diameters of all the circular disks should be the same so as to show the equivalence of certain fractional parts.

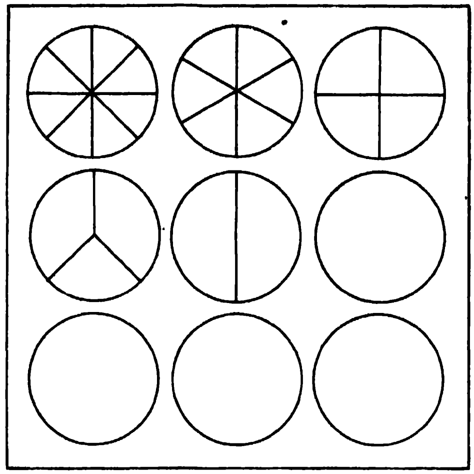


Figure 13.1

The teacher's number ray (see Fig. 13.2) may be made of plastic or some other type of material or it may be a drawing on a chalkboard to serve as a visual aid. The ray should include at least two arrows of each dimension shown in Figure 13.2. The arrows on a number ray may be used either as a manipulative instrument or as a visual aid. A number ray with arrows is one of the most effective ways of showing the meaning of a fractional number and its relationship to other numbers.

A fraction chart (Fig. 13.3) shows a unit segment of a number ray, as the

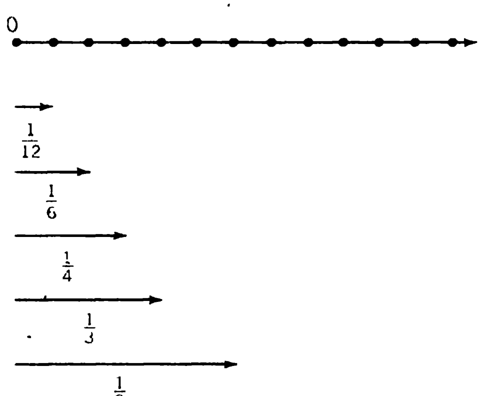


Figure 13.2

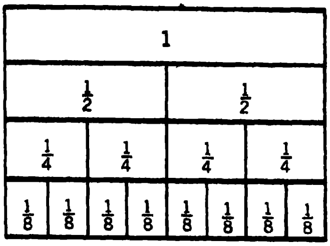


Figure 13.3

segment from 0 to 1. This unit is divided into *congruent* (same size and shape) parts. In one chart the unit may be divided to show halves, fourths, and eighths. Another chart may have the unit divided to show thirds, sixths, and twelfths. Each of these charts is inexpensive and easy to make.

A more desirable fraction chart than the poster type is a chart with movable parts. The frame of the chart includes panels for cards, as shown in Figure 13.4. The cards in each of the panels are interchangeable. Thus the pupil can remove a card showing a half and replace it with a card showing two fourths or four eighths. There should be a set of cards to correspond to the two sets of fractions mentioned for the poster charts.

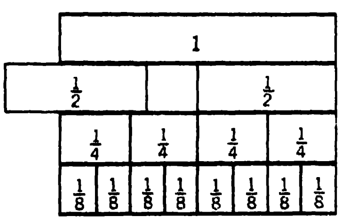


Figure 13.4

OPERATIONS WITH RATIONAL NUMBERS

The sequence for introducing the four operations will be the same for rational numbers as for whole numbers. Since a positive rational number is the quotient of two natural numbers, the quotient may be interpreted as a *rate* or a *ratio* if the physical situation leading to division is known. The quotient of two numbers without any reference to a specific situation cannot be identified as either a rate or a ratio. Beginning work with fractional numbers should, however, be identified with situations that can be recognized as a rate or a ratio. The ratio of the lengths of one pair of line segments may be $\frac{5}{6}$ and of another pair $\frac{2}{9}$. These two ratios would not be added, since the answer would be meaningless. Therefore only fractional numbers that represent a rate are added in solving a problem having social significance.

The pupil usually experiences a minimum of difficulty in renaming two fractional numbers expressed with unlike denominators if one of the denominators is a multiple of the other. If the denominators are not related as multiples, the problem of renaming such numbers is difficult for many pupils. Not only is the example difficult but for many pupils it has no applications except as a computational procedure they do not understand.

Differentiation of the curriculum

The curriculum dealing with fractional numbers should be differentiated to meet the needs of two unlike groups. One group, consisting of slow learners, should deal with those fractional numbers that have applications in everyday life; the other group, consisting of more able learners, should deal with any fractional numbers that may be useful in illustrating the mathematical principles and properties of numbers.

The group that understands the characteristics of prime numbers and the properties of addition may solve an ex-

ample of the type $\frac{2}{3} + \frac{3}{8} + \frac{9}{11}$ and profit from the experience. A pupil in this group would understand how to find a common denominator and how to rename any fractional number. The example does not represent a problematic situation that usually has an application in daily affairs. Slow learners should not be required to add fractional numbers that illustrate only mathematical structure and principles but rather those that occur in problematic situations. The more able learners, on the other hand, would add fractional numbers with emphasis on structure and properties of the operation.

CONCEPTS CONVEYED BY FRACTIONAL NUMBERS

Renaming numbers

Fractional cutouts for both teacher and pupil are effective instructional aids for modeling fractional numbers. The teacher holds up before the class a large disk and a half disk and asks what each represents. The pupil then selects similar materials from his kit. He demonstrates two things with the cutouts. First, he arranges the halves on the disk to show that two halves make one whole. Second, he shows that the two halves are equal. A more precise term is "congruent." Congruent cutouts have the same size and shape and match throughout when superimposed on each other. The pupil demonstrates the congruence of halves of disks by showing that they match throughout.

The teacher now has a pupil demonstrate on the flannel board that two halves make one whole. The teacher shows the class how to symbolize this fact, as 1. The class reads the number sentence as, "Two halves are equal to one." The number sentence shows that $\frac{2}{2}$ is another name for 1 and 1 is another number name for $\frac{2}{2}$. In the same way, the pupil demonstrates with thirds, fourths, and sixths. He symbolizes each experience as shown:

$$\frac{2}{2} = 1 \qquad \frac{4}{4} = 1$$

$$\frac{3}{3} = 1 \qquad \frac{6}{6} = 1$$

The cutout demonstrations should enable most pupils to discover the pattern for renaming 1. The teacher has the pupil give other number names for 1 when the whole is not divided into more than 10 equal parts. The class should be able to rename 1 as $\frac{5}{5}$, $\frac{7}{7}$, $\frac{8}{8}$, $\frac{9}{9}$, or $\frac{10}{10}$. To challenge the more able pupil, have him rename 1 as $\frac{a}{a}$ when 1 is divided into equal parts.

Next, the teacher helps the class to understand the numeral that names a fractional number. The pupil learned that a half is one of the two equal parts into which a whole is divided. The symbolization is as follows:

$$\text{Fractional numeral } \frac{1}{2} \begin{array}{l}\text{Numerator}\\ \text{Denominator}\end{array}$$

The use of a number ray should follow the work with cutouts for representing fractional numbers. Begin with a ray scaled to represent whole numbers, as in Figure 13.5. Each point identified is expressed as a whole number and as a rational number. Thus, 2 and $\frac{2}{1}$ name the same number. Now have the class tell how to represent a half on the ray. Most pupils should be able to state that a point placed on the ray midway between 0 and 1 will represent a half, as shown in Figure 13.6. Be sure the class can explain why 0, $\frac{0}{1}$, and $\frac{0}{2}$ name the same number.

Another ray, Figure 13.7 should be divided to show fourths. The same scale

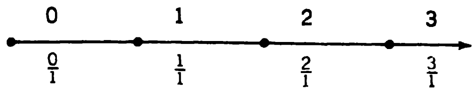


Figure 13.5

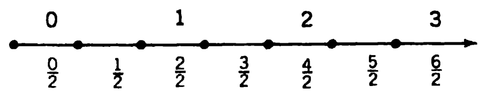


Figure 13.6

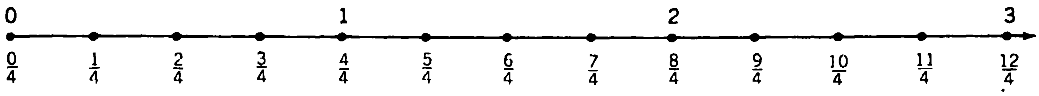


Figure 13.7

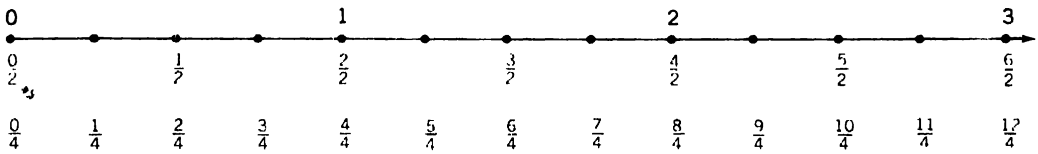


Figure 13.8

should be used on each ray, and the rays in Figures 13.6 and 13.7 should be combined as in Figure 13.8.

The ray in Figure 13.8 shows how a fractional number may have different names. The first pair identified is $\frac{0}{2}$ and $\frac{0}{4}$. In a similar way other whole numbers and fractional numbers have different names.

The pupil should use the number ray and his cutouts, if needed, to answer questions such as the following:

1. How many halves in 1? in 2? in 3?
2. How many quarters (or fourths) in 1? in 2? in 3?
3. How many quarters make a half?
4. Give another number name for $\frac{1}{2}$; for $\frac{1}{2}$.
5. Select the larger number named: a. $\frac{1}{2}$, $\frac{1}{3}$; b. $\frac{1}{3}$, $\frac{1}{4}$; c. $\frac{1}{2}$, $\frac{3}{4}$.

Next, the teacher should have the class scale a ray to show eighths.

The teacher explains to the class how to read a fraction. The fraction $\frac{1}{4}$ is read, "one fourth" or "one quarter." The fraction $\frac{1}{4}$ may be interpreted as one of the four equal parts of a whole or unit.

Modeling fractional numbers

The pupil has identified fractional numbers represented on a flannel board or on a number ray. The next step in the presentation consists in modeling them with geometric figures. The pupil should model such fractions as halves, fourths, and eighths. He may represent these numbers by folding paper or by making drawings or by both means. He divides a square to show fourths. The usual ways to divide a square to show fourths are depicted in Figure 13.9.

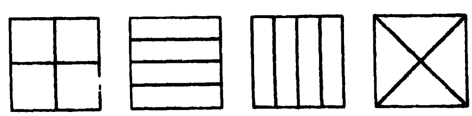


Figure 13.9

Working with eighths

An effective way to introduce eighths is to have the pupil fold a rectangular sheet of paper to show eighths. He should discover that an eighth is half

of a fourth and that a fourth is equal to two eighths. Similarly, he should discover other relationships among halves, fourths, and eighths.

Next, the pupil should model eighths by dividing a square or a rectangle into eighths. The teacher should challenge the class to demonstrate as many ways as possible to represent eighths in a square. Most pupils discover the five ways shown in Figure 13.10. All the parts in each diagram are congruent.

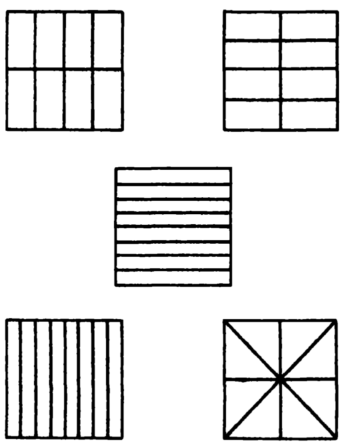


Figure 13.10

It is possible for a square to be divided into eight equal parts that are not congruent as shown in Figure 13.11. Equal parts, as used here, refers to parts with equal measure (see p. 347). Each of the parts of the square in Figure 13.11 encloses the same amount of space, but all the parts do not have the same shape. The pupil who discovers that equal parts do not necessarily have to be congruent understands that an eighth of a square encloses space equal to one eighth of the space included within the square. One boy expressed the idea of eighths as meaning equal spaces in the following descriptive phrasing: "Each part contains the same amount of room."

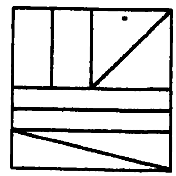


Figure 13.11

The pupil should discover that an eighth is half of a quarter and/or that an eighth is a fourth of a half. He would divide the square into halves or quarters. From that point he would divide each of these parts so as to form eighths (Fig. 13.12). He could then be certain that one of the small parts would be one fourth of a half, ($\frac{1}{8}$), or half of a quarter, ($\frac{1}{8}$). The three parts represent some of the ways a pupil may discover to represent an eighth.

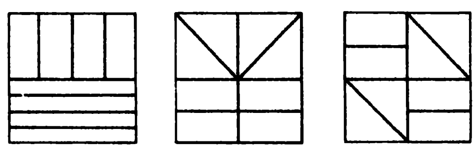


Figure 13.12

All pupils who understand the meaning of an eighth should be able to discover four of the five representations shown in Figure 13.10. The more able pupils should be able to discover at least three representations of eighths of the type shown in Figure 13.12. One of the writers observed a grade 5 pupil discover 20 different ways to divide a square to represent eighths. This pupil completed the task in approximately 40 minutes, or an average of 2 minutes per square. In order to understand an accomplishment of this kind, the reader should try to duplicate the achievement of this pupil.

After the pupil becomes familiar with the set of fractions having denominators of 2, 4, and 8, he should deal with thirds

and sixths. The same activities used for teaching halves and fourths apply to teaching thirds and sixths. The pupil uses his kit material and a number ray to discover relationships among fractional numbers and to rename these numbers as $\frac{1}{3}$ and $\frac{2}{6}$, $\frac{3}{3}$ and 1, $\frac{6}{6}$ and 1, $\frac{6}{3}$ and 2, and the like.

Identifying parts

As soon as the pupil develops an understanding of the fractions in the set of halves and of thirds, he should be ready to identify different fractional parts in other geometric figures. Workbooks that contain geometric designs of the kind shown in Figure 13.13 are effective for determining the pupil's ability to identify different fractional parts. He should color one part of each figure and write this part as a fraction. He should then have another set of the same figures. In this set the part of each figure to be colored would be indicated in symbolic form. If he were to color two thirds of the rectangle divided into sixths, he should color any four of the six parts to represent two thirds of the rectangle shown. The pupil who colors any named fractional part of a figure demonstrates that he is able to model that fraction.

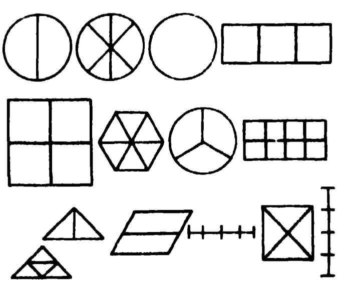


Figure 13.13

It is also possible and desirable to model fractions by using the concept of a set and its subset, as illustrated by the following:

{○, □, ○, □, □} Help pupils recognize that $\frac{2}{5}$ of the members are circles and $\frac{3}{5}$ are squares

{●, ●, ●, ●, ○, ●, ○, ●} Help pupils recognize that $\frac{6}{8}$ of the circles are solid and $\frac{2}{8}$ are not

Other geometric examples may be given, but there are also many classroom situations that should be used. Use fractional numbers to describe the number of red books in a set of books, the number of boys in a set of pupils, the number of empty chairs in a set of chairs, and so on.

Braumfeld and Wolfe have described the use of stretchers and shrinkers to help pupils understand fractional numbers.[2] Whole numbers greater than 1 are called stretchers, while unit fractions (of form $\frac{1}{n}$) are called shrinkers. The two may be interpreted geometrically (see Table 13.2).

The fraction $\frac{2}{5}$ may then be interpreted as a stretcher of 2 applied to a shrinker of $\frac{1}{5}$ $(2 \times \frac{1}{5})$.

Apply a shrinker of $\frac{1}{5}$ to 1 and get

Apply a stretcher of 2 to $\frac{1}{5}$

and get $\frac{1}{5} + \frac{1}{5}$, or $\frac{2}{5}$.

In terms of modeling fractions, there is little involved in this method that is not treated by the number ray or fraction chart, but as an additional means of helping pupils understand multiplication, this method deserves serious investigation.

[2] Peter Braumfeld and Martin Wolfe, "Fractions for Slow Learners," *The Arithmetic Teacher*, December 1966, 14:647–655.

TABLE 13.2
Graphic Interpretation of Stretchers and Shrinkers

Start with Unit Length	*Apply*	*To Get*
	Stretcher of 2	
	Stretcher of 3	
	Stretcher of 4	
	Shrinker of $\frac{1}{2}$	
	Shrinker of $\frac{1}{4}$	
	Shrinker of $\frac{1}{5}$	

Comparison of fractional numbers

The final activity in developing the meaning of fractional numbers consists in comparing such numbers. A pupil often finds it difficult to realize that a fourth is less than a third. Since 4 is greater than 3, he assumes that the same relationship exists between a fourth and a third. Adults have been known to make irrational business transactions because they incorrectly interpreted the value of a rational number. When the first successful oil well in this country was discovered in Titusville, Pennsylvania, in 1859, oil companies leased the land in the region and usually offered the owner of the land a royalty of a quarter of the value of the oil marketed. One writer has noted that "several farmers greedily refused to lease their lands to hurriedly formed companies for one-fourth royalty, and cunningly held out for one-eighth or even one-twelfth because it sounded bigger."[3]

The pupil uses his cutouts to compare halves, thirds, fourths, sixths, and eighths. He shows that a third is less than a half by superimposing one fractional part on the other. Similarly, he shows that a fourth is less than a third. He should then write the unit fractions in order of value as follows:

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{6} \quad \frac{1}{8}$$

The series should enable the pupil to discover that when the numerators are the same, the larger the denominator the smaller the number named. The pupil also should discover that when the denominators are the same, the larger the numerator the larger the number named.

If two fractions do not have the same numerator, as $\frac{2}{3}$ and $\frac{3}{5}$, the pupil should change the fractions to fractions having like denominators and then compare the numbers named. If a, b, and c are whole numbers and $b \neq 0$, any two fractional numbers expressed with like denominators may be compared as follows:

$$\frac{a}{b} = \frac{c}{b} \quad \text{if} \quad a = c$$
$$\frac{a}{b} > \frac{c}{b} \quad \text{if} \quad a > c$$
$$\frac{a}{b} < \frac{c}{b} \quad \text{if} \quad a < c$$

The pupil learns to change fractional numbers expressed with unlike denomi-

[3]Hildegarde Dolson, *The Great Oildorado* (New York: Random House, Inc., 1959), p. 6.

nators to fractional numbers expressed with like denominators by applying the identity element for multiplication (see p. 235).

Fast learners may discover a short way to compare two fractional numbers by comparing the products resulting from multiplying the numerator of one fraction and the denominator of the other. The arrows connecting the fractions in the example at the right indicate the factors of each product. The numerator 3 is a factor of 27 and the numerator 7 is a factor of 28. Since 27 is less than 28, $\frac{3}{4}$ is less than $\frac{7}{9}$.

$$\begin{matrix} 3 & \searrow & 7 \\ 4 & & 9 \end{matrix}$$

If $\frac{3}{4}$ and $\frac{7}{9}$ were renamed with numerals having the lowest common denominator, each denominator would be expressed as thirty-sixths. The numerator of the numeral $\frac{3}{4}$ would be expressed as 3×9 and the numerator of the numeral $\frac{7}{9}$ would be expressed as 4×7. In general terms, if $\frac{a}{b}$ and $\frac{c}{d}$ represent any two fractional numbers, these numbers may be compared by comparing *ad* with *bc*.

The teacher should be certain that the pupil discovers and understands the short-cut procedure for comparing two fractional numbers. It should be pointed out that use of the rule of cross-multiplying a numerator and a denominator to compare two rational numbers can be the epitome of rote learning.

The steps in teaching the pupil to compare two fractional numbers are as follows:

1. Have him use cutouts.
2. Show the numbers on a number ray.
3. Rename the numbers with fractional numerals having like denominators and then compare the numerators.
4. Compare the products that result from multiplying the numerator of one fraction and the denominator of the other. This procedure is for the more able pupil.

The incomplete number sentences that follow illustrate the type of example to give the class to compare rational numbers. The pupil writes in the circle the symbol, =, >, or <, that makes the statement true.

a. $\frac{1}{2} \bigcirc \frac{1}{3}$ d. $\frac{1}{4} \bigcirc \frac{2}{8}$

b. $\frac{1}{3} \bigcirc \frac{2}{6}$ e $\frac{1}{4} \bigcirc \frac{1}{6}$

c $\frac{3}{4} \bigcirc \frac{7}{8}$ f $\frac{2}{3} \bigcirc \frac{3}{4}$

An effective challenge for the more able pupil consists in having him write examples or fractions that meet conditions pertaining to an inequality. The following are samples of the type of problem to be given:

1. A fractional number that has a value less than a fourth but greater than a sixth, as $\frac{1}{6} < \frac{1}{5} < \frac{1}{4}$.
2. Several numbers, each of which has a value less than an eighth but greater than a twelfth, as $\frac{1}{12} < \frac{1}{10} < \frac{1}{8}$.
3. A fractional numeral expressed with a numerator of 2 that has a value less than a half but greater than a third, as $\frac{1}{3} < \frac{2}{5} < \frac{1}{2}$.

A fractional number as part of a group

A fractional number may represent a part of a unit or group. Thus $\frac{1}{4}$ of an orange means one of the four equal parts of the orange. One fourth describes a subset of one member with respect to an original set of four members. Also $\frac{1}{4}$ of a dozen oranges means one of the four equal parts into which 12 oranges have been divided. Finding a fractional part of a group, such as $\frac{1}{4}$ of a dozen oranges, is the same as sorting 12 oranges into four equivalent subsets of three oranges each.

One of the two uses of division is to find the size of one of *n* equivalent sets into which a given set has been parti-

tioned. Partitive division is the same as finding a fractional part of a number. Thus finding $\frac{1}{4}$ of 12 is the same as dividing 12 by 4. The problem of finding a fractional part of a number is discussed in detail in Chapter 14. The pupil should discover that multiplying by $\frac{1}{4}$ is the same as dividing by 4. This is one of the basic mathematical results of the inverse relation between multiplication and division.

A fractional number as a ratio

A *ratio* is the quotient of two numbers used to compare sets with similar members. In Figure 13.14, the ratio of (a) to (b) is indicated. The diagrams show that the ratio of two numbers is a rational number. The 3 disks in (a) may be compared by division with the 4 disks in (b). The quotient of the number in (a) divided by the number in (b) is $\frac{3}{4}$. If we interchange the divisor and the dividend the quotient is $\frac{4}{3}$. The quotients $\frac{3}{4}$ and $\frac{4}{3}$ represent the ratio of 3 to 4 and 4 to 3, respectively.

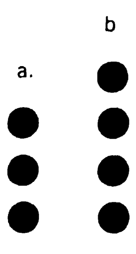


Figure 13.14

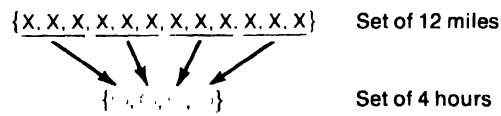
The pattern for introducing the ratio concept of a rational number is similar to the plan used to introduce the partitive concept. The sequence of steps for the ratio concept is as follows:

1. Compare two groups with markers, such as 2 disks compared with 3 disks.
2. Change the size of the two groups but keep the ratio between them constant. Thus add 2 disks to the first group in item (1) and 3 disks to the second group. Then $\frac{2}{3}$ and $\frac{1}{6}$ are different names for the same number.
3. Have the class model a ratio by a drawing, such as represented by $\frac{2}{3}$ or $\frac{3}{4}$.
4. Express the ratio of the number in each of two groups of objects in the classroom, such as books or pupils.

Rate, ratio, sets, and rational numbers

The quotient $\frac{2}{3}$ cannot be identified as a rate or ratio until the situation described by the quotient is known. A ratio compares sets of like objects, while a rate compares sets of unlike objects.[4] Both a rate and a ratio of 3 indicate that the cardinal number of one set is 3 times that of the other. In the same manner, a rate and a ratio of $\frac{1}{3}$ indicate that the cardinal number of one set is $\frac{1}{3}$ that of the other.

A rate of 3 miles per hour indicates that for each subset of 1 hour obtained (by a partition into equivalent subsets) from a set of hours, there is a subset of 3 miles obtained (by a partition into equivalent subsets) from a set of miles. The following diagram indicates how the comparison of a set of 12 miles with a set of 4 hours leads to a rate of 3 miles per hour.



The quotient of 3 indicates the number of miles in each of the 4 equivalent subsets obtained by the partition of the

[4] For a ratio, one of the two sets may be a subset of the other.

set of 12 miles. The quotient of 3, therefore, represents partitive division.

The breaking of an orange into quarters can be visualized in terms of a comparison of sets by interpreting it as the result of distributing 1 orange equally among 4 people. For each subset of 1 person (from the set of people) there is a subset containing $\frac{1}{4}$ of an orange (obtained from the set resulting when the orange was divided into 4 equal portions). The quotient of $\frac{1}{4}$ indicates the number of oranges in each subset obtained by partitioning the set of 4 quarters of an orange into 4 equivalent subsets and therefore represents partitive division. The answer, $\frac{1}{4}$ orange, would be impossible without fractional (rational) numbers.

If a set of 3 oranges is compared with a set of 12 people, the quotient $\frac{1}{4}$ may be interpreted as indicating that for each subset of 4 people there is a subset containing 1 orange.

A rate such as 3 miles per hour is sometimes called a one-to-many correspondence, since each hour corresponds to more than one (3) mile. A rate of $\frac{1}{4}$ orange per person may then be referred to as a many-to-one correspondence, since 4 people correspond to 1 orange.

That both the situation involving miles and hours and that involving people and oranges represent partition division can also be illustrated by the following set sentences.

A set of 12 miles may be broken into 4 sets of n miles ($n = 3$).

A set of 1 orange may be broken into 4 sets of n oranges ($n = \frac{1}{4}$).

A ratio of 3 is obtained when set A (with 12 members) is compared with set B (with 4 similar members), where set B may or may not be a subset of set A. The quotient 3 indicates that set A may be partitioned into 3 subsets, each of which is equivalent to set B. Since the members are similar, it is difficult to distinguish these 3 sets from set B. For this reason it is sometimes said that set B is contained 3 times in set A (even though the sets may be disjoint).

In this case, set A is broken into subsets of 4 members, and the quotient 3 indicates that there are 3 such subsets, the characteristic of a comparison, measurement, or quotitive division. The descriptive set sentence is: A set of 12 members may be broken into n sets of 4 members ($n = 3$). This sentence also indicates comparison division. If the sets are compared in the opposite order, the ratio is 1 to 3, or $\frac{1}{3}$. A set sentence may help to make this fact clear: A set of 4 members is broken into n sets of 12 members ($n = \frac{1}{3}$). The term "$\frac{1}{3}$ of a set" is not common usage in the mathematics of sets Some students may prefer to say $\frac{1}{3}$ of a collection or group of 12 members.

Levels of maturity in renaming fractions

The fractional number $\frac{2}{3}$ is expressed in *simplest form*, in *standard form*, or in *lowest terms*. The fractional number $\frac{6}{9}$ can be renamed in standard form as $\frac{2}{3}$. A fractional numeral is in standard form when 1 is the largest natural number factor that is common to both terms. When the fractional numeral $\frac{2}{3}$ is replaced by $\frac{6}{9}$, the number is renamed in *higher terms*. The pair $\frac{2}{3}$ and $\frac{6}{9}$ represents *equivalent fractional numerals* because they name the same number.

There are three levels of maturity in dealing with the renaming of fractional numbers in higher or lower terms. The first level may be designated the *exploratory level*. At this level the pupil uses supplementary aids to show the equivalence of fractional numerals. He operates at this level when he uses cutouts, a fraction chart, or a number ray to show that $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{4}{8}$ are different names

for the same number. Number rays (A–C) of Figure 13.15 show the equivalence of these numerals.

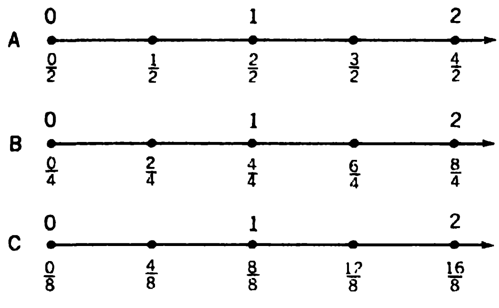


Figure 13.15

Geometric figures may also be used to rename fractional numbers on the exploratory level, as illustrated by the two rectangles in Figure 13.16. The class should identify the shaded area as $\frac{1}{2}$ of the rectangle. An additional line should then be drawn, as illustrated, which divides the rectangle into fourths. The pupils can then recognize that the shaded portion is now $\frac{2}{4}$ of the rectangle and that $\frac{1}{2}$ and $\frac{2}{4}$ name the same fractional number and therefore are equivalent fractional numerals.

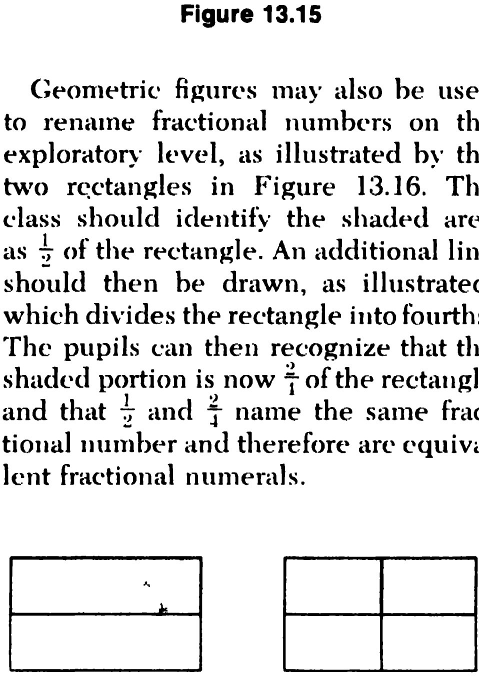


Figure 13.16

The subset concept may also be used on the exploratory level. From early work in fractional numbers pupils should recognize that $\frac{4}{10}$ of the members of the following set are solid:

{●, ●, ●, ●, ○, ○, ○, ○, ○, ○}

The set may then be rearranged into a set of pairs of members, as shown below. The class should then recognize that $\frac{2}{5}$ of the pairs contain solid members, or that $\frac{2}{5}$ of the members of the set are solid.

{●●, ●●, ○○, ○○, ○○}

Since $\frac{2}{5}$ and $\frac{4}{10}$ name the same number, they are therefore equivalent fractional numerals. Further work on the exploratory level should help the pupil discover that $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{4}{8}$ are names (fractional numerals) for the same fractional number. The following set of equivalent numerals for $\frac{1}{2}$ may then be written:

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \ldots\right\}$$

The pupil should recognize the pattern in the above set and continue it to produce as many additional numerals of the set as the teacher requests. Similar exploratory activities should help pupils write and recognize the pattern for sets of equivalent fractional numerals similar to the following:

$$\left\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}, \ldots\right\}$$

$$\left\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \frac{5}{20}, \ldots\right\}$$

$$\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \ldots\right\}$$

$$\left\{\frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \frac{10}{25}, \ldots\right\}$$

With this background the pupil should then be able to work with open sentences of the following type:

$$\frac{1}{2} = \frac{\square}{6} \qquad \frac{3}{4} = \frac{6}{\square} \qquad \frac{3}{\square} = \frac{12}{16}$$

$$\frac{2}{3} = \frac{\square}{12} \qquad \frac{\square}{5} = \frac{4}{10} \qquad \frac{3}{4} = \frac{\triangle}{\square}$$

(many possible answers)

In conjunction with this work, frequent brief renaming sessions should be held in which the pupil renames $\frac{1}{2}$ as

$\frac{2\times 1}{2\times 2}$ or $\frac{3\times 1}{3\times 2}$ as well as $\frac{3\times 2}{3\times 5}$ as $\frac{2}{5}$. Pupils should be able to rename as a result of writing sets of equivalent fractional numerals, as illustrated above. If a pupil has trouble with a particular example, have him write the appropriate set of equivalent fractional numerals. Pupils should then be able to deal effectively with open sentences of the following types:

$$\frac{1}{2}=\frac{4\times 1}{4\times \square} \qquad \frac{3}{4}=\frac{5\times 3}{\square\times 4} \qquad \frac{2\times 3}{2\times 4}=\frac{\square}{8}$$

$$\frac{2}{3}=\frac{3\times 2}{3\times \square} \qquad \frac{2}{5}=\frac{\square\times 2}{2\times 5} \qquad \frac{3\times 4}{3\times 5}=\frac{12}{\triangle}$$

This type of activity should enable pupils to rename fractions on the second level, the *operational level*, as follows:

$$\frac{1}{2}=\frac{3\times 1}{3\times 2}=\frac{3}{6} \qquad \frac{8}{12}=\frac{4\times 2}{4\times 3}=\frac{2}{3}$$

$$\frac{3}{4}=\frac{5\times 3}{5\times 4}=\frac{15}{20} \qquad \frac{15}{18}=\frac{3\times 5}{3\times 6}=\frac{5}{6}$$

While the above procedure can be described in terms of multiplying and dividing the numerator and denominator, the modern tendency is to use the nonverbal-pattern approach, as outlined above. The teacher must learn which pupils will benefit from verbalization and provide such help as needed. Able pupils should be encouraged to interpret results verbally.

The third level of renaming fractions is the *structural level*, in which the identity concept provides the mathematical basis, as illustrated below.

$$\frac{1}{2}=1\times\frac{1}{2}=\frac{5}{5}\times\frac{1}{2}=\frac{5\times 1}{5\times 2}=\frac{5}{10}$$

$$\frac{3}{4}=1\times\frac{3}{4}=\frac{4}{4}\times\frac{3}{4}=\frac{4\times 3}{4\times 4}=\frac{12}{16}$$

$$\frac{9}{12}=\frac{3\times 3}{3\times 4}=\frac{3}{3}\times\frac{3}{4}=1\times\frac{3}{4}=\frac{3}{4}$$

The pupils will quickly abbreviate the above to the following:

$$\frac{1}{2}=\frac{5\times 1}{5\times 2}=\frac{5}{10}$$

$$\frac{3}{4}=\frac{4\times 3}{4\times 4}=\frac{12}{16}$$

$$\frac{9}{12}=\frac{3\times 3}{3\times 4}=\frac{3}{4}$$

It should be recognized that the abbreviated procedure based on structure is identical to the operational procedure previously described. However, the method now has a sound mathematical basis, which will be stressed in the junior and senior high school as well as in the upper elementary grades.

The pattern approach to the operational level is required because it is needed at a time when multiplication of fractions has not yet been studied. It should be clear that the structural approach to renaming fractions cannot be introduced until multiplication of fractions is understood.

Steps in renaming fractions

Fractional numbers may be renamed in one sequence of steps, as shown in example (a). This plan is based on the fact that the pupil selects the largest factor common to both numerator and denominator in order to express a fraction in its simplest form or in lowest terms. When the terms of a fraction contain more than one common factor, the pupil may not be able to discover the highest common factor. In that event there would be two or more sequences of steps in renaming the fraction in simplest form, as in example (b).

a $\frac{16}{24}=\frac{8\times 2}{8\times 3}=\frac{2}{3}$

b $\frac{16}{24}=\frac{4\times 4}{4\times 6}=\frac{4}{6}$

$\frac{4}{6}=\frac{2\times 2}{2\times 3}=\frac{2}{3}$

In (a) the pupil identified the highest common factor of 16 and 24 as 8 and in

one sequence of steps renamed $\frac{16}{24}$ as $\frac{2}{3}$. In (b) the pupil did not identify the highest common factor of both terms. Two sequences of steps were then necessary to rename $\frac{16}{24}$ as $\frac{2}{3}$. If a pupil is not able to factor a pair of numbers as shown in (a), he should use the procedure shown in (b). The more able pupils should be encouraged to use the procedure illustrated in (a). The teacher should have these pupils find the highest common factor of such number pairs as (18, 30), (24, 36), and the like.

ADDITION OF FRACTIONAL NUMBERS

Fractional numbers expressed with like denominators

Addition of fractional numbers is most readily performed when the numbers are expressed with numerals having common (like) denominators. Addition of fractional numbers expressed with like denominators can also be performed on three levels – the exploratory, operational, and structural.

The exploratory phase of adding fractional numbers relies on manipulative materials such as the number ray and other geometric representations. A different type of model may be used to find the sum of the rational numbers in each of the following problems (see Fig. 13.17):

1. How much is one quarter of a pie and two quarters of a pie?
2. The length of one piece of ribbon is $\frac{5}{8}$ yard and the length of another piece is $\frac{1}{8}$ yard. What is the length of both pieces?
3. Ruth walked $\frac{1}{5}$ mile from her home to the post office and then walked $\frac{3}{5}$ mile to school. How far did she walk from her home to school?

The teacher writes an open-number sentence on the chalkboard for each problem. The pupil finds the number named by the missing numeral in each problem by reference to models. He uses his cutouts to find the sum in (1), a number ray in (2), and a diagram in (3).

The number sentences are as follows:

1. $\frac{1}{4} + \frac{2}{4} = n$
2. $\frac{5}{8} + \frac{1}{8} = n$
3. $\frac{1}{5} + \frac{3}{5} = n$

After sufficient work on the exploratory level, pupils should learn to rename examples like $\frac{2}{5} + \frac{1}{5}$ as $\frac{2+1}{5}$. More able pupils may be able to rename $\frac{a}{c} + \frac{b}{c}$ as $\frac{a+b}{c}$.

When the basic pattern indicated above is mastered, pupils can add frac-

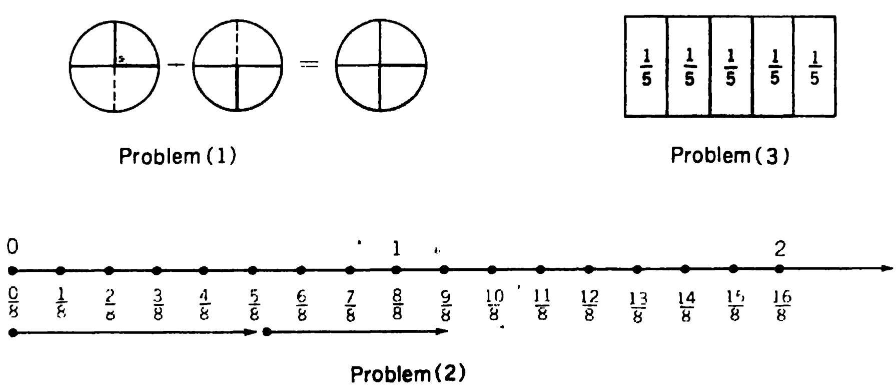


Figure 13.17

tional numbers with like denominators on the operational level; for example:[5]

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3} \qquad \frac{2}{5} + \frac{3}{5} = \frac{5}{5} = 1$$

$$\frac{3}{8} + \frac{4}{8} = \frac{7}{8} \qquad \frac{5}{6} + \frac{2}{6} = \frac{7}{6}$$

On the operational level, it is desirable that the pupil rename the sum $\frac{2}{6} + \frac{3}{6}$ as $\frac{5}{6}$ without writing the intermediate expression $\frac{2+3}{6}$. On the other hand, it is desirable that the pupil frequently rename $\frac{2+3}{6}$ as $\frac{2}{6} + \frac{3}{6}$. Such renaming is essential for a pupil to understand how to rename the fractional number $\frac{5}{3}$ with the mixed numeral $1\frac{2}{3}$. This procedure can be understood by performing the following steps (see p. 243):

$$\frac{5}{3} = \frac{3+2}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{2}{3} = 1\frac{2}{3}$$

The structural level of adding fractional numbers expressed with like denominators depends on two ideas:

1. Multiplication and division are inverse operations; in place of dividing by 6, multiplication by $\frac{1}{6}$ may be performed. When this idea is understood, the fraction $\frac{2}{6}$ may be renamed as $2 \times \frac{1}{6}$, since the fractional number $\frac{2}{6}$ is the quotient of 2 and 6 (and also of 4 and 12, and so on). It is equally important that pupils be able to rename $4 \times \frac{1}{5}$ as $\frac{4}{5}$ if the structural level of adding fractional numbers is to be understood.

2. Multiplication is distributive with respect to addition. This statement means that $2(3 + 4)$ can be renamed as $2 \times 3 + 2 \times 4$. It is important to recognize that the distributive property also indicates that $3 \times 10 + 4 \times 10$ can be renamed as $(3 + 4) \times 10$. In a similar manner, the distributive property indicates that $3 \times \frac{1}{6} + 2 \times \frac{1}{6}$ can be renamed as $(3 + 2) \times \frac{1}{6}$.

[5]In the last example listed it should not be necessary to rename $\frac{7}{6}$ with the mixed fractional numeral $1\frac{1}{6}$ in early additions (see p. 241).

These two basic ideas lead to addition of fractional numbers on the structural level, as follows:

$$\frac{2}{6} + \frac{3}{6} = 2 \times \frac{1}{6} + 3 \times \frac{1}{6} = (2 + 3) \times \frac{1}{6} = 5 \times \frac{1}{6} = \frac{5}{6}$$

Not all elementary school mathematics programs introduce the structural method of adding fractional numbers, so that this method may be delayed until the junior high school. The method does offer another way in which the curriculum may be differentiated to challenge more able pupils. Addition of fractional numbers cannot have a sound mathematical foundation until the structural level of adding such numbers is understood.

Work with open sentences of the following type may be useful even if the structural method of adding fractional numbers is not introduced:

$$\frac{2}{3} = \square \times \frac{1}{3}$$

$$\frac{2}{5} = 2 \times \frac{\square}{5}$$

$$\frac{4}{6} = 4 \times \square$$

$$3 \times \frac{1}{6} = \frac{\square}{6}$$

$$\square \times \frac{1}{5} = \frac{4}{\square}$$

$$2 \times \frac{1}{6} + 3 \times \frac{1}{6} = (\square + 3) \times \frac{1}{6}$$

$$3 \times \frac{1}{8} + 2 \times \frac{1}{8} = (3 + 2) \times \frac{1}{\square}$$

$$2 \times \frac{1}{8} + 5 \times \frac{1}{8} = (2 + 5) \times \square$$

$$\frac{1}{6} + \square \times \frac{1}{6} = (1 + 2) \times \frac{1}{6}$$

Able pupils should make the following generalizations:

1. Fractional numbers to be added should be expressed with like denominators.

2. Add the numerators to find the numerator of the fraction in the sum.

3. The denominator of the fraction in the sum is the same as the denominator of each fraction.

Fractional numbers expressed with unlike denominators

If two fractional numbers are to be added, either the denominators involved are like or they are not. The procedure with like denominators was outlined in the previous section. If the denominators are unlike, the numbers are renamed with numerals having like denominators.[6] Addition can then proceed as previously indicated.

Addition of fractional numbers expressed with unlike denominators can be performed on the exploratory, operational, and structural levels. However, work on the exploratory level should be very brief. Probably only the number ray should be used at this stage. Adding such pairs of numbers as $\frac{1}{2} + \frac{1}{3}$ and $\frac{2}{3} + \frac{3}{4}$ on the number ray is good readiness activity for introducing the operational level.

On the operational level, the pupil replaces numerals having unlike denominators with equivalent numerals having like denominators; for example:

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$
$$\frac{1}{6} + \frac{3}{4} = \frac{2}{12} + \frac{9}{12} = \frac{11}{12}$$

The structural level[7] of adding numbers expressed with unlike denominators combines the ideas outlined in the discussions of renaming fractional numbers on the structural level and adding fractional numbers expressed with like denominators) is the lowest common denominator (see p. 210).

$\frac{2}{3} + \frac{3}{4} = \frac{2}{3} \times 1 + \frac{3}{4} \times 1$	Identity for multiplication
$= \frac{2}{3} \times \frac{4}{4} + \frac{3}{4} \times \frac{3}{3}$	Rename 1 as $\frac{3}{3}$ and $\frac{4}{4}$
$= \frac{2 \times 4}{3 \times 4} + \frac{3 \times 3}{4 \times 3}$	Multiplication property of fractional numbers
$= \frac{8}{12} + \frac{9}{12}$	Multiplication of whole numbers
$= 8 \times \frac{1}{12} + 9 \times \frac{1}{12}$	Multiplication by 12 in place of division by 12
$= (8 + 9) \times \frac{1}{12}$	Distributive property
$= 17 \times \frac{1}{12}$	Addition of whole numbers
$= \frac{17}{12}$	Division by 12 in place of multiplication by $\frac{1}{12}$

Addition on a structural basis, as illustrated above, is usually performed in the junior high school and not in the elementary school. A combination of the operational and the structural levels, as illustrated below, however, may be appropriate for many elementary school pupils.

$$\frac{1}{2} + \frac{1}{3} = \frac{1 \times 3}{2 \times 3} + \frac{1 \times 2}{3 \times 2} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$
$$\frac{1}{6} + \frac{3}{4} = \frac{1 \times 2}{6 \times 2} + \frac{3 \times 3}{4 \times 3} = \frac{2}{12} + \frac{9}{12} = \frac{11}{12}$$

The preceding discussion is oversimplified to the extent that it gives no indication as to how the pupils are to find the common denominator (preferably the lowest common denominator) and then rename the given numbers with numerals having the required common denominator. Renaming fractional

[6] On the scientific and engineering level, fractional numbers with unlike denominators are frequently expressed as decimals to facilitate the operation of addition (as well as of other operations). This is particularly true when computing equipment is used.

[7] It should be noted that the structural-level approach cannot be performed until multiplication of fractions is understood.

numbers at random is a useful activity, but renaming a number to obtain a numeral having a specified denominator is more difficult. Activities with open sentences similar to the following may be useful for this purpose:

$$\frac{1}{2} = \frac{\square \times 1}{\square \times 2} = \frac{6}{12}$$

$$\frac{2}{3} = \frac{\square \times 2}{4 \times 3} = \frac{\triangle}{12}$$

$$\frac{3}{4} = \frac{\triangle \times 3}{12}$$

$$\frac{2}{5} = \frac{\square \times 2}{\square \times 5} = \frac{\triangle}{10}$$

$$\frac{3}{8} = \frac{\square}{16}$$

$$\frac{4}{5} = \frac{\square}{5}$$

$$\frac{4}{5} = \frac{8}{\square}$$

Some renaming sessions should be specifically directed at renaming numbers with numerals having a specified denominator.

Finding a common denominator

As discussed in Chapter 12, there are several plans that may be used to find the lowest common denominator of two fractions of the type $\frac{1}{3} + \frac{1}{4}$. One method is to find the intersection set of the set of multiples of the denominators. The teacher should have the pupil write at least six elements of the set of multiples of 3 and of 4, as shown.

Set A {3, 6, 9, 12, 15, 18, · · ·}
Set B · {4, 8, 12, 16, 20, 24, · · ·}
Set C · {12, 24, 36, · · ·}

Set $C = A \cap B$. Set C contains the elements that are multiples of numbers in A and B, hence these elements are common denominators of fraction numerals having 3 and 4 as denominators. The smallest number in the intersection set is the lowest common denominator (12).

An able pupil may abbreviate the above method by examining only the set of multiples of the largest denominator. The first such multiple that is also a multiple of the other denominator (or denominators) is the lowest common denominator (see p. 210).

The method of intersection of sets of multiples or its abbreviated form is probably the most efficient method of determining the lowest common denominator for most sets of denominators encountered in the elementary school and can be done mentally by many pupils. When the numbers are larger than 20, the method of intersection of set of multiples may be too cumbersome for practical purposes. Under these conditions, the method of prime factors is probably the most efficient. In this method the lowest common denominator is the product of all the prime factors, where each prime number is used as a factor the greatest number of times it occurs in any one denominator (see p. 211).

Problems in the addition (or subtraction) of fractional numbers may be classified according to the relationship among the denominators of the fractional numerals in the following manner:

1. Fractions having *like denominators*, as $\frac{1}{3} + \frac{1}{3}$
2. Fractions having *unlike but related denominators*, as $\frac{1}{2} + \frac{5}{8}$
3. Fractions having *unlike and unrelated denominators*, as (a) $\frac{1}{3} + \frac{3}{4}$ or (b) $\frac{1}{4} + \frac{1}{6}$.

For sums of the type described in item (1), addition can be performed immediately. For sums of the type shown in item (2), the largest denominator is a multiple of the other denominator (or all the denominators) and is therefore the lowest common denominator. For sums of the type shown in item (3a),

the unrelated denominators have no natural number common factor greater than 1 and the lowest common denominator is the product of the denominators. For sums shown in item (3b), a natural number common factor greater than 1 exists and the lowest common denominator is less than the product of the denominators. In this case, the method of intersection of sets of multiples or the prime factor method may be used if the pupil is not able to determine the lowest common denominator by inspection.

The preceding analysis suggests the following as a logical sequence of steps which a pupil might take in determining the lowest common denominator of two or more denominators that are not equal:

Step 1. Examine the denominators to determine if one denominator is a multiple of the other (or others). If such a denominator exists, it is the lowest common denominator. If such a denominator does not exist, go on to step 2.

Step 2. Examine the denominators to determine if they have a natural number common factor greater than 1. If no such number exists, the product of the denominators is the lowest common denominator. If such a number does exist, proceed as in step 3.

Step 3. Use the method of the intersection of sets of multiples, the abbreviated multiple method, or the method of prime factors, depending upon the size of the numbers and personal preference.

The new element in the addition of fractional numbers expressed with unlike denominators is to rename the numerals with like denominators.

Fractional numerals

Initial problems of adding fractional numbers usually involve those that are less than 1. It is not practical, however, to limit addition of fractional numbers to those with sums less than 1 for any length of time. In traditional arithmetic the impression was frequently given that an answer of $\frac{7}{3}$ is wrong and must be given as $2\frac{1}{3}$. Such an impression should be avoided. There are many situations in which the numeral $\frac{7}{3}$ is preferable to $2\frac{1}{3}$.

The fractional numeral $3\frac{1}{2}$ may be expressed as $3 + \frac{1}{2}$ or in fractional form as $\frac{7}{2}$. It is possible to change a mixed fractional numeral to a fractional numeral by apply the identity element of 1. The numeral $3 + \frac{1}{2}$ may be written as $(3 \times 1) + \frac{1}{2}$. The 1 may be renamed as $\frac{2}{2}$, and the numeral may then be expressed as $(3 \times \frac{2}{2}) + \frac{1}{2}$, which is the equivalent of $\frac{6}{2} + \frac{1}{2}$, or $\frac{7}{2}$. The numerals $3\frac{1}{2}$ and $\frac{7}{2}$ name the same number.

A fractional numeral naming a number greater than 1 may be expressed as a mixed fractional numeral by reversing the procedure described. The numeral $\frac{7}{2}$ names the same number as $\frac{2}{2} + \frac{2}{2} + \frac{2}{2} + \frac{1}{2}$. If we replace $\frac{2}{2}$ by its equivalent 1, the numeral $\frac{2}{2} + \frac{2}{2} + \frac{2}{2} + \frac{1}{2}$ may be written as $1 + 1 + 1 + \frac{1}{2}$, or $3 + \frac{1}{2}$, which is the same as $3\frac{1}{2}$.

The pupil uses the long form until he discovers a short way of renaming $\frac{7}{2}$ as a mixed fractional numeral. The fractional numeral $\frac{7}{2} = \frac{6}{2} + \frac{1}{2}$, in which $\frac{6}{2}$ is another name for 3, hence $\frac{7}{2}$, may be written as $3 + \frac{1}{2}$, or $3\frac{1}{2}$. The work may be simplified as in (a):

$$a \quad \frac{7}{2} = \frac{6+1}{2} = 3 + \frac{1}{2}, \text{ or } 3\frac{1}{2}$$

When a mixed numeral is written in the form given in (a), the numerator of the fraction is expressed as the sum of two addends. One of the addends is the largest multiple of the denominator that is contained in the numerator. Renaming a fraction expressed in this form as a mixed fractional numeral involves the distributive property of division over addition, as shown in (b):

b.	$\frac{15}{4} = \frac{12+3}{4}$	12 is the largest multiple of 4 in 15
	$\frac{12+3}{4} = 3 + \frac{3}{4}$	Distributive property
	$\frac{15}{4} = 3\frac{3}{4}$	Renaming numbers

Ways of expressing the sum

Only fractions having like denominators can be added in symbolic form. The sum may be expressed by fractional numerals as follows:

1. *In simplest form:* highest common factor of both terms is 1, as $\frac{3}{5}$, $\frac{7}{2}$
2. *In standard form:* in simple form and has a value less than 1, as $\frac{1}{4}$
3. *Not in simplest form:* fractions can be renamed as follows:
   a. In standard form as $\frac{4}{6} = \frac{2}{3}$
   b. As a whole number, as $\frac{6}{3} = 2$
   c. As a mixed fractional numeral in simplest form as $\frac{12}{8} = 1\frac{1}{2}$

The teacher decides the way to express the sum of two or more fractional numbers. For the past several decades most teachers have had pupils express the sum in simplest form. Simplest form was interpreted to mean two things: (1) one is the greatest common factor of both terms of the fraction, and (2) the numerator is always less than the denominator. "Simplest form" as used in this text does not imply that the numerator must be less than the denominator. It is difficult to see how $1\frac{2}{3}$ represents a simpler form that $\frac{5}{3}$, since both name the same number.

Since many different numerals can express a sum, the teacher must indicate how pupils should express the answer in addition and subtraction of fractional numbers. We shall assume that the sum should be in simplest form. This means that both the numerator and the denominator have no common natural number factor greater than 1. If the fraction in simplest form is greater than 1, the teacher may ask the pupil to rename the number as a mixed fractional numeral.

Kinds of examples in addition of rational numbers expressed with like denominators

There are many structural types of examples in addition of rational numbers expressed with like denominators, as illustrated in examples (a–e).

a. $\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$ c. $\frac{1}{3} + \frac{2}{3} = \frac{3}{3}$

b. $\frac{3}{5} + \frac{4}{5} = \frac{7}{5}$ d. $\frac{1}{8} + \frac{3}{8} = \frac{4}{8}$

e. $\frac{7}{8} + \frac{5}{8} = \frac{12}{8}$

The sums in examples (a) and (b) are in simplest form while those in (c–e) are not. In (b) and (d) the sums may be changed to mixed fractional numerals.

Mixed fractional numerals

The same patterns for adding rational numbers expressed with like denominators apply to adding numbers named by mixed fractional numerals. The mixed fractional numeral may be expressed as the sum of a whole number and a fraction:

$$\begin{array}{rcl} 2\frac{1}{8} & = & 2 + \frac{1}{8} \\ +\,3\frac{5}{8} & = & 3 + \frac{5}{8} \\ \hline & & 5 + \frac{6}{8} = 5 + \frac{3}{4} \text{ or } 5\frac{3}{4} \end{array} \qquad \begin{array}{rcl} 23 & = & 20 + 3 \\ +\,41 & = & 40 + 1 \\ \hline & & 60 + 4 = 64 \end{array}$$

The similarity of this process to early addition of whole numbers should be recognized. The pupil adds the fractional numbers and the whole numbers and then adds the two sums. The fraction in the sum is renamed in the same way as in addition of fractions. The pupil will not continue to write a mixed fractional numeral such as $2\frac{3}{4}$ in the long form as $2 + \frac{3}{4}$. He uses the expanded notation when he first learns to add numbers expressed with mixed fractional numerals.

The pupil may add numbers named by mixed fractional numerals when the addends are written in horizontal form. He should then rearrange or regroup the numerals to simplify the computation. The regrouping for finding the sum of $4\frac{2}{3}$ and $1\frac{2}{3}$ is as follows:

$4\frac{2}{3} + 1\frac{2}{3} = (4 + \frac{2}{3}) + (1 + \frac{2}{3})$	Expanded notation
$= (4 + 1) + (\frac{2}{3} + \frac{2}{3})$	Consequence of commutative and associative properties
$= 5 + \frac{4}{3}$	Renaming numbers
$= 5 + (\frac{3}{3} + \frac{1}{3})$	Renaming numbers
$= (5 + 1) + \frac{1}{3}$, or $6\frac{1}{3}$	Renaming numbers and associative property

In the same manner,

$$23 + 41 = (20 + 3) + (40 + 1)$$
$$= (20 + 40) + (3 + 1) = 60 + 4 = 64$$

The example illustrates the principle that the way numbers are rearranged or regrouped does not affect their sum. The sequence of steps applies to introductory work involving addition of numbers of this type. As the pupil becomes more familiar with the pattern for adding these numbers, he will use a shortened form.

SUBTRACTION OF FRACTIONAL NUMBERS

The pupil who can find n in example (a) can find n in example (b).

a $\frac{1}{3} + \frac{1}{3} = n$ b. $\frac{2}{3} - \frac{1}{3} = n$

After a pupil learns how to add fractional numbers having a sum less than 1, he should be able to subtract in the corresponding examples with fractions and mixed fractional numerals provided no regrouping is needed in the latter group. As with whole numbers, the teacher should emphasize the inverse relationship between addition and subtraction of fractional numbers.

The sequence of units of work in subtraction of rational numbers is as follows:

1. Subtraction of rational numbers expressed with like denominators, no regrouping
2. Subtraction of rational numbers expressed with like denominators, regrouping
3. Subtraction of rational numbers expressed with unlike but related denominators, all types
4. Subtraction of rational numbers expressed with unlike and unrelated denominators, all types.

As a pupil adds each of the four types of examples described he should be able to solve the corresponding examples in subtraction provided the sum is not expressed as a mixed fractional numeral. The class may need special help to subtract with numbers named by mixed fractional numerals. The following four kinds of examples are named by numbers of this kind. Similarity to subtraction of whole numbers should be noted.

a.	$4\frac{1}{3}$ -2	Subtracting a whole number from a number expressed by a mixed fractional numeral	41 -20
b.	$6\frac{3}{4}$ $-2\frac{1}{4}$	No regrouping needed to subtract	63 -21
c	4 $-2\frac{1}{3}$	Interchange of types of numbers in (a)	40 -21
d	$5\frac{1}{3}$ $-2\frac{2}{3}$	Regrouping needed to subtract	51 -22

The examples involving subtraction of whole numbers at the right above are similar in type to those on the left involving mixed numerals. Examples of types (a) and (b) present no new difficulty to the pupil because it is not necessary to regroup the numbers in

order to subtract. Examples (c) and (d) cannot be solved until the minuends (sums) are regrouped or renamed.

Renaming fractional numbers in subtraction

A visual representation with cutouts or a number ray may be used to introduce regrouping in subtraction of fractional numbers. Figure 13.18 uses cutouts to give a visual representation of renaming numbers in the example $1 - \frac{1}{4}$. The steps visualized are as follows: (A) shows 1 regrouped as $\frac{4}{4}$; (B) shows $\frac{1}{4}$ taken from $\frac{4}{4}$; (C) shows the answer, or $\frac{3}{4}$.

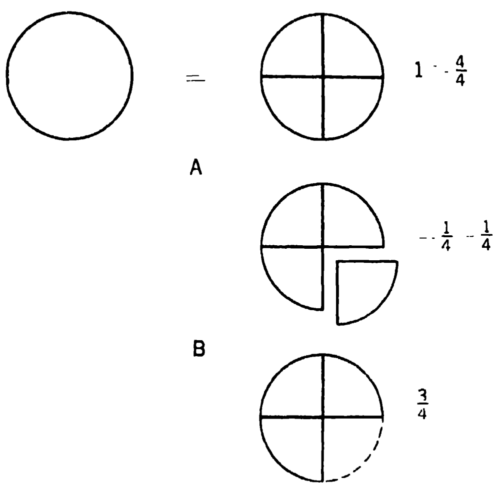


Figure 13.18

The graphic representation on the number ray in Figure 13.19 shows $\frac{1}{4}$ subtracted from $\frac{4}{4}$. It is advisable to present both the horizontal and vertical notations for the algorism.

Subtracting a mixed fractional number from a whole number

A teacher introduced subtraction involving a whole number and a mixed fractional number with the following problem: A piece of string $1\frac{3}{4}$ yards

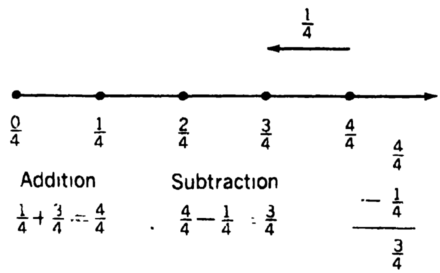


Figure 13.19

long was cut from a string 3 yards long. What was the length of the piece remaining?

The number sentence for the problem is $3 - 1\frac{3}{4} = n$. The following steps are recommended for finding n:

1. Use cutouts or a number ray.
2. Make a visual representation, as in Figure 13.20. Diagram (A) shows 3 regrouped as 2 and $\frac{4}{4}$; (B) shows $1\frac{3}{4}$ subtracted from $2\frac{4}{4}$; (C) shows the answer, or $1\frac{1}{4}$. The pupil should identify each step in the representation.
3. Have the class explain the steps in the following solution:

$$\begin{array}{rcccc} 3 & = & 2+1 & = & 2+\frac{4}{4} \\ -1\frac{3}{4} & = & 1+\frac{3}{4} & = & 1+\frac{3}{4} \\ \hline & & & & 1+\frac{1}{4} = 1\frac{1}{4} \end{array}$$

4. Have the class explain the steps in the textbook presentation.
5. Help the pupils to recognize the similarity and difference between the previous example and the following familiar work with whole numbers:

$$\begin{array}{rcc} 70 & 70+0 & 60+10 \\ -23 & 20+3 & 20+\ \ 3 \\ \hline & & 40+\ \ 7=47 \end{array}$$

6. In frequent and brief renaming sessions, the following types of renaming should be stressed:

a. Rename 5 as 4 + 1 and then as 5 + $\frac{1}{4}$, and so on

b. Rename 7 as $6\frac{5}{5}$ or as $5\frac{6}{3}$

c. Rename $1\frac{1}{2}$ as $\frac{2}{2} + \frac{1}{2} = \frac{3}{2}$

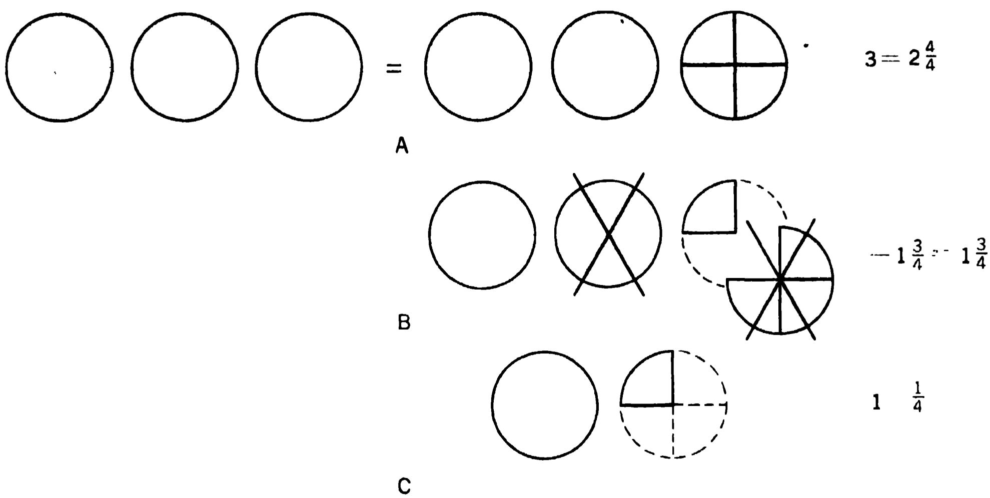


Figure 13.20

d. Rename $\frac{7}{3}$ as $\frac{6+1}{3} = \frac{6}{3} + \frac{1}{3} = 2 + \frac{1}{3}$, or $2\frac{1}{3}$

e. Rename $5\frac{1}{3}$ as $(4 + 1) + \frac{1}{3} = 4 + (1 + \frac{1}{3}) = 4\frac{1}{3}$

f. Rename $6\frac{11}{6}$ as $7\frac{5}{6}$

Some pupils encounter difficulty in regrouping a whole number, as in subtraction of fractional numbers. All regrouping the pupil did before was in base ten. Regrouping now involves expressing 1 as a fraction. It may be necessary for some pupils to write the different steps in the solution, as shown in item (3) above. The pupil who must write out each step of a solution operates at a low level of maturity in dealing with rational numbers. He should be able to eliminate the intermediate step in renaming 6 as $5\frac{3}{3}$.

Mixed fractional numerals classified according to denominators

There are three kinds of examples in subtraction of numbers named by mixed fractional numerals when regrouping is involved. The numerals may have like denominators, unlike but related denominators, and unlike and unrelated denominators. Example (a) contains two unlike but related denominators; (b) contains two unlike and unrelated denominators. By the time the pupil is ready to subtract in an example of either kind, he should not find it necessary to write all the intermediate steps to regroup the larger number, as $7\frac{1}{2}$ in (a) and $6\frac{1}{3}$ in (b). The mixed fractional numerals should have the fractional parts with like denominators. Example (a) would then be expressed as:

a $7\frac{1}{2} - 4\frac{5}{6}$

b $6\frac{1}{3} - 3\frac{1}{2}$

$$\begin{array}{r} 7\frac{1}{2} \\ -4\frac{5}{6} \\ \hline \end{array} = \begin{array}{r} 7\frac{3}{6} \\ 4\frac{5}{6} \\ \hline \end{array} = \begin{array}{r} 6\frac{9}{6} \\ 4\frac{5}{6} \\ \hline \end{array}$$

Subtracting numbers named by mixed fractional numerals

The example $7\frac{1}{4} - 5\frac{3}{4}$ illustrates subtraction of two numbers named by mixed fractional numerals. The $7\frac{1}{4}$ must be regrouped before the numbers can

be subtracted by the decomposition method. Before introducing an example of this kind, the class should subtract in an example of the type $1\frac{1}{4} - \frac{3}{4}$. Figure 13.21 is a visual representation of this example. The steps involved are as follows: (A) shows $1\frac{1}{4}$ regrouped as $\frac{5}{4}$; (B) shows $\frac{3}{4}$ subtracted from $\frac{5}{4}$; (C) shows that the answer is $\frac{2}{4}$, or $\frac{1}{2}$.

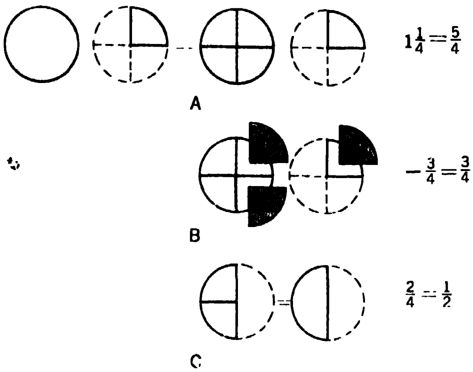


Figure 13.21

The symbolic representation of the subtraction operation is as follows:

$$1[illegible] = 1 + [illegible] \quad [illegible]$$

The representation shown is a long procedure. As the pupil becomes familiar with the operation, he can short cut many of the written steps and think $1\frac{1}{4}$ as $\frac{5}{4}$. If the introductory work with a new procedure is written in full as shown, the need for exploratory and visual materials is greatly reduced. The number line may be preferable (see Fig. 13.22).

Similarity to early work with whole numbers should also be stressed:

31	30 + 1	20 + 11
− 8	− 8	− 8
		20 + 3 = 23

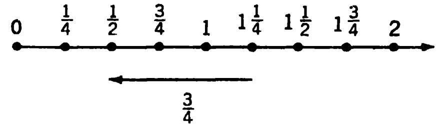


Figure 13.22

The example $1\frac{1}{4} - \frac{3}{4}$ can be subtracted by the equal-additions method by changing $1\frac{1}{4}$ to $1\frac{5}{4}$ and $\frac{3}{4}$ to $1\frac{3}{4}$. The transformation is difficult to rationalize. For that reason, the method of equal additions is not recommended for a modern program.

As soon as the pupil understands how to subtract in an example of the type $1\frac{1}{3} - \frac{2}{3}$, he should experience little difficulty in learning to subtract two numbers of the type $6\frac{1}{3} - 2\frac{2}{3}$. For introductory work, the symbolic representation of the solution should be written as follows:

$$\begin{array}{r} 6[illegible] \\ -2[illegible] \\ \hline \end{array} = \begin{array}{r} 6 + [illegible] \\ -2 + [illegible] \\ \hline \end{array} = \begin{array}{r} (5 + 1) + [illegible] \\ -\quad 2 + [illegible] \\ \hline \end{array}$$

$$= \begin{array}{r} 5 + ([illegible] + [illegible]) \\ -2 + [illegible] \\ \hline \end{array} = \begin{array}{r} 5 + [illegible] \\ -2 + [illegible] \\ \hline 3 + [illegible] = 3[illegible] \end{array}$$

The solution shown illustrates the following properties of number or of an operation:

1. Expanded notation for a numeral, as $6\frac{1}{3} = 6 + \frac{1}{3}$

2. Renaming a number, as $6 = 5 + 1$ or $1 = \frac{3}{3}$

3. The associative property of addition, as $4 + (1 + \frac{1}{3}) = (4 + 1) + \frac{1}{3}$. A fourth property not shown by the sample involves the identity element for multiplication. This property is used in expressing an answer in simplest form, as $3\frac{2}{4} = 3\frac{1}{2}$.

EXERCISES

1. State two needs that were met by expanding the number system to include fractions. Make a Venn diagram to show the subsets of the set of rational numbers.
2. A teacher remarked as follows: "If a pupil writes out all the steps involved in an operation, supplementary aids are not necessary." Evaluate this statement.
3. Give problems to illustrate the different concepts conveyed by a fraction.
4. Give six illustrations to show how a square may be divided into eighths that represent congruent regions.
5. Show how fractional numbers may be used effectively in teaching inequalities.
6. Describe the different levels of maturity in the renaming of fractions.
7. Show how the identity element of 1 is applied in changing a mixed fractional numeral to a fractional numeral.
8. Identify all the properties of addition that are applied in finding the sum of $2\frac{7}{8}$ and $5\frac{3}{8}$.
9. Evaluate the different plans that may be used to find a common denominator of fractions having unlike and unrelated denominators.
10. Use the factor method and find the lowest common denominator of fractions having denominators of:
    a. 8, 12, and 15 b. 6, 10, and 18
11. Write the four examples that use the rational numbers in the following sets:
    A: $\{\frac{1}{2}, \frac{1}{3}, \frac{5}{6}\}$ *B:* $\{\frac{2}{3}, \frac{1}{2}, \frac{1}{6}\}$

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MULTIPLICATION AND DIVISION OF RATIONAL NUMBERS

Chapter 13 demonstrated that the pattern for addition of whole numbers does not apply in full for addition of rational numbers. The pattern for multiplication of these two types of numbers is also different. The pupil learned that 3×5 may be expressed as $5 + 5 + 5$. Multiplication of whole numbers is a short form of addition of equal addends. This definition will not apply to rational numbers. In the example $\frac{3}{4} \times 5$, the 5 is not added $\frac{3}{4}$ times. In the example $\frac{2}{3} \times \frac{4}{5}$, it is not possible to express multiplication as the product of an array. The product of the numerators (2×4) and the product of the denominators (3×5) may be interpreted as arrays, but not the product of $\frac{2}{3}$ and $\frac{4}{5}$.

Two ways of finding the product of two whole numbers are by repeated addition and by an array. Neither of these methods may be used to multiply two rational numbers. However, it is necessary to devise a method of multiplying two rational numbers.

The relationship between multiplication and division holds for all types of numbers. Each operation is the inverse

of the other. Thus once the pattern is established for multiplying two rational numbers, the pattern for division will be the inverse or undoing procedure.

This chapter discusses the following topics: multiplying a whole number and a rational number; multiplying two rational numbers, dividing a whole number and a rational number; three types of problems involving rational numbers; properties of whole numbers and rational numbers.

The term "rational number" as used in this chapter refers only to the nonnegative rational numbers.

MULTIPLYING A WHOLE NUMBER AND A RATIONAL NUMBER

Types of examples in multiplication of rational numbers

There are three types of examples in multiplication of rational numbers (fractional numbers).

1. Multiplying a fractional number by a whole number, as $3 \times \frac{3}{4}$
2. Finding a fractional part of a number, as $\frac{2}{3} \times 6$ or $\frac{2}{3}$ of 6
3. Multiplying a fractional number by a fractional number, as $\frac{1}{2} \times \frac{3}{4}$.

The first and second types are the same from a mathematical point of view, since multiplication of rational numbers is commutative. Therefore the order of the factors does not affect the product, hence $3 \times \frac{3}{4} = \frac{3}{4} \times 3$. In verbal problems or statements, however, there is a difference in usage between the two types of examples. The symbolic representation, or *arithmetic expression*, $3 \times \frac{3}{4}$ indicates that $\frac{3}{4}$ is to be used as an addend three times. The expression $\frac{3}{4} \times 3$ implies that $\frac{3}{4}$ of 3 is to be found. Two verbal problems will help to differentiate between the types.

A ribbon is $\frac{3}{4}$ yard long. What will be the length of three of these ribbons?

A ribbon is 3 yards long. What will be the length of a piece $\frac{3}{4}$ as long?

The answer to the first problem can be found by finding the sum of $\frac{3}{4} + \frac{3}{4} + \frac{3}{4}$. Since the addends are the same, the answer can also be found by multiplication. The second problem implies that a length of 3 yards of ribbon is to be divided into 4 equal parts and three of these parts are to be considered. Finding a fractional part of a number illustrates partitive division. Although the algorisms for finding the answers to the two problems are the same, the situations represented are different. The two forms should be taught simultaneously because the same numbers are involved in both problems. Both problems involve *n* groups of *m* objects. The first is 3 groups of $\frac{3}{4}$ yard and the second is $\frac{3}{4}$ of a group of 3 yards.

A ribbon is $\frac{3}{4}$ yard long. How much ribbon is needed for two pieces of this length?

The teacher should have the class discover ways to find the answer. The following are some of the procedures the class should suggest:

1. Measure a string or ribbon $\frac{3}{4}$ yard long and find the length of two of these pieces.
2. Use fractional cutouts (Fig. 14.1) to find the answer.
3. Find the answer from a number ray (Fig. 14.2).

2 – 1 1

4. Find the answer by addition.
5. Think, "$\frac{3}{4}$ yard is $\frac{1}{4}$ less than 1 yard. The two pieces of $\frac{3}{4}$ yard each would be $\frac{1}{2}$ yard less than 2 yards, or $1\frac{1}{2}$ yards."
6. Think, "$\frac{3}{4}$ yard is equal to $\frac{1}{2}$ yard and $\frac{1}{4}$ yard. $\frac{1}{2} + \frac{1}{2} = 1$; $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$; 1

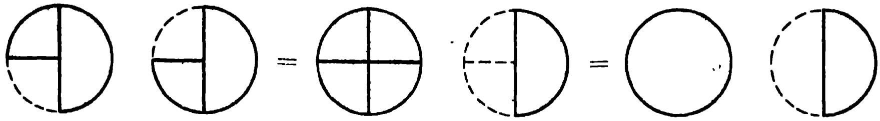

Figure 14.1

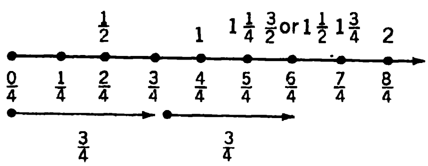

Figure 14.2

+ $\frac{1}{2}$ = $1\frac{1}{2}$. The sum of $\frac{3}{4}$ yard and $\frac{3}{4}$ yard is $1\frac{1}{2}$ yards."

The teacher should have the entire class perform the activities called for in items (2–4). The class should also read and explain each step given in the development in the textbook. Slow learners should also find the answer to the problem by measurement, as suggested in item (1).

The teacher should then show the conventional notation for multiplying a rational number by a whole number. The algorism is as follows:

a $2 \times \frac{3}{4} = \frac{3}{4} + \frac{3}{4} = \frac{3+3}{4}$

$= \frac{2 \times 3}{4} = \frac{6}{4} = \frac{4}{4} + \frac{2}{4} = 1\frac{1}{2}$

b $2 \times \frac{3}{4} = \frac{2 \times 3}{1 \times 4} = \frac{6}{4}$

$= \frac{4}{4} + \frac{2}{4} = 1\frac{1}{2}$

c $2 \times \frac{3}{4} = \frac{2 \times 3}{4} = \frac{6}{4} = 1\frac{2}{4}$, or $1\frac{1}{2}$

d $2 \times \frac{3}{4} = \frac{6}{4} = 1\frac{2}{4}$, or $1\frac{1}{2}$

Example (a) shows that multiplying by a whole number is equivalent to repeated addition. In example (b), 2 is renamed as $\frac{2}{1}$ and then each factor is a fraction. The product of the factors in the numerator is the numerator of the fraction in the product; the product of the factors in the denominator is the denominator of the fraction in the product. The fraction in the product is $\frac{6}{4}$, which may be expressed a $1\frac{2}{4}$, or $1\frac{1}{2}$. The factors 2×3 and 1×4 are whole numbers, hence the order of the factors does not affect the product. This fact shows that multiplication of rational numbers is commutative. Now the pupil writes the example $\frac{3}{4} \times 2$ as follows:

$$\frac{3}{4} \times 2 = \frac{3 \times 2}{4 \times 1} = \frac{6}{4} = 1\frac{2}{4}, \text{ or } 1\frac{1}{2}$$

In initial work in multiplying a fractional number and a whole number, the solutions represented by (a) and (b) are recommended. As the pupil's understanding of the work increases, solutions (c) and (d) are recommended. After he discovers the pattern for finding the product of a rational number and a whole number, the pupil should generalize as follows:

1. Rename the whole number with a fractional numeral with a denominator of 1. Then find the product of the numerators and the product of the denominators.
2. Express the product in (1) in standard form. The generalization in 1 may be shortened as follows:
3. Multiply the numerator by the whole number and retain the given denominator in the product. Statement (2) remains unchanged.

Multiplication of a whole number and a unit fraction can be performed on the basis of the renaming discussed above.

$$2 \times \frac{1}{5} = \frac{2}{5} \qquad 3 \times \frac{1}{7} = \frac{3}{7} \qquad c \times \frac{1}{n} = \frac{c}{n}$$

The product of $2 \times \frac{3}{8}$ may then be obtained as follows:

$2 \times \frac{3}{8} = 2 \times (3 \times \frac{1}{8})$	Rename $\frac{3}{8}$
$= (2 \times 3) \times \frac{1}{8}$	Associative property
$= 6 \times \frac{1}{8}$	Rename 2×3
$= \frac{6}{8}$ (or $\frac{3}{4}$)	Rename $6 \times \frac{1}{8}$

In a similar manner:

$a \times \frac{b}{c} = a \times (b \times \frac{1}{c})$	Rename $\frac{b}{c}$
$= (a \times b) \times \frac{1}{c}$	Associative property
$= \frac{a \times b}{c}$	Rename $(a \times b) \times \frac{1}{c}$

At this stage, renaming sessions should stress the following:

Rename $\frac{3}{4}$ as $3 \times \frac{1}{4}$
Rename $5 \times \frac{1}{3}$ as $\frac{5}{3}$
Rename $5 \times \frac{2}{7}$ as $\frac{5 \times 2}{7}$
Rename $4 \times 3\frac{1}{2}$ as $4 \times \frac{7}{2}$ *and* $4 \times 3 + 4 \times \frac{1}{2}$
Rename $2 \div 3$ as $\frac{2}{3}$
Rename $\frac{3}{4}$ as $3 \div 4$
Rename $\frac{7}{4}$ as $\frac{4+3}{4}$ and then as $\frac{4}{4} + \frac{3}{4}$, or $1\frac{3}{4}$

"Of" is associated with multiplication

The phraseology for reading the expression $2 \times \frac{3}{4}$ is "two *times* three fourths." The reading for $\frac{3}{4} \times 2$ is "three fourths *times* two" or "three fourths *of* two." In Figure 14.3, (A) shows that the product of $2 \times \frac{3}{4}$ is $\frac{3}{2}$; (B) shows that the product of $\frac{3}{4}$ of 2 is $\frac{3}{2}$. Each diagram uses congruent regions to show that the two answers are the same. We read the sign $\times$ in the first expression as *times* and in the second expression as *of* or *times*. Rational numbers are commutative for multiplication, and the models show that $2 \times \frac{3}{4} = \frac{3}{4} \times 2$.

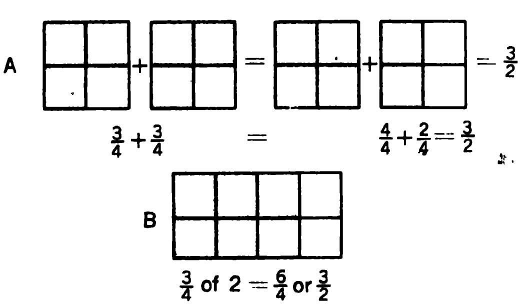

Figure 14.3

Is the answer sensible?

The teacher should be careful to keep the work in multiplication of fractional numbers from becoming mechanical. It is possible for a pupil to develop skill in multiplying a fractional number by a whole number and not know whether the answer is sensible. In the example $3 \times \frac{3}{4}$, the pupil may give the solution

$$3 \times \frac{3}{4} = 3 \times \frac{3}{4} = \frac{9}{4}, \text{ or } 2\frac{1}{4}$$

To show that the solution is sensible, the pupil should think as follows: "The number $\frac{3}{4}$ is more than $\frac{1}{2}$ but less than 1. $3 \times \frac{1}{2} = \frac{3}{2}$, or $1\frac{1}{2}$, and $3 \times 1 = 3$. Therefore the product of 3 and $\frac{3}{4}$ must be more than $1\frac{1}{2}$ but less than 3. Since $2\frac{1}{4}$ is between these two numbers, the answer is sensible."

Most pupils who have the necessary background to deal with multiplication of fractional numbers are able to multiply by $\frac{1}{2}$ or 1. In almost all cases the pupil can multiply these numbers mentally. Most pupils at the grade at which the topic is taught can find the product of 8 and $\frac{1}{2}$ without a written solution. It should never be necessary to write the work to find the product of a number and 1. To provide a check to see if an answer is sensible, round off a fractional number less than 1 either as $\frac{1}{2}$ or as 1. Assign one of these values to the rational number, depending upon the value of the given fraction. Then multiply by using the rational number having its assigned value. The pupil can then determine if a fractional number has a value greater or less than $\frac{1}{2}$ by dividing the denominator by 2 and comparing that quotient with the numerator of the fraction. Thus $\frac{4}{9}$ is less than $\frac{1}{2}$ because 4 is less than half of 9.

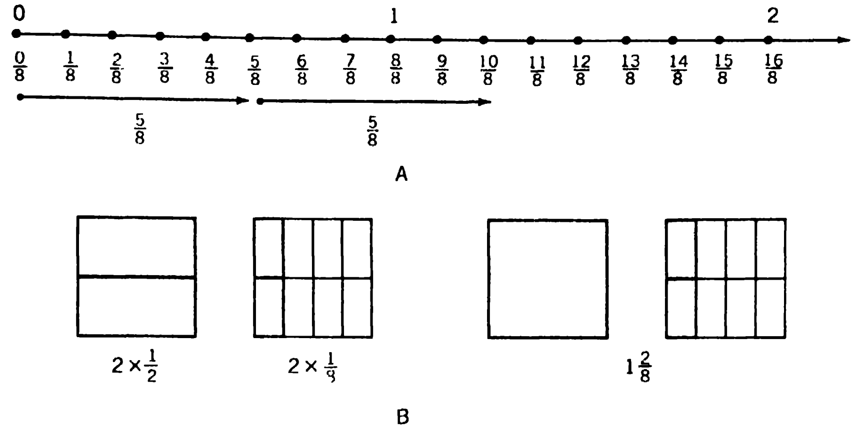

Figure 14.4

The thought pattern for determining whether the answer to an example involving the multiplication of a fractional number and a whole number is sensible is illustrated in the following examples:

a. $3 \times \frac{7}{8} = 2\frac{5}{8}$ Think: "$\frac{7}{8}$ as 1, the product must be a little less than 3. (3×1)"

b. $4 \times \frac{2}{5} = 1\frac{3}{5}$ Think: "$\frac{2}{5}$ as $\frac{1}{2}$, the product must be a little less than 2. $(4 \times \frac{1}{2})$"

c. $\frac{3}{5} \times 6 = 3\frac{3}{5}$ Think: "$\frac{3}{5}$ as $\frac{1}{2}$; the product must be a little more than 3 $(\frac{1}{2} \times 6)$"

d. $\frac{5}{6} \times 8 = 6\frac{4}{6}$ Think: "$\frac{5}{6}$ as 1; the product must be less than 8 but more than 4. (1×8) or $(\frac{1}{2} \times 8)$"

The plan of checking to determine if an answer is sensible is strongly recommended for superior pupils.

Distributive property with fractional numbers

We found that the commutative property of multiplication applies to that operation when the factors are a fractional number and a whole number.[1] The distributive property of multiplication over addition applies to these operations when dealing with a fractional number and a whole number. A few illustrations will demonstrate that this assumption is reasonable. The teacher should have the class use diagrams or some other type of model to show that this property applies to these numbers. In Figure 14.4 (A) and (B) show the product of $2 \times (\frac{1}{2} + \frac{1}{8})$. The number ray in (A) shows $2 \times \frac{5}{8} = \frac{10}{8}$. In this case the number named by $\frac{1}{2} + \frac{1}{8}$ is expressed as $\frac{5}{8}$. Diagram (B) shows that $2 \times \frac{1}{2} + 2 \times \frac{1}{8} = 1\frac{2}{8}$, or $\frac{10}{8}$. Examples of this kind should illustrate that the distributive property of multiplication with respect to addition is applicable.

The superior pupil should be encouraged to devise methods of checking his work in multiplication of rational numbers. The use of the distributive property of multiplication over addition and subtraction affords a good check on the computation. To show that $3 \times \frac{3}{4} = \frac{9}{4}$, the pupil can rename $\frac{3}{4}$ as $(\frac{2}{4} + \frac{1}{4})$ or as $(\frac{4}{4} - \frac{1}{4})$ and then multiply as shown in (a) and (b).

a. $3 \times (\frac{2}{4} + \frac{1}{4}) = \frac{6}{4} + \frac{3}{4} = \frac{9}{4}$

b. $3 \times (\frac{4}{4} - \frac{1}{4}) = \frac{12}{4} - \frac{3}{4} = \frac{9}{4}$

[1]A whole number is a rational number. The whole number 2 is the quotient of 2 and 1. The discussion here deals with the special case in which one of the rational numbers is also a whole number.

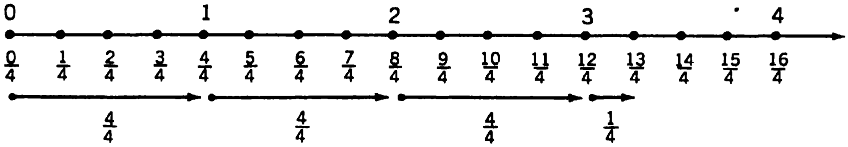

Figure 14.5

The pupil follows the same pattern in similar examples to check the solution. An exercise of this kind serves two functions. First, it provides a good check on the work, and second, it enables a pupil to discover relationships among numbers and to apply the properties of the operations.

Multiplying a number named by a mixed fractional numeral and a whole number

There are two effective ways to multiply a whole number and a number named by a mixed fractional numeral.[2] One method is to rename the number with a fraction[3] and then multiply. The second way is to express the number in long form as the sum of two addends and then multiply by the whole number. The example $2 \times 3\frac{1}{5}$ may be expressed as $2 \times (3 + \frac{1}{5})$. Performing the indicated operations illustrates the distributive property.

In order to multiply by the first procedure, it is necessary to rename the number with a fraction. The steps in changing a mixed numeral to a fraction are as follows:

1. Use cutouts to find the fractional equivalents of such numbers as $1\frac{1}{2}$, $1\frac{2}{3}$, and $1\frac{3}{4}$.
2. Use a number ray (Fig. 14.5) to show how the same number may have different names. Thus, $3\frac{1}{4}$ and $\frac{13}{4}$ are different numerals for the same number.
3. Write the example in long form for changing a mixed numeral to a fraction. To express $3\frac{1}{4}$ in fractional form, proceed as follows:

$$\begin{aligned} 3\tfrac{1}{4} &= 3 + \tfrac{1}{4} = (3 \times 1) + \tfrac{1}{4} \\ &= (3 \times \tfrac{4}{4}) + \tfrac{1}{4} \\ &= \tfrac{12}{4} + \tfrac{1}{4} = \tfrac{13}{4} \end{aligned}$$

The method of changing a mixed numeral to a fraction illustrates the use of

[2]The term "mixed number" has been used traditionally to refer to a number and a numeral. Which usage is intended must be determined by the context as illustrated by the following:

1. Multiply a mixed number by a whole number. In this common expression the term "mixed number" refers to a number because numbers (not numerals) are multiplied.
2. Change the mixed number to a fraction (fractional numeral). In this example the term "mixed number" refers to a mixed fractional numeral (sometimes called a mixed numeral), since a number cannot be changed to a numeral.

In this chapter the term "mixed number" will be used as an abbreviation for "rational number represented by a mixed numeral." In this context every mixed number is a rational number, but a rational number is referred to as a mixed number only when represented by a mixed numeral. Every mixed number, as $3\frac{1}{4}$, may be renamed by replacing its mixed numeral by a fractional numeral, as $\frac{13}{4}$. The term "improper fraction" is no longer in general use as a description for $\frac{13}{4}$. Some modern programs no longer use the term "mixed number."

As used in this chapter, mixed numbers may be added, multiplied, or divided, but they cannot be changed to fractions (fractional numerals). Mixed numerals may be changed to fractional numerals but cannot be added.

[3]In Chapter 13 care was taken to distinguish between a fractional number and its fractional numeral. In this chapter the common convention of using "fraction" as an abbreviation for "fractional numeral" is used. The reader should understand that some other programs use "fraction" as an abbreviation for "fractional number." It is not important which usage is accepted for a given program, but it is important that the usage be consistent. If confusion results, it is probably best to use no abbreviations and refer only to fractional numbers and fractional numerals.

the identity element of 1. In the illustration, 1 is expressed as $\frac{4}{4}$. The expression $3\frac{1}{4} = 3 + \frac{1}{4} = 3 \times 1 + \frac{1}{4} = 3 \times \frac{4}{4} + \frac{1}{4}$ also illustrates the order of performing the operations. The multiplication operation with the numeral $3 \times \frac{4}{4} + \frac{1}{4}$ is performed before the operation of addition. To the product of $3 \times \frac{4}{4}$, or $\frac{12}{4}$, add $\frac{1}{4}$, giving a sum of $\frac{13}{4}$. Thus the two numerals $3\frac{1}{4}$ and $\frac{13}{4}$ represent the same number.

When a mixed number is greater than 10, the usual plan to multiply by a whole number is to apply the distributive property. Example (a) shows a conventional notation for multiplying a mixed number by a whole number:

$$\begin{array}{lrl} \text{a} & 15\frac{3}{4} & \\ & \times\ 6 & \\ \hline & 4\frac{1}{2} & (6 \times \frac{3}{4}) \\ & 90 & (6 \times 15) \\ \hline & 94\frac{1}{2} & \end{array}$$

In example (a), first multiply $\frac{3}{4}$ by 6 and then 15 by 6. The sum of the two partial products is the product. Although the vertical notation in (a) is different from the horizontal notation in (b), the procedure used is the same:

$$\begin{array}{ll} \text{b} & 6 \times (15 + \frac{3}{4}) = \\ & 6 \times 15 + 6 \times \frac{3}{4} = \\ & 90 + 4\frac{1}{2} = 94\frac{1}{2} \end{array}$$

The teacher should have the pupil tell why the order of finding the products could be interchanged. In (a) the product 90 may be written first and followed by the product $4\frac{1}{2}$. Similarly, the products of the addends in (b) may be interchanged because the commutative property applies to addition and multiplication.

Example (c) shows the vertical notation for multiplying a whole number by a mixed number. Example (d) shows the horizontal notation for expressing the same set of factors. Just as $3 \times \frac{3}{8}$ and $\frac{3}{8} \times 3$ are the same from the mathematical viewpoint, so $3 \times 15\frac{1}{2}$ and $15\frac{1}{2} \times 3$ are the same.

$$\begin{array}{lrl} \text{c} & 38 & \\ & \times\ 3\frac{3}{4} & \\ \hline & 28\frac{1}{2} & (\frac{3}{4} \times 38) \\ & 114 & (3 \times 38) \\ \hline & 142\frac{1}{2} & \end{array}$$

$$\begin{array}{ll} \text{d} & 38 \times (3 + \frac{3}{4}) = \\ & 38 \times 3 + 38 \times \frac{3}{4} = \\ & 114 + 28\frac{1}{2} = 142\frac{1}{2} \end{array}$$

Each expression may represent a different social application of number, but both expressions represent the same mathematical situation of finding the product of two factors. The pupil should be taught to use the notation that makes the computation easier for him.

The usual classroom procedure for multiplying a mixed number and a whole number when each factor is less than 10 is to change the mixed numeral to a fraction. Thus, to find the product of $3 \times 5\frac{1}{2}$, the procedure to use is as follows:

$$3 \times 5\frac{1}{2} = 3 \times \frac{11}{2} = \frac{33}{2}, \text{ or } 16\frac{1}{2}$$

This form is satisfactory provided the pupil understands that the same principle which applies to an example of the type $6\frac{2}{3} \times 25$ also applies to an example of the type $3 \times 5\frac{1}{2}$. Examples (e) and (f) show the horizontal and vertical notations for finding the product without changing the mixed numeral to a fraction.

$$\begin{array}{lrl} \text{e} & 5\frac{1}{2} & \\ & \times\ 3 & \\ \hline & 1\frac{1}{2} & (3 \times \frac{1}{2}) \\ & 15 & (3 \times 5) \\ \hline \end{array}$$

$$\begin{array}{ll} \text{f} & 3 \times (5 + \frac{1}{2}) = \\ & 15 + 1\frac{1}{2} \end{array}$$

The size of the number is not a completely satisfactory guide for deciding whether to use a mixed numeral or fraction; for example,

$$\begin{aligned} 4 \times 5\tfrac{1}{2} &= 4 \times 5 + 4 \times \tfrac{1}{2} \\ &= 20 + 2, \text{ or } 22 \end{aligned}$$

In this case use of the distributive property is probably preferable even though $5\frac{1}{2}$ is less than 10.

Is the answer sensible?

The pupil should check to determine if the product of a whole number and a mixed number is sensible. This is especially true for the more able pupil. He should round off the mixed number both upward and downward. The products of the whole number and each rounded number represent the *limits* of the product of the given factors. In the example $7 \times 23\frac{2}{3}$, the number formed by rounding off $23\frac{2}{3}$ upward is 24 and by rounding off $23\frac{2}{3}$ downward the number formed is 23. Therefore the products of 7×23 and 7×24 are the limits for the product of $7 \times 23\frac{2}{3}$. Since $23\frac{2}{3}$ is nearer to 24 than to 23, the answer to $7 \times 23\frac{2}{3}$ must be nearer to 168 than to 161. The product of $7 \times 23\frac{2}{3}$ is $165\frac{2}{3}$. An answer of $166\frac{1}{3}$ to the example would be incorrect but sensible. An answer of $171\frac{2}{3}$ to the example would be neither correct nor sensible.

MULTIPLYING TWO RATIONAL NUMBERS

Discovering the procedure

An adequate readiness program dealing with multiplication of rational numbers should enable the pupil to answer questions such as the following:

How much is $\frac{1}{2}$ of $\frac{1}{2}$?

How much is $\frac{1}{2}$ of $\frac{1}{3}$?

How much is $\frac{1}{2}$ of $\frac{2}{3}$?

The pupil can use his cutouts to find the answer in case he does not know, or he can use these aids to verify his answer. The conventional notation for finding the answers to the above problems is as follows:

$$\frac{1}{2} \text{ of } \frac{1}{2} = \frac{1}{2} \times \frac{1}{2} = \frac{1 \times 1}{2 \times 2} = \frac{1}{4}$$
$$\frac{1}{2} \text{ of } \frac{1}{3} = \frac{1}{2} \times \frac{1}{3} = \frac{1 \times 1}{2 \times 3} = \frac{1}{6}$$
$$\frac{1}{2} \text{ of } \frac{2}{3} = \frac{1}{2} \times \frac{2}{3} = \frac{1 \times 2}{2 \times 3} = \frac{2}{6} = \frac{1}{3}$$

The numerals $\frac{1}{2} \times \frac{1}{2}$ and $\frac{1}{4}$ represent the same number. Similarly, the numerals $\frac{1}{2} \times \frac{1}{3}$ and $\frac{1}{6}$ represent the same number and the numerals $\frac{1}{2} \times \frac{2}{3}$ and $\frac{1}{3}$ represent the same number. In each case the numeral on the right of the equal sign represents the product of two fractional numbers. The illustrations show that the product of two fractional numbers may be found as follows:

1. Write the product of the numerators as the numerator of the fraction in the answer.

2. Write the product of the denominators as the denominator of the fraction in the answer.

The procedure for finding the products in the illustrations follows the pattern the pupil used for multiplying a fractional number and a whole number. In the example $4 \times \frac{2}{9}$, the pupil would rename 4 as $\frac{4}{1}$ and multiply, as shown at the right.

$$\frac{4 \times 2}{1 \times 9} = \frac{8}{9}$$

The terms of the fraction in the product of the two rational numbers are the product of the numerators and the product of the denominators.

The pupil should be able to explain why the product of two fractional numbers, such as named by $\frac{1}{2}$ and $\frac{5}{8}$, is less than either factor. He should be able to give answers to questions of the following type:

1. Does interchanging the factors in the example $\frac{1}{2} \times \frac{5}{8}$ affect the product?
2. Is the value of the fraction in the product greater or less than either factor?
3. If either factor is multiplied by 1, what will be the product?
4. Since each factor is less than 1, will $\frac{5}{8}$ multiplied by a number less than 1 give a product more than or less than $\frac{5}{8}$?
5. Multiplying a fractional number by $\frac{1}{2}$ is the same as dividing by what number?

A pupil who knows how to multiply numbers expressed as unit fractions, such as $\frac{1}{2} \times \frac{1}{4}$, can discover how to multiply any two fractional numbers by applying the principle of regrouping of factors. We can illustrate the procedure by multiplying $\frac{2}{3}$ and $\frac{4}{5}$.

[illegible]	The factors
[illegible] 2 × [illegible] 4 × [illegible]	Renaming each factor
[illegible] (2 · [illegible]) (4 · [illegible])	Replacing $\frac{2}{3}$ and $\frac{4}{5}$ with their equals
[illegible] (2 [illegible] 1) · ([illegible])	Regrouping of factors has no effect on the product
[illegible] 8 [illegible] or [illegible]	Renaming numbers
[illegible]	Renaming numbers

The last step shows that the product of the numerators is equal to the numerator of the fraction in the answer and the product of the denominators is equal to the denominator of the fraction in the answer.

In general terms, $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ provided $b \neq 0$ and $d \neq 0$. The product $\frac{a \times c}{b \times d}$ shows that multiplication of any two rational numbers is commutative. Both the numerator $a \times c$ and the denominator $b \times d$ are the products of two whole numbers. Since the order of multiplying two whole numbers does not affect the product, $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$.

Slow learners may not be able to understand the sequence of steps given for multiplying two rational numbers. The teacher should use a diagram to show the product of any two fractions, for example, $\frac{a}{b} \times \frac{c}{d}$ (b and $d \neq 0$). A model will not demonstrate the mathematical reason for the procedure for multiplying two rational numbers. On the other hand, a diagram will show that the terms obtained by multiplying the numerators and the denominators are correct. To find the product of $\frac{2}{3} \times \frac{4}{5}$, use congruent regions, as shown in Figure 14.6. First, divide a rectangle into 5 congruent regions and shade 4 of them. Next, divide each region into 3 congruent regions to form 15 congruent regions. Now color 2 of each 3 shaded regions. The 8 regions in color show the number of congruent regions that are in $\frac{2}{3}$ of $\frac{4}{5}$ of the rectangle, hence the product of $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$. The same answer can be found by multiplying the numerators and the denominators as $\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$.

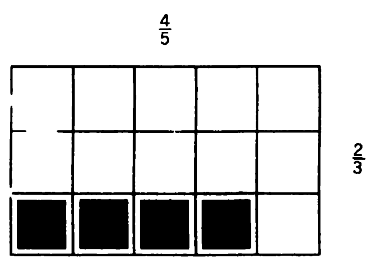

Figure 14.6

The use of a model does not constitute a proof, but a few illustrations of this kind will make it seem reasonable that the pattern of multiplying numerators and denominators applies to all rational numbers. A plan of this kind is satisfactory for pupils who are unable to understand a rigorous mathematical proof.

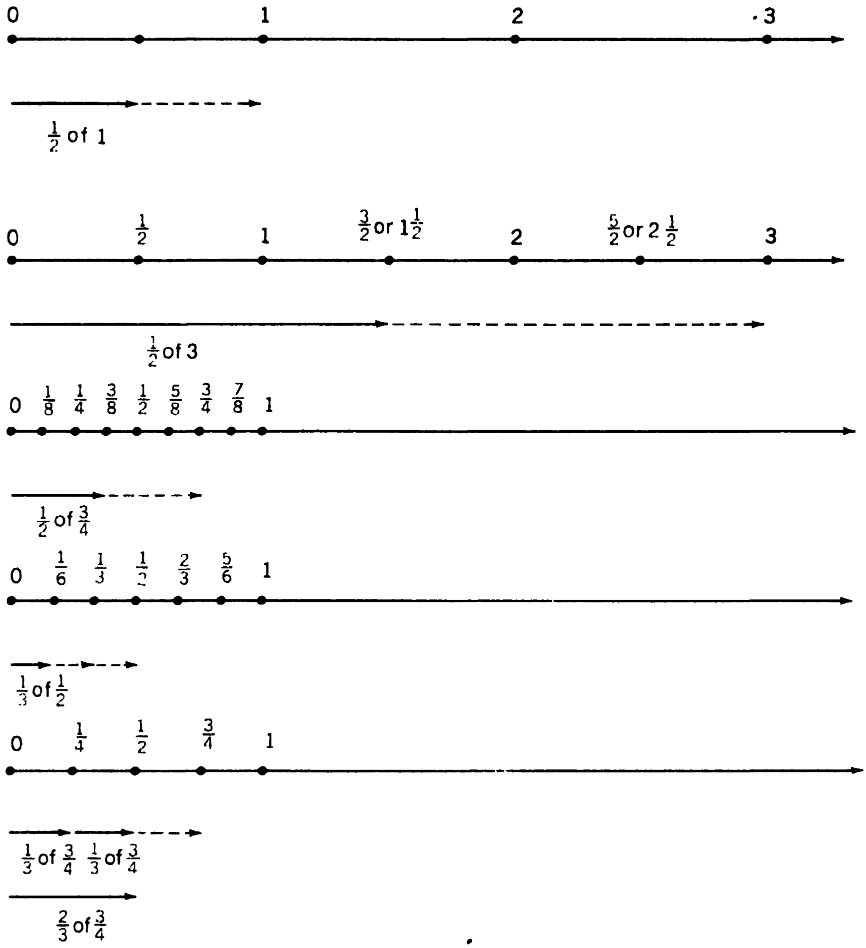

Figure 14.7

The number ray is also useful in visualizing multiplication of fractional numbers (see Fig. 14.7). To find an arrow one-half of a given segment, the length of the arrow must be such that two such arrows "add" to give the original segment. The shrinker idea may also be used (see p. 229).

Multiply or divide first?

Examples (a–c) show three ways to solve the example $\frac{2}{3} \times \frac{5}{6}$.

a $\frac{2}{3} \times \frac{5}{6} = \frac{2 \times 5}{3 \times 6} = \frac{10}{18}$, or $\frac{5}{9}$

b $\frac{\overset{1}{\cancel{2}}}{3} \cdot \frac{5}{\underset{3}{\cancel{6}}} = \frac{1}{3} \cdot \frac{5}{3} = \frac{5}{9}$

c $\frac{2}{3} \times \frac{5}{6} = \frac{2 \times 5}{3 \times 6} = \frac{2 \times 5}{3 \times 2 \times 3} = \frac{2}{2} \times \frac{5}{3 \times 3} = \frac{5}{9}$

In (a) the answer, $\frac{10}{18}$, is renamed in standard form as $\frac{5}{9}$. In (b) the product is renamed in standard form before the two given fractions are multiplied. This is an abbreviated form for renaming $\frac{2 \times 5}{3 \times 6}$, as indicated in (c).

The term "cancellation" has been used to designate the principle of renaming the product in standard form in

multiplication of rational numbers. Example (b) shows that cancellation is another way to indicate renaming of fractions. The teacher should not use the term in the classroom to designate the procedure illustrated in example (b). Rather, the pupil should discover that the procedure shown in (b) represents a different sequence of operations from that shown in (a).

It is important to recognize that (b) is the final adult form and should not be introduced until the sequence in (c) is fully understood. Rote performance of the form indicated in (b) can lead to serious mistakes and misunderstandings. One of the most common errors in early algebra is to "cancel" the x's in the expression $\frac{a+x}{b+x}$. Indiscriminate use of the word "cancel" and use of form (b) without adequate preparation contribute to such errors.

It may be of value to stress the following types of renaming when multiplication of two fractional numbers is being introduced:

Rename $\frac{6}{8}$ as $\frac{3}{4} \times \frac{2}{2}$ and then as $\frac{3}{4}$

Rename $\frac{6 \times 9}{15 \times 8}$ as $\frac{2}{2} \times \frac{3}{3} \times \frac{3 \times 3}{5 \times 4}$ and then as $\frac{9}{20}$

Rename $\frac{4}{7} \times \frac{5}{6}$ as $\frac{2}{2} \times \frac{2 \times 5}{7 \times 3}$ and then as $\frac{10}{21}$

Multiplying mixed numbers

Two procedures or methods may be used to introduce multiplication of mixed numbers. According to one plan, the mixed numerals are changed to fractions. Example (a) illustrates this procedure. This is the conventional algorism for multiplying two mixed numbers.

a. $2\frac{1}{3} \times 5\frac{3}{4} = \frac{7}{3} \times \frac{23}{4} = \frac{7 \times 23}{3 \times 4} = \frac{161}{12}$, or $13\frac{5}{12}$

According to the second plan for multiplying two mixed numbers, each mixed number is expressed as an indicated sum and then multiplied, as shown in example (b).

The two factors $2\frac{1}{3}$ and $5\frac{3}{4}$ would be written as follows:

$$2\tfrac{1}{3} \times 5\tfrac{3}{4} = (2 + \tfrac{1}{3}) \times (5 + \tfrac{3}{4})$$

According to the distributive property, each addend of the indicated sum $5 + \frac{3}{4}$ is to be multiplied by 2 and then by $\frac{1}{3}$, or each addend of the indicated sum $2 + \frac{1}{3}$ is to be multiplied by 5 and then by $\frac{3}{4}$. Therefore the above expression may be written as follows:

b $(2 + \frac{1}{3}) \times (5 + \frac{3}{4}) = 2 \times (5 + \frac{3}{4}) + \frac{1}{3} \times (5 + \frac{3}{4})$

$= 10 + 1\frac{1}{2} + 1\frac{2}{3} + \frac{1}{4}$,

or $13\frac{5}{12}$

or

$(2 + \frac{1}{3}) \times (5 + \frac{3}{4}) = (2 + \frac{1}{3}) \times 5 + (2 + \frac{1}{3}) \times \frac{3}{4}$

$= 10 + 1\frac{2}{3} + 1\frac{1}{2} + \frac{1}{4}$,

or $13\frac{5}{12}$

Understanding the steps given in (b) will enable the pupil to deal intelligently with the distributive property in algebra.

Multiplying three fractions

If three fractional numbers are to be multiplied, two of them must be grouped because the operation of multiplication is binary. The example $\frac{2}{3} \times \frac{3}{5} \times \frac{7}{9}$ may be written as $\frac{2 \times 3 \times 7}{3 \times 5 \times 9}$. Since both numerator and denominator contain whole numbers as factors, the properties of multiplication of whole numbers will apply to multiplication of fractional numbers. Therefore the associative property of multiplication applies to multiplication of rational numbers. The factors in the numerator may be grouped as $(2 \times 3) \times 7$ or as $2 \times (3 \times 7)$. Similarly, the factors in the denominator may be grouped. We may then conclude that the fractions may be grouped similarly, hence $(\frac{2}{3} \times \frac{3}{5}) \times \frac{7}{9} = \frac{2}{3} \times (\frac{3}{5} \times \frac{7}{9})$.

In general terms, $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = (\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f} = \frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f})$.

We found in dealing with multiplication of whole numbers that the way factors are arranged or regrouped does not affect the product. This generalization is a consequence of the commutative and associative properties. These same two properties of multiplication of whole numbers hold true for multiplication of rational numbers. The sequence of steps to show that rational numbers may be rearranged without affecting the product is as follows:

$(\frac{3}{8} \times 9) \times \frac{8}{11}$	The three factors
$\frac{3}{8} \times (9 \times \frac{8}{11})$	Associative property
$\frac{3}{8} \times (\frac{8}{11} \times 9)$	Commutative property
$(\frac{3}{8} \times \frac{8}{11}) \times 9$	Associative property
$\frac{3}{11} \times 9$	Renaming $\frac{3}{8} \times \frac{8}{11}$

An effective type of exercise for the more able pupil consists in finding the product of three or more factors without the use of paper and pencil (mental arithmetic). He should be able to make the groupings that will simplify the work. Such examples as follow are typical of the kind to use for this purpose.

a. $\frac{5}{3} \times \frac{7}{8} \times 1\frac{1}{5}$

b. $\frac{5}{6} \times \frac{3}{4} \times \frac{4}{3}$

c. $\frac{7}{8} \times 9 \times 1\frac{1}{7}$

For ease in computation, these factors should be regrouped as follows:

a. $(\frac{5}{3} \times 1\frac{1}{5}) \times \frac{7}{8}$

b. $\frac{5}{6} \times (\frac{3}{4} \times \frac{4}{3})$

c. $(\frac{7}{8} \times 1\frac{1}{7}) \times 9$

DIVIDING A WHOLE NUMBER BY A RATIONAL NUMBER

Kinds of examples in division of fractional numbers

There are only two types of examples in multiplication because of the commutative property of multiplication. Thus, $\frac{2}{3} \times 4$ and $4 \times \frac{2}{3}$ are the same from the standpoint of multiplication. Since division is not commutative, the corresponding examples in division are not the same as for multiplication. Therefore there are three types of examples in division of fractional numbers:

1. Dividing a whole number by a fractional number, as $4 \div \frac{2}{3}$
2. Dividing a fractional number by a whole number, as $\frac{2}{3} \div 4$
3. Dividing a fractional number by a fractional number, as $\frac{1}{2} \div \frac{2}{3}$.

Dividing a whole number by a unit fraction

A unit fraction has a numerator of 1. The pupil experiences little difficulty in dividing a whole number by a unit fraction. He intuitively knows the answer when a whole or two wholes are divided by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and so forth, and he can check the answer by using his cutouts. A number ray also provides a good check for examples of this type:

$1 \div \frac{1}{2} = 2$ $\quad 1 \div \frac{1}{3} = 3$ $\quad 1 \div \frac{1}{4} = 4$

$2 \div \frac{1}{2} = 4$ $\quad 2 \div \frac{1}{3} = 6$ $\quad 2 \div \frac{1}{4} = 8$

A number ray shows that the quotient of 2 divided by $\frac{1}{4}$ is 8 (see Fig. 14.8).

The pupil used a model of some type to find the quotient of a whole number divided by a unit fraction. He can check the work by multiplication because division is the inverse of multiplication. If $n \times \frac{1}{2} = 3$, then $\frac{1}{2} \times n = 3$. If half a number is 3, the number is 6. The statement $\frac{1}{2} \times 6 = 3$ is true, therefore the corresponding equation in division must be true. The pupil may also use repeated subtraction as indicated at the right. Four subtractions indicate there are four $\frac{1}{2}$'s in 2.

$$\begin{array}{r} 2 \\ -\frac{1}{2} \\ \hline 1\frac{1}{2} \\ -\frac{1}{2} \\ \hline 1 \\ -\frac{1}{2} \\ \hline \frac{1}{2} \\ -\frac{1}{2} \\ \hline 0 \end{array}$$

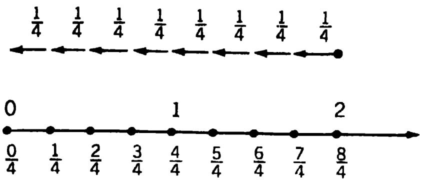

Figure 14.8

Every example in division may be changed to an example in multiplication. In any example in division the product of the divisor and the quotient is equal to the dividend. The equation $6 \div 2 = n$ may be expressed as $2 \times n = 6$. Similarly, the equation $2 \div \frac{1}{3} = n$ may be expressed as $\frac{1}{3} \times n = 2$. Therefore the quotient of every example having a fractional divisor can be found by solving the corresponding equation using multiplication. The pupil experiences little difficulty in solving an equation of the type $\frac{1}{2} \times n = 6$. If the divisor is not a unit fraction, as in the equation $27 \div \frac{3}{8} = n$, the solution of the corresponding equation using multiplication may be difficult for the pupil to understand. He must understand the mathematical basis for dealing with a fractional divisor.

Identity element for multiplication

Dividing by a unit fraction is an application of the identity element for multiplication. The pupil discovered that multiplying or dividing a whole number by 1 does not change the value of that number.

The number named by $\frac{2}{\frac{1}{4}}$ can be multiplied by 1 without changing the number. Therefore we may multiply $\frac{2}{\frac{1}{4}}$ by 1 and rename 1 as $\frac{4}{4}$. The work may then be indicated as follows:

$$2 \div \frac{1}{4} = \frac{2}{\frac{1}{4}} \times 1 = \frac{2}{\frac{1}{4}} \times \frac{4}{4} = \frac{2 \times 4}{\frac{1}{4} \times 4} = \frac{8}{1}, \text{ or } 8$$

In a similar manner the pupil can find the quotient in each example in which he found the answer by using models or by solving the corresponding example in multiplication. The use of the identity element in the solution of examples in division is illustrated as follows:

$$1 \div \frac{1}{2} = \frac{1}{\frac{1}{2}} \times 1 = \frac{1}{\frac{1}{2}} \cdot \frac{2}{2} = \frac{2}{1}, \text{ or } 2$$

$$2 \div \frac{1}{3} = \frac{2}{\frac{1}{3}} \times 1 = \frac{2}{\frac{1}{3}} \cdot \frac{3}{3} = \frac{6}{1}, \text{ or } 6$$

In each illustration the number is multiplied by 1 renamed so that the product in the denominators (divisors) of the fractions will be 1. The product of each pair of factors, such as $2 \times \frac{1}{2}$, $3 \times \frac{1}{3}$, and $4 \times \frac{1}{4}$ is 1. Two numbers that have a product of 1 are *multiplicative inverses* or *reciprocals*. Each number is called the reciprocal or inverse of the other. Every number except 0 has a reciprocal. One is the only number that is equal to its reciprocal, as $1 \times 1 = 1$. Since the reciprocal of a number is also the multiplicative inverse of that number, multiplying by the reciprocal of a number is the same as multiplying by the multiplicative inverse of that number.

The quotient in each of the illustrations was found by multiplying the dividend by the inverse of the divisor. The quotient of $2 \div \frac{1}{4}$ is equal to 4×2, or 8. The inverse or reciprocal of a unit fraction is the denominator of that fraction. Therefore dividing by a unit fraction is the same as multiplying by the denominator of that fraction. If a is any whole number and $\frac{1}{b}$ is any rational number, $a \div \frac{1}{b} = a \times \frac{b}{1}$, or $a \times b$.

The pupil learned that dividing by 2 is the same as multiplying by $\frac{1}{2}$. Conversely, dividing by $\frac{1}{2}$ is the same as multiplying by 2. The numeral for 2 may be expressed as $\frac{2}{1}$, hence the two

reciprocals named by $\frac{1}{2}$ and $\frac{2}{1}$ have their terms interchanged. Teachers frequently have designated the interchanging of the numerator and denominator of a fraction as *inverting the fraction.* Usually the pupil learned the following rule for dividing by a fraction: Invert the fractional divisor and multiply. Often he did not understand the procedure but was able to find the correct quotient when the divisor was a fraction. As long as emphasis was placed on computational proficiency, this procedure was satisfactory.

The teacher should not use the term "invert" to describe interchanging the terms of a fraction. The pupil must learn the essential vocabulary of mathematics. The term "reciprocal" or "inverse" expresses the relationship between two numbers that have a product of 1, such as $\frac{1}{4}$ and $\frac{4}{1}$ or $\frac{2}{3}$ and $\frac{3}{2}$. The correct generalization for division by a fractional divisor is: *multiply by the reciprocal (multiplicative inverse) of the divisor.* Of greater importance than terminology is the basic understanding that multiplying by the reciprocal of the divisor is an application of the identity property of 1. (See the discussion of the mathematical basis for division, p. 261.)

Dividing a whole number by any fraction

The pupil learned to divide a whole number by a unit fraction. The procedure for dividing by a unit fraction applies to any fractional divisor. The teacher should have the class engage in a variety of activities involving dividing by a fraction, as $\frac{a}{b}$. A problem of the following type may be used to introduce the topic: How many $\frac{2}{3}$ yard pieces can be cut from a string 4 yards long?

The class should suggest different ways to find the answer to this problem. Some of the ways mentioned should be the following:

1. Cut a string 4 yards long and measure off $\frac{2}{3}$ yard pieces.
2. Draw four circles and divide them into thirds and find how many $\frac{2}{3}$ parts there are in the four circles (see Fig. 14.9).

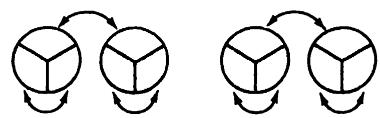

Figure 14.9

3. Find the answer from a number ray, as shown in Figure 14.10.

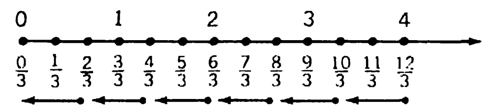

Figure 14.10

4. Make repeated subtractions of $\frac{2}{3}$ from 4, or add enough $\frac{2}{3}$'s to have a sum equal to 4.
5. Think: "$4 \div \frac{1}{3}$ is 12, hence 4 divided by $\frac{2}{3}$ will be half of 12, or 6."
6. Think: "$3 \times \frac{2}{3} = 2$, hence there are three $\frac{2}{3}$'s in 2. In 4 there will be twice as many $\frac{2}{3}$'s as in 2, or 6."

The class should perform either items (2) or (3) or both, as well as items (1) and (4).

The experiments that the class performs indicate that the answer to the problem is 6. This answer can be found by changing the example in division to an example in multiplication. The divisor $\frac{2}{3}$ in the division example is changed to the factor $\frac{3}{2}$ in the multiplication example.

It should be emphasized that this type of problem is basically no different from any other involving division by a fraction.

The teacher should help the pupils recognize that number sentences involving fractions belong in sets in the same way as do whole numbers. If $8 \times \frac{1}{2} = 4$, then $\frac{1}{2} \times 8 = 4$, $4 \div \frac{1}{2} = 8$, and $4 \div 8 = \frac{1}{2}$. In a similar manner, each of the multiplication examples listed on the left is a member of the same set of number sentences as its corresponding division example on the right.

$f_1 \times f_2 = p$	$p \div f_1 = f_2$
a $6 \times \frac{2}{3} = 4$	a′ $4 \div \frac{2}{3} = 6$
b $8 \times \frac{3}{4} = 6$	b′. $6 \div \frac{3}{4} = 8$
c. $12 \times \frac{5}{6} = 10$	c′. $10 \div \frac{5}{6} = 12$

[4] Most of the generalizations already learned pertaining to multiplication and division of fractional numbers apply to this situation. The teacher should have the class identify the following characteristics.

1. One factor in each multiplication example is a rational number less than 1 and the other factor is a whole number.
2. The product is less than the whole number.
3. The answer in item (2) is sensible because the product is the same as the whole number when the other factor is 1. Since this factor is less than 1, the product must be less than the whole number.
4. The dividend in division corresponds to the product in multiplication.
5. The divisor and quotient correspond to factors in multiplication.
6. The quotient of a whole number divided by a rational number less than 1 is greater than the number divided.
7. The answer in item (6) is sensible. When the divisor is 1, the quotient is the same as the dividend. Since the divisor is less than 1, the quotient must be greater than the dividend. Division is repeated subtraction. If the number subtracted is less than 1, the number of subtractions must be greater than the number divided to have a final remainder of 0.
8. Multiplication and division are inverse operations. If the product of a whole number and a fractional number is less than the whole number, then the quotient of the whole number divided by that fractional number must be greater than the given whole number.

The number of pupils who will be able to discover all of the generalizations enumerated is limited. Only the more capable learners will discover the last item. The teacher should be certain that the class is encouraged to make as many discoveries as possible about the two operations dealing with rational numbers.

Mathematical basis for division by a fractional number

The mathematical basis for dividing one fractional number by another fractional number involves four fundamental ideas:

1. Recognition of 1 as the multiplicative identity. This is one of the basic properties of a number field.
2. Every rational number except 0 has a multiplicative inverse (reciprocal), or for every $\frac{a}{b} \neq 0$, there is a $\frac{b}{a}$ such that $\frac{a}{b} \times \frac{b}{a} = 1$. This is also a basic property of a number field.
3. The product of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ac}{bd}$. This can be proved as a theorem based on field postulates, but it is not appropriate to do so on the elementary school level where it must be accepted on the basis of models such as the rectangles in Figure 14.3 or the number ray in Figure 14.7.

[4](a) and (a′), (b) and (b′), and (c) and (c′) each show two sentences of the four sentences in the multiplication-division pattern. The complete set for (a) and (a′) is $6 \times \frac{2}{3} = 4$, $\frac{2}{3} \times 6 = 4$, $4 \div \frac{2}{3} = 6$; $4 \div 6 = \frac{2}{3}$ (see p. 158).

4. The quotient $a \div b$ can be renamed as $\frac{a}{b}$ purely by definition. Pupils should be able to rename $\frac{2}{3}$ as $2 \div 3$ and $4 \div 5$ as $\frac{4}{5}$. In the same manner, pupils should be able to rename $\frac{2}{3} \div \frac{3}{4}$ as $\frac{\frac{2}{3}}{\frac{3}{4}}$ and finally to rename $\frac{a}{b} \div \frac{c}{d}$ as $\frac{\frac{a}{b}}{\frac{c}{d}}$.

With the above ideas, the quotient $\frac{2}{3} \div \frac{3}{4}$ may be obtained as follows:

$$\frac{2}{3} \div \frac{3}{4} = \frac{\frac{2}{3}}{\frac{3}{4}} = \frac{\frac{2}{3}}{\frac{3}{4}} \times 1 = \frac{\frac{2}{3}}{\frac{3}{4}} \times \frac{\frac{4}{3}}{\frac{4}{3}}$$

$$= \frac{\frac{2}{3} \times \frac{4}{3}}{\frac{3}{4} \times \frac{4}{3}} = \frac{\frac{2}{3} \times \frac{4}{3}}{1} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$$

Applying the same ideas in general, one obtains:

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} \times \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}}$$

$$= \frac{\frac{a}{b} \times \frac{d}{c}}{1} = \frac{a}{b} \times \frac{d}{c}$$

The above proves that in place of dividing a number by $\frac{c}{d}$, this number may be multiplied by $\frac{d}{c}$. The division $\frac{2}{3} \div \frac{3}{4}$ may also be performed as follows:

$$\frac{\frac{2}{3}}{\frac{3}{4}} = \frac{\frac{2}{3}}{\frac{3}{4}} \cdot \frac{12}{12} = \frac{\frac{2}{3} \times 12}{\frac{3}{4} \times 12} = \frac{8}{9}$$

The latter method is useful in early division of fractional numbers but does require the determination of the lowest common denominator to be efficient. In this method, 1 is renamed as $\frac{x}{x}$, where x is the lowest common denominator of the fraction in both the numerator and the denominator.

It can be shown that the following pattern, used frequently with whole numbers, also applies to rational numbers.

$a \times b = c$ $\quad\quad$ $c \div a = b$

$b \times a = c$ $\quad\quad$ $c \div b = a$

When a pupil determines that $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$, he should be encouraged to write the following pattern and to determine whether all of the sentences are true:

$\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$ $\quad\quad$ $\frac{2}{3} = \frac{3}{4} \times \frac{8}{9}$

$\frac{2}{3} \div \frac{8}{9} = \frac{3}{4}$ $\quad\quad$ $\frac{2}{3} = \frac{8}{9} \times \frac{3}{4}$

This pattern should be accepted and readily understood if sufficient use has been made of the corresponding pattern for whole numbers. Just as the pattern for whole numbers is useful in solving open sentences that involve multiplication and division, the above pattern is useful in dealing with open sentences involving multiplication and division of fractional numbers, as illustrated below:

If $\frac{3}{4} \times n = \frac{2}{3}$ $\quad\quad$ If $n \div \frac{3}{4} =$ [illegible]

Then $n = \frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$ $\quad\quad$ Then $n =$ [illegible]

Dividing a whole number by a mixed number

The pupil follows the same pattern for dividing in an example of the type $4 \div 2\frac{3}{4}$ that he would use to divide in the example $5 \div \frac{4}{1}$. The divisor $2\frac{1}{4}$ is expressed in fractional form as $\frac{11}{4}$. The example then becomes $5 \div \frac{11}{4}$, which involves dividing a whole number by a fractional number. The solution would be as follows:

$5 \div 2\frac{1}{4} = 5 \div \frac{11}{4}$

$5 \div \frac{11}{4} = 5 \times \frac{4}{11} = \frac{20}{11}$, or $1\frac{9}{11}$

Dividing a fractional number by a whole number

The pupil learned that finding half a number, as $\frac{1}{2}$ of 6, is the same as dividing 6 by 2. Similarly, finding a third of a number is the same as dividing the number by 3. If a fractional number is to be divided by a whole number, the situation is reversed. If multiplying a

number by $\frac{1}{2}$ is the same as dividing by 2, dividing a number by 2 is the same as multiplying by $\frac{1}{2}$. Dividing a rational number $\frac{a}{b}$ by a whole number n is the same as multiplying $\frac{a}{b}$ by $\frac{1}{n}$.

The pupil who does not understand the relationship just described as part of the basic pattern for division by a fractional number must be helped to discover how to divide a fractional number by a whole number. He knows the answers to examples of the kind that follow and he can verify the quotients by models.

$$\tfrac{1}{2} \div 2 = \tfrac{1}{4} \qquad \tfrac{1}{4} \div 2 = \tfrac{1}{8}$$
$$\tfrac{1}{2} \div 3 = \tfrac{1}{6} \qquad \tfrac{1}{3} \div 2 = \tfrac{1}{6}$$

Since dividing by 2 is the same as finding half a number, the examples shown can be solved as follows:

$$\tfrac{1}{2} \div 2 = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$$
$$\tfrac{1}{2} \div 3 = \tfrac{1}{2} \times \tfrac{1}{3} = \tfrac{1}{6}$$
$$\tfrac{1}{4} \div 2 = \tfrac{1}{4} \times \tfrac{1}{2} = \tfrac{1}{8}$$
$$\tfrac{1}{3} \div 2 = \tfrac{1}{3} \times \tfrac{1}{2} = \tfrac{1}{6}$$

The illustrations show that dividing by a whole number is the same as multiplying by the reciprocal of that number. Conversely, multiplying by a whole number is the same as dividing by the reciprocal or multiplicative inverse of that number.

What is the next fractional number?

We found that the operations are performed differently with whole numbers than with fractional numbers. Another difference between these two kinds of numbers depends upon their sequence. It is always possible to name the next larger whole number. If n is any whole number, the next larger whole number is $n + 1$. It is not possible to name the next larger fractional number in the set of rational numbers. We can illustrate by considering the fractions $\frac{1}{2}$ and $\frac{3}{4}$. The segment on a number ray between two points corresponding to $\frac{1}{2}$ and $\frac{3}{4}$ can be divided into two congruent parts.

There is always a point on a number ray that is midway between two points which correspond to two numbers. The number that corresponds to the midpoint on the line segment connecting two numbers is their average. The average of two numbers is their sum divided by 2. The point on a number ray that is midway between the points identified with $\frac{1}{2}$ and $\frac{3}{4}$ corresponds to the number $(\frac{2}{4} + \frac{3}{4}) \div 2 = \frac{5}{4} \times \frac{1}{2}$, or $\frac{5}{8}$. Similarly, there is a point on a number ray midway between the points identified with $\frac{1}{2}$ and $\frac{5}{8}$ that corresponds to the number $\frac{9}{16}$. In the same way it is possible to identify points that correspond to the numbers $\frac{17}{32}$, $\frac{33}{64}$, $\frac{65}{128}$, and the like. There is always another number between $\frac{1}{2}$ and the last designated number. Similarly, it may be shown that between any two rational numbers there is an infinite number of rational numbers. Mathematicians describe this property of rational numbers as *denseness*. We say that the set of rational numbers is dense.

The teacher should have the superior pupils demonstrate that there is no next rational number in the sense that there is a regular sequence in the set of whole numbers. The pupil who does well in this activity should find the number corresponding to the midpoint of the segment of a number ray between the points that correspond to two rational numbers, as $\frac{1}{2}$ and $\frac{2}{3}$. He should then continue the operation until he discovers the pattern for identifying the next number in a given sequence. Three benefits to be derived from an exercise of this kind are as follows:

1. The pupil receives practice in dividing the sum of two rational num-

bers by a whole number. An example of this kind illustrates the distributive property of division over addition in dealing with rational numbers.

2. The experience should enable the pupil to develop insight into the concept of denseness of rational numbers about a given point.

3. The pupil should discover the pattern for writing the terms of a particular number series.

Divisor and dividend a fractional number

Is the quotient sensible? If the divisor is a fractional number, the dividend may be a whole number, a fractional number that is less than 1 or greater than 1, or a mixed number. In each case the division operation is performed by multiplying by the inverse of the divisor. The pupil readily learns the algorism for dividing by a fractional number, but the work may be mechanical. It is important, therefore, for him to be able to determine whether the quotient of two rational numbers is sensible. The reasonableness of the quotient depends upon the pupil's understanding of the following three generalizations:

1. The quotient of a number divided by itself is 1, as $7\overline{)7}$.

2. The quotient of a number divided by a smaller number is greater than 1, as $6\overline{)24}$.

3. The quotient of a number divided by a larger number is less than 1, as $12\overline{)5}$.

These three principles apply to a fractional divisor as well as to a divisor that is a whole number. The pupil who understands these principles and the relative value of the two numbers divided should be able to generalize about the quotient, as shown in the following examples:

$\frac{3}{4} \div \frac{1}{2}$	The quotient is greater than 1 (Principle 2)
$\frac{2}{3} \div \frac{5}{6}$	The quotient is less than 1 (Principle 3)
$2\frac{1}{2} \div 3\frac{1}{4}$	The quotient is less than 1 (Principle 3)
$\frac{[illegible]}{6} \div \frac{[illegible]}{8}$	The quotient is 1 (Principle 1)

The ability to determine whether an answer is sensible enables the pupil to correct errors that otherwise would not be detected. In the example $\frac{2}{3} \div \frac{3}{4}$, the pupil may multiply by $\frac{3}{2}$ instead of $\frac{1}{3}$ because he interchanged (inverted) the terms of the wrong fraction. This is a common type of error. If the terms of the fraction $\frac{2}{3}$ are interchanged instead of the terms of the fraction $\frac{3}{4}$, the solution becomes $\frac{3}{2} \times \frac{3}{4} = \frac{9}{8}$, or $1\frac{1}{8}$. The pupil who knows that $\frac{2}{3}$ is less than $\frac{3}{4}$ and understands principle (3) discovers that the answer $1\frac{1}{8}$ is not sensible. He then looks for the source of the error in the solution.

An effective means of challenging the fast learner in division of rational numbers is to have him decide if the quotient will be equal to, less than, or greater than 1 before dividing in an example. The examples may be written in the following form. The pupil inserts in the circles the correct symbol, =, >, or <, that will make each mathematical sentence true.

[illegible] ÷ [illegible] ○ 1	[illegible] ÷ $\frac{[illegible]}{6}$ ○ 1	$\frac{4}{5} \div \frac{5}{[illegible]}$ ○ 1
$\frac{[illegible]}{4} \div \frac{[illegible]}{5}$ ○ 1	$\frac{[illegible]}{4} \div \frac{7}{10}$ ○ 1	$\frac{[illegible]}{8} \div \frac{[illegible]}{3}$ ○ 1

The pupil solves the examples to check his selection of symbols. This kind of work in division of rational numbers is intended primarily for the pupil of superior ability.

The common denominator method

The traditional way to divide by a fractional divisor was to invert the divi-

sor and multiply. Now it is conventional to use the more sophisticated phraseology "multiply by the reciprocal or multiplicative inverse of the divisor." Regardless of the phraseology, many pupils do not understand the mathematical basis of the operation.

A procedure that may be easier to understand than the methods described consists in renaming fractional numbers expressed with unlike denominators as fractional numbers expressed with like denominators and then divide numerators and denominators. A limited investigation by Brownell showed that slow learners preferred this method to the more conventional method.[5]

Another investigation by Capps compared the inversion method with the common denominator method. In a controlled experiment with approximately 560 pupils in the group using the common denominator method and a smaller number of pupils in the control group using the inversion method, the results from the two groups were not significantly different. Capps's concluding statement is appropriate in interpreting the results of the experiment: "One thing is evident. There are many facets of the two methods which need further investigation before either can be condemned or commended."[6]

According to the common denominator method, the numbers to be divided are expressed as having like denominators, as shown below. In multiplication both numerators and denominators are multiplied. Then in division both numerators and denominators are

$$\frac{1}{2} \div \frac{3}{8} = \frac{4}{8} \div \frac{3}{8} = \frac{4 \div 3}{8 \div 8} = \frac{4}{3}, \text{ or } 1 \text{ [illegible]}$$

[5]W. A. Brownell, "Two Kinds of Learning in Arithmetic," *Journal of Educational Research*, March 1938, 31:656–664.

[6]Lelon R. Capps, "Division of Fractions," *The Arithmetic Teacher*, January 1962, 9:16.

divided. Since the denominators are equal, the quotient of the denominators is 1. To divide fractions having like denominators, then, one divides the numerators.

The following method, used by many modern programs for dividing two fractional numbers, is a direct application of the identity property of 1 in multiplication:

$$\frac{2}{3} \div \frac{3}{4} = \frac{\frac{2}{3}}{\frac{3}{4}} = \frac{\frac{2}{3}}{\frac{3}{4}} \cdot \frac{12}{12} = \frac{8}{9}$$

The procedure is perfectly general and may be applied to any two fractions:

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} \times \frac{bd}{bd} = \frac{\frac{a}{b} \times bd}{\frac{c}{d} \times bd} = \frac{ad}{bc}$$

In this situation multiplication by 1 occurs in the form of multiplication by $\frac{bd}{bd}$ because bd is the lowest common denominator of the two fractions. Of course, we also know that to divide by a fraction we may multiply by its reciprocal. Thus:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Dividing mixed numbers

Mixed numbers are divided in the same way that fractional numbers are divided. The mixed numerals should be changed to fractions. The illustrations below show the form to be used in dealing with mixed numbers:

a 6 ÷ 2 [illegible] = 6 [illegible] = 6 [illegible], or 2 [illegible]

b 4 [illegible] 3 [illegible] 3 [illegible] or 1 [illegible]

c 1 [illegible] ÷ 3 [illegible]

d 3 [illegible] ÷ 1 [illegible], or $2\frac{4}{5}$

The division of mixed numbers may also be performed in a manner similar to that illustrated on page 262.

$$\frac{3\frac{1}{2}}{1\frac{1}{4}} \times \frac{4}{4} = \frac{(3 + \frac{1}{2}) \times 4}{(1 + \frac{1}{4}) \times 4} = \frac{12 + 2}{4 + 1} = \frac{14}{5}, \text{ or } 2\tfrac{4}{5}$$

Note that the distributive property is also used here.

THREE TYPES OF PROBLEMS INVOLVING RATIONAL NUMBERS

Relating fractions, decimals, and per cent

There are three types of problems in multiplication and division of fractional numbers. Problems representative of these patterns may use decimals or per cents. The familiar "three cases of per cent" illustrate the three kinds of problems in multiplication and division of fractional numbers.

The three uses of fractional numbers in multiplication and division are as follows:

1. Finding a fractional part of a number
2. Finding the ratio of two numbers
3. Finding a number when a fractional part of it is given.

The three uses of fractional numbers may be illustrated by the following problems:

1. A team won $\frac{4}{5}$ of the basketball games played. If the team played 20 games, how many games did it win?

$$n = \tfrac{4}{5} \text{ of } 20 = \tfrac{4}{5} \times 20 = 16$$

The team won 16 games.

2. A team played 20 games and won 16 of them. What part of the games played did the team win?

$$n \times 20 = 16, \text{ or } n = 16 \div 20 = \tfrac{4}{5}$$

The team won $\frac{4}{5}$ of its games.

3. A team won $\frac{4}{5}$ of the games it played. If the team won 16 games, how many games did it play?

$$\tfrac{4}{5} \text{ of } n = 16$$
$$\tfrac{4}{5} \times n = 16$$
$$n = 16 \div \tfrac{4}{5}$$
$$n = 16 \times \tfrac{5}{4}, \text{ or } 20$$

The team played 20 games.

In problem (3) the product of two factors and one factor are given. To find the factor n, divide the product by the known factor.

If .8 replaces the fraction in the above problems, the problems then represent the types of examples found in multiplication and division of decimals. If 80% replaces the fraction $\frac{4}{5}$, the problems represent the types of examples in per cent.

The numbers in the three problems may be arranged as shown in Table 14.1. In each problem two of the three numbers involved are known. The third number is expressed as a variable n.

TABLE 14.1
Problems with Open Sentences

	Fractional Part	*Base Number*	*Product*	*Open Sentence*
1.	$\frac{4}{5}$	20	n	$\frac{4}{5} \times 20 = n$
2.	n	20	16	$n \times 20 = 16$
3.	$\frac{4}{5}$	n	$16n$	$\frac{4}{5} \times n = 16$

Table 14.1 illustrates the value of insisting that pupils interpret verbal problems with open sentences. The table shows that the three examples are essentially three different forms of the same problem. In the past, these three examples have all too often been treated as three separate, unrelated problems.

A *formula*, such as the percentage formula $p = br$, may be an effective means to enable the pupil to solve for the variable. The number multiplied by the fractional part is frequently referred to as the *base*. Identification of the base is a key to successful interpretation (the writing of an open sentence) of the three types of problems described above.

TABLE 14.2

Properties of Whole Numbers and Positive Rational Numbers

Property	*Set of W*	*Set of Positive Ra*
Commutative for A	$a + b = b + a$	$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$
Commutative for M	$a \times c = c \times a$	$\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$
Associative for A	$(a + b) + c = a + (b + c)$	$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)$
Associative for M	$(a \times b) \times c = a \times (b \times c)$	$\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right)$
Distributive of M over A	$a \times (b + c) = ab + bc$	$\frac{a}{b} \times \left(\frac{b}{c} + \frac{e}{f}\right) = \frac{ab}{bc} + \frac{ae}{bf}$
D over A	$(a + b) \div c = \frac{a}{c} + \frac{b}{c}$ if a and b are multiples of c	$\left(\frac{a}{b} + \frac{c}{d}\right) \div \frac{e}{f} = \frac{af}{be} + \frac{cf}{de}$
Identity element for A	$a + 0 = a$	$\frac{a}{b} + \frac{0}{b} = \frac{a}{b}$
for M	$a \times 1 = a$	$\frac{a}{b} \times 1 = \frac{a}{b}$
Zero in M	$a \times 0 = 0$	$\frac{a}{b} \times 0 = 0$
Reciprocal or inverse	None (except for 1)	$\frac{a}{b} \times \frac{b}{a} = 1$
		$a \times \frac{1}{a} = 1$
Closure	$a + b$ is a W	$\frac{a}{b} + \frac{c}{d}$ is a Ra
	$a \times b$ is a W	$\frac{a}{b} \times \frac{c}{d}$ is a Ra
		$\frac{a}{b} \div \frac{c}{d}$ is a Ra (divisor not 0)
Density	None	Between the endpoints of any line segment there is an infinite number of points that correspond to Ra

PROPERTIES OF WHOLE NUMBERS AND RATIONAL NUMBERS

A tabular representation

Chapters 9 and 11 discuss whole numbers and the properties that govern operations with these numbers. Chapter 13 together with the present chapter deal with rational numbers and the principles underlying operations with these numbers. A comparative study of the properties of each set of numbers will show points of likeness and difference (see Table 14.2).

We shall assume that a, b, and c are whole numbers and that $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are positive rational numbers. In Table 14.2, W = a whole number, Ra = a rational number, A = addition, D = division, M = multiplication, and S = subtraction.

EXERCISES

1. Show the relationship between partitive division and multiplication by a fractional number.
2. Make a diagram to show that the product of $\frac{3}{4} \times \frac{5}{6}$ is $\frac{5}{8}$.
3. Show by approximation that the product in the example $\frac{7}{8} \times 3\frac{1}{4} = 2\frac{7}{8}$ is sensible. Solve the example and show that the product is not correct.
4. Give the mathematical basis of cancellation as it refers to multiplication of rational numbers.
5. Arrange the factors in the example $\frac{7}{8} \times 3\frac{1}{3} \times 1\frac{1}{7}$ so as to make the computation as easy as possible. What properties are involved in the solution?
6. Show how you would have a pupil understand that the product of two rational numbers each having a value less than 1 is less than either number.
7. Let $\frac{m}{n}$ represent any positive rational number. Compare the value of m with n when the reciprocal (multiplicative inverse) of $\frac{m}{n}$ is:
   a. A rational number having a value less than 1.
   b. A rational number having a value greater than 1.
   c. A whole number greater than 1.
8. Write the equation $\frac{3}{4} \div \frac{2}{3} = n$ as an equation involving multiplication. What is the mathematical basis of the transformation?
9. Find the number on a segment of a number ray that is midway between the points corresponding to the numbers $\frac{1}{3}$ and $\frac{1}{2}$. Then identify the point on the segment midway between the point corresponding with that number and the point corresponding with $\frac{1}{2}$. Continue the procedure in the same way until you discover the pattern for naming the next number in the sequence.
10. Use the common denominator method to find the quotient in the following:
    a. $\frac{2}{3} \div \frac{5}{6}$ c. $2\frac{1}{2} \div \frac{7}{12}$
    b. $\frac{5}{6} \div \frac{3}{8}$ d. $\frac{5}{8} \div 3\frac{1}{2}$
11. By inspection determine which symbol, $<$, $>$, or $=$, should be inserted in the circle to make each of the following mathematical sentences true:

    $\frac{7}{8} \div \frac{5}{6} \bigcirc 1$ $\frac{11}{12} \div \frac{9}{10} \bigcirc 1$
    $\frac{2}{3} \div \frac{3}{4} \bigcirc 1$ $\frac{5}{6} \div \frac{30}{36} \bigcirc 1$
    $\frac{3}{4} \div \frac{8}{9} \bigcirc 1$ $\frac{5}{8} \div \frac{7}{15} \bigcirc 1$
12. Use the numbers in each set. Write the four examples that may be made with the elements of each set of numbers.
    A: $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\}$ C: $\{12, \frac{3}{4}, 9\}$
    B: $\{\frac{2}{3}, \frac{1}{4}, \frac{1}{6}\}$ D: $\{\frac{8}{9}, \frac{2}{3}, \frac{4}{3}\}$

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RATIONAL NUMBERS EXPRESSED AS DECIMALS

Chapters 13 and 14 discussed rational numbers that are expressed as fractions. This chapter deals with rational numbers that are expressed as *decimals*. The numerals $\frac{2}{5}$, $\frac{3}{4}$, and $\frac{1}{8}$ name the same numbers as .4, .75, and .125, respectively. The numerals $\frac{2}{5}$ and .4 are different ways of expressing the same rational number.

Historically, decimals are a rather recent innovation in naming rational numbers. In the latter part of the sixteenth century, a Dutch mathematician named Stevin gave the first systematic treatment of decimals, but not until two centuries later did decimals come into popular use. Even today the symbolism used for decimals is not universal. In England the *decimal point* is positioned higher on the line than is the practice in the United States. For example, the numeral 3.14 in England is written 3·14. In some European countries a comma is used instead of a decimal point to designate a decimal, as 3,14.

The introduction of decimals made it easier both to obtain and to interpret the answer to problems that previously had to be solved with fractions. Fractions having unwieldly denominators, such as $\frac{723}{2468}$, made computation difficult. Computations in the four operations with decimals and whole numbers are performed in a similar way.

The way to express a rational number less than 1 but greater than 0 follows the pattern of that for whole numbers. In a whole number each place to the left of ones' place is a power of 10. In a rational number expressed as a decimal, each place to the right of ones' place is a power of the reciprocal of ten, or one tenth,[1] as shown by the following expanded notation for the numeral 72.45:

$$72.45 = 7 \times 10^1 + 2 \times 10^0 + 4 \times (\tfrac{1}{10})^1 + 5 \times (\tfrac{1}{10})^2$$

The illustration shows that the introduction of decimals extended place value to the right of ones' place.

Decimals represent another way of expressing rational numbers. Therefore the same properties that apply to positive rational numbers apply to decimals.

The following topics are included in this chapter: introducing the decimal concept; addition and subtraction of decimals; multiplication of decimals, division of decimals.

INTRODUCING THE DECIMAL CONCEPT

Materials for introducing decimals

Each pupil should have a kit of squares and rectangular strips similar to the materials he used in dealing with positive integers. One side of the large square in his kit is divided into 100 small squares representing 100. The reverse side of the square is unruled. The large square can represent 1. The same type of square is used in introducing decimals. An unruled strip having an area of a tenth of the large square represents .1, and a small square represents .01. The number represented in Figure 15.1 is 1.34.

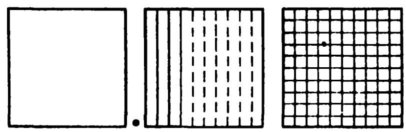


Figure 15.1

A number ray is an effective teaching aid for introducing decimals (see Fig. 15.2). Both fractional and decimal numerals may be used to express the numbers corresponding to points on the number ray. The fractional numerals are shown above the ray and the corresponding decimal numerals are shown below the ray.

For demonstration purposes, the classroom should contain a place-value chart to represent ones', tenths', and hundredth places, as shown in Figure 15.3.

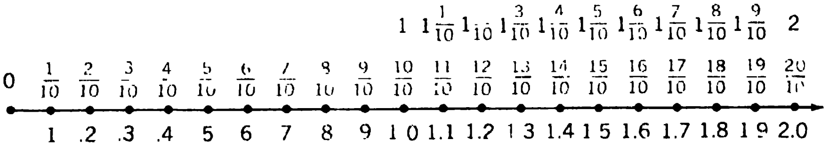


Figure 15.2

Ones	Tenths	Hundredths

Figure 15.3

[1] The number named by $\frac{1}{10}$ may be expressed as 10^{-1}. Then $.45 = 4 \times 10^{-1} + 5 \times 10^{-2}$. The pupil will learn in algebra that 10^{-1} and $\frac{1}{10}$ are different names for the same number.

A fraction chart is an effective classroom teaching aid. The chart may be made of oak tag or it may consist of a frame with movable parts, as shown in Figure 15.4. The strips in the movable parts should show the equivalence of fractional numbers and decimals. One strip should show some number, as a half expressed as a fractional numeral. The adjoining strip should show the same number expressed as a decimal numeral.

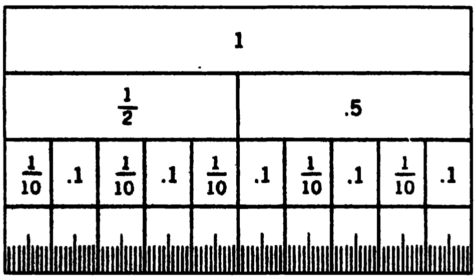


Figure 15.4

Teaching the meaning of a decimal

Each pupil selects a large square from his kit and identifies the square as representing 1. He then selects a rectangular strip from his kit and determines the value of this strip by placing enough strips on the square to represent a whole. The pupil writes the fractional numeral to show the number represented. Then the teacher shows the decimal numeral to represent the same number:

Fraction	*Decimal*
$\frac{1}{10}$	.1
$\frac{2}{10}$	2
$\frac{3}{10}$	3
.	.
.	.
.	.
$\frac{10}{10} = 1$	1 0 = 1

The demonstration with the strips is continued until all ten strips are used. The left-hand column of the example shows the fractional numeral for $\frac{10}{10}$, which is the same number as represented by 1. The column at the right shows the decimal numeral representing the same number. The decimal numeral is 1.0, or 1. A number ray divided as illustrated on page 271 is also an effective means for showing that the numerals $\frac{10}{10}$, 1, and 1.0 represent the same number.

It is important for the pupil to discover the relationship between fractions and decimals. The teacher should stress the fact that the fraction $\frac{7}{10}$ and the decimal .7 represent the same number, but the numerals used to represent that number are different.

Next the teacher has the class discover how it is possible to represent 11 tenths. The pupil uses his kit material to show that this number may be represented in either the ungrouped or the grouped form by using both fractions and decimals. Using fractions, he finds that the ungrouped form is $\frac{11}{10}$ and the grouped form is $1\frac{1}{10}$. Figure 15.5, containing 11 strips, shows the ungrouped number. Notice that tenths' place is overloaded and therefore it is necessary to regroup the number as a whole and a tenth. The numeral for this number is 1.1.

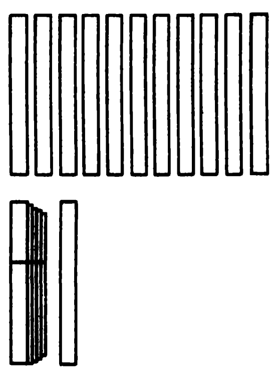


Figure 15.5

It is not necessary to use a marker to symbolize the decimal point as the pupil works with his kit material to show a number. He should indicate a break of approximately two inches between the squares representing the number in ones' place and the strip representing tenths. Figure 15.6 shows how to represent the numeral 2.3.

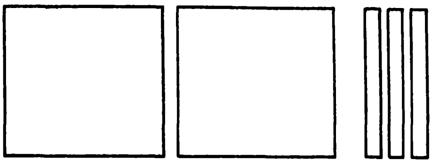


Figure 15.6

After the pupil discovers the meaning of tenths, he should find the missing numerals in an exercise of the type that follows:

a. 3 5 7 9
b 2 5 8 11
c 12 14 17 20
d 1 9 17 25 33

It is equally important for the pupil to discover how to change a grouped number, as 2.1, to an ungrouped number, or 21 tenths, as it is for him to make the change in the reverse order. Some pupils may need many experiences using kit materials in order to discover the equivalence between the two symbolic representations. Thus 2 is equal to 20 tenths. The number named by 2 may be named as 1 and 10 tenths. Accordingly, it is clear that there are different ways of naming a decimal or any other rational number. At initial instruction the pupil should learn how to regroup or rename a number as the next equivalent smaller or larger unit. Th s 2 ones are equal to 20 tenths, and vice versa.

An exercise of the following type is representative of the kind of written work to give to the class. The pupil should complete each number sentence to make it a true statement.

a $.3 = \frac{\square}{10}$, $.4 = \frac{4}{\square}$, $.7 = \frac{\triangle}{\square}$

b $3 = 2\frac{\square}{10}$, $2.7 = 1\frac{\triangle}{10}$; $1\frac{13}{10} = \square.3$

c Rename as a decimal:
$\frac{3}{10} = \square$; $\frac{10}{10} = \square$; $\frac{13}{10} = \square$

When a pupil demonstrates that he understands the meaning of tenths, he is ready to explore the meaning of hundredths. The teacher has the pupil select a small square from his kit material (Fig. 15.7). The pupil estimates its value and verifies the estimation by showing

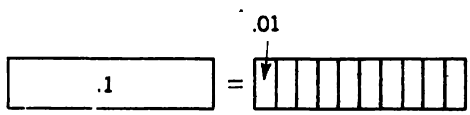


Figure 15.7

that 10 of the smaller squares are equal to 1 strip. Therefore the value of a small square is $\frac{1}{10}$ of the value of 1 strip, or $\frac{1}{100}$ of the value of a large square. Then the teacher represents 1 hundredth by using a place-value chart (Fig. 15.8). The empty pocket in tenths' place shows

Ones		Tenths	Hundredths
	•		
0		0	1

Figure 15.8

that there are no tenths in that place. The 0 in .01 indicates the number of tenths in tenths' place.

$$\frac{1}{10} \times \frac{1}{10} = \frac{1}{100} = 01$$

After a pupil has had a variety of insightful experiences with exploratory and visual materials, he should be able

to make the following generalizations pertaining to decimals:

1. One place to the right of ones' place represents tenths.

2. Two places to the right of ones' place represent hundredths.

3. If a decimal numeral and a fractional numeral represent the same number, the denominator of the fractional numeral is 10 or a power of 10.

4. The number of decimal places in a decimal numeral is the same as the number of zeros in the denominator of an equivalent fractional numeral in which the denominator is a power of 10.

It is important for the pupil to discover the relationship described in the fourth generalization. The distinguishing feature of a rational number expressed with a decimal numeral is the use of an unwritten denominator and a decimal point. Every decimal can be expressed as a fraction with a denominator that is a power of ten. Because the number of places or digits in the numerator corresponds to the power of 10 or to the number of zeros in the denominator, it is not necessary to write the denominator. The numeral .27 contains two *decimal places*, hence the corresponding numeral expressed as a fraction is $\frac{27}{100}$. "A fraction with a denominator that is a power of 10, for example 1/100, 1/10, or $1/10^2$, might for brevity be called a *basic fraction*."[2] The term "basic fraction" is descriptive of a rational number expressed as a decimal, but the term "decimal" will be used in this text.

Since the denominator of a decimal expressed as a fraction is a power of 10, it is easy to rename a decimal. The number expressed as .3 may be written as $\frac{3}{10}$. To rename $\frac{3}{10}$, multiply this fraction by 1 expressed as $\frac{10}{10}$. The number represented by .3 may be renamed as follows:

$$\frac{3}{10} = \frac{3}{10} \times 1 = \frac{3}{10} \times \frac{10}{10} = \frac{30}{100} = .30$$

In a similar way, .3 may be expressed as $\frac{300}{1000}$ by multiplying $\frac{30}{100}$ by $\frac{10}{10}$ or $\frac{3}{10}$ by $\frac{100}{100}$. Therefore, to rename a decimal, annex one or more zeros to the decimal numeral, as .4 = .40 or .4 = .400.[3]

Introducing thousandths

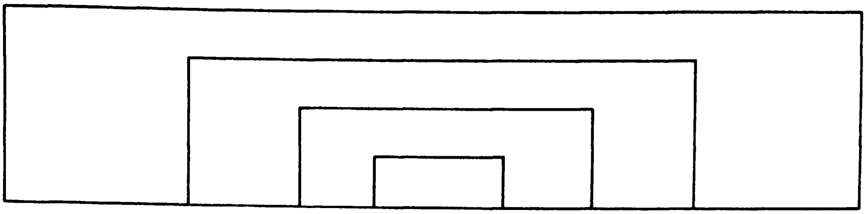
The teacher should not find it necessary to use exploratory materials to introduce the meaning of thousandths. There are two reasons for this statement. First, the value of 1 is 1000 times the value of .001. Materials to show this relationship will be of little value in enabling the pupil to discover the comparative value of these numbers. Second, the pupil should be able to discover the pattern of the relationship existing between consecutive places in the number system from his knowledge of this system in dealing with whole numbers. The pupil who does not discover this pattern from dealing with ones, tenths, and hundredths very probably will not be helped in this respect by further use of exploratory materials.

Decimal point identifies ones' place

Teachers frequently have a pupil identify the place a digit occupies in a numeral from the position of the decimal point. Thus the 3 in hundredths' place in the numeral 21.03 is two places to the

[2] *Topics in Arithmetic*. Twenty-ninth Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: The Council, 1964), p. 303.

[3] The numerals .4, .40, and .400 do not name the same number when measurements are involved (see p. 278). A measurement expressed as .4 in. has a much greater error than a measurement expressed as .400 in.



	Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
A.	$(10)^3$	$(10)^2$	$(10)^1$	$(10)^0$	$\left(\frac{1}{10}\right)^1$	$\left(\frac{1}{10}\right)^2$	$\left(\frac{1}{10}\right)^3$
B.	1000	100	10	1	.1	.01	.001

Figure 15.9

right of the decimal point. The 2 in tens' place is two places to the left of the decimal point. Both the 2 and the 3 are the same number of places from the decimal point. However, the 2 holds *tens'* place and the 3 holds *hundredths'* place. Since there is a distinct relationship between corresponding places on each side of the decimal point, as tens and ten*ths*, hundreds and hundred*ths*, these corresponding places should be the same number of places from the point of reference.

Figure 15.9 gives the names and the numerals for the first three places to the left and right of ones' place. The numerals in (A) show the value of each place expressed as a power of 10 or $\frac{1}{10}$.[4] The numerals in (B) give the standard number names of each place.

The figure shows that ones' place is the point of reference from which to identify a place or a digit in a numeral. The exponent of base 10 or $\frac{1}{10}$ is equal to the number of places a digit is removed from ones' place. Since 10^0 or $(\frac{1}{10})^0$ is equal to 1, the 0 corresponds to ones' place, which is the starting point of identification of a digit in a numeral. In the numeral 723.17, the exponent of 10 and $\frac{1}{10}$ is 2. This exponent indicates that 7 is two places both to the left and right of ones' place.

[4]The numerals $(\frac{1}{10})$ and $(10)^{-1}$ name the same number. The use of the reciprocal of base 10, $\frac{1}{10}$, makes it possible to eliminate negative exponents in naming numbers that are powers of .1.

Writing decimals from dictation

The pupil should spend a minimum of class time in practicing to write decimals from dictation for the following reasons. First, it is difficult to achieve accuracy in writing decimals from dictation because of the promiscuous use of the term "and" in reading a whole number, as "four hundred and fifty" for the number 450. Second, it is difficult to distinguish between the endings of corresponding places on each side of ones' place, as hundreds and hundred*ths*. Third, business procedure does not follow this plan. In business practice each digit is read by giving its face value. The number 2.375 is read as 2-point 3 7 5. A familiar usage is the conventional reading of the approximate value of π as 3-point 1 4 1 6. Fourth, writing decimals from dictation does not increase a pupil's understanding of them. A pupil should, however, be able to write from dictation a decimal expressed as tenths, hundredths, and perhaps as thousandths.

Extending place value to the right of ones' place

By the time the pupil begins a systematic study of decimals, he should

know that moving a digit in a numeral one place to the left multiplies the cardinal value of that digit by 10. Similarly, moving a digit one place to the right divides the value of that digit by 10. The more able pupil should know that if a digit is moved more than one place to the left, the cardinal value of that digit is multiplied by a power of 10. The power of 10 is the same as the number of places the digit is moved. Also, moving a digit more than one place to the right divides the total value of that digit by 10^n if n represents the number of places the digit is moved. This plan applies to whole numbers. The pupil should discover that this plan also applies to decimals. The pupil should compare the total values of a digit as it occupies different places in a numeral, as illustrated in the numeral 222.22. The total value of each 2 is shown at the right. Beginning with .02, each succeeding number named in the table is 10 times the preceding number. Similarly, beginning with 200, each number named above 200 in the table is one tenth the preceding number. Therefore, moving a digit has the same effect in a numeral naming both a whole number and a decimal. The arrows in the diagram below show the effect of moving a digit to the left or right in a numeral.

02
2
2
20
200

Multiplies by a power of 10

$\longleftarrow$

2 2 2 . 2 2

$\longrightarrow$

Divides by a power of 10

The teacher should be sure the pupil discovers that moving a digit in opposite directions involves opposite operations—namely, multiplication to the left and division to the right.

The pupil learned by grade 3 that the standard numeral for a number can be expressed in expanded notation, as $723 = 7 \times 100 + 2 \times 10 + 3$. In grade 5 or 6 he learned to express the value of a place in a numeral as a power of 10. The expanded notation for 723 may then be expressed as follows:

$$\begin{aligned} 723 &= 7 \times 100 + 2 \times 10 + 3 \\ &= 7 \times (10)^2 + 2 \times (10)^1 + 3 \end{aligned}$$

The pattern for expressing whole numbers in expanded form applies to decimals. The reciprocal of the base ten is used in the expanded notation in decimals in order to eliminate negative exponents. The expanded notation for the numeral .275 may be expressed as follows:

$$\begin{aligned} 275 &= 2 \times \tfrac{1}{10} + 7 \times \tfrac{1}{100} + 5 \times \tfrac{1}{1000} \\ &= 2 \times (\tfrac{1}{10})^1 + 7 \times (\tfrac{1}{10})^2 + 5 \times (\tfrac{1}{10})^3 \end{aligned}$$

The following open-number sentences are representative of the kind the class should complete in dealing with the expanded notation of a decimal. The pupil should make each statement a true statement.

a, $7 \times \frac{1}{10} + 3 \times (\frac{1}{10})^2 + 4 \times (\frac{1}{10})^3 = \square$

b $5 \times 10 + 4 \times (\frac{1}{10})^1 + 6 \times (\frac{1}{10})^2 = \square$

c $.25 = \square \times \frac{1}{10} + 5 \times \triangle$

d $342 = 3 \times \square + \triangle \times (\frac{1}{10})^2 + 2 \times (\frac{1}{\square})^3$

ADDITION AND SUBTRACTION OF DECIMALS

The chief reason for the introduction of the decimal notation is to simplify computation with rational numbers. The procedure for adding and subtracting decimals is the same as for whole numbers. In order to deal effectively with addition and subtraction of decimals, the pupil should discover the following:

1. We add and subtract decimals in the same way as we add and subtract whole numbers.

2. The properties of rational numbers are the same regardless of the set of numerals used.

The pupil learned in addition and subtraction of whole numbers that ones are combined with ones, tens with tens, and the like. This same pattern applies to decimals. Therefore, only numbers named in corresponding places can be combined, as tenths and tenths, hundredths and hundredths, and the like. The pupil may use his rectangular strips to find the sum of two decimals, such as .5 and .4 or .7 and .8. A number ray, as shown in Figure 15.10, is also effective for representing the sum. The pupil writes the decimal numerals in both vertical and horizontal forms to show the sum of each pair of numbers.

$$\begin{array}{r} 5 \\ +\ 4 \\ \hline 9 \end{array} \qquad \begin{array}{r} 7 \\ +\ 8 \\ \hline 15 \end{array}$$

$5 + 4 = 9 \qquad 7 + 8 = 15$

$4 + 5 = 9 \qquad 8 + 7 = 15$

The examples indicate that the commutative law applies to addition of decimals as well as to addition of whole numbers. The pupil should check the work by finding the sum with fractional numerals.

The use of a number ray reemphasizes how subtraction is related to addition. To find the sum of two decimals, such as .5 and .4, on a number ray, find the point corresponding to .5 and then move 4 spaces to the right and end at the point corresponding to .9. To subtract .5 from .9, begin at the point corresponding to .9 and move 4 spaces to the left. The terminal point is the number represented by .5. This is the starting point for the corresponding example in addition.

The number of addends should be increased from two to three or more to show how grouping affects the sum. By referring to a number ray, the pupil can demonstrate that the sum of .3 + .4 + .5 is the same regardless of the way in which the addends are grouped.

After the pupil understands how to express rational numbers in decimal form, he should solve open sentences of the following type:

$\square = 2 + 5 \qquad \square + \square = 6$

$7 + \square = 14 \qquad \triangle + \square = 5$

$3 = 3 + \square \qquad 18 = \triangle + \square$

$3 + 4 = \square + 3$

$7 + 9 = 7 + (4 + \square)$

$3 + \square = 8 + 3$

$(3 + 5) + 2 = 3 + (\square + 2)$

$(4 + \square) + 7 = (5 + 6) + 7$

The number sentences illustrate the commutative and associative properties in addition of rational numbers. The teacher should also have the pupil discover how the distributive property ap-

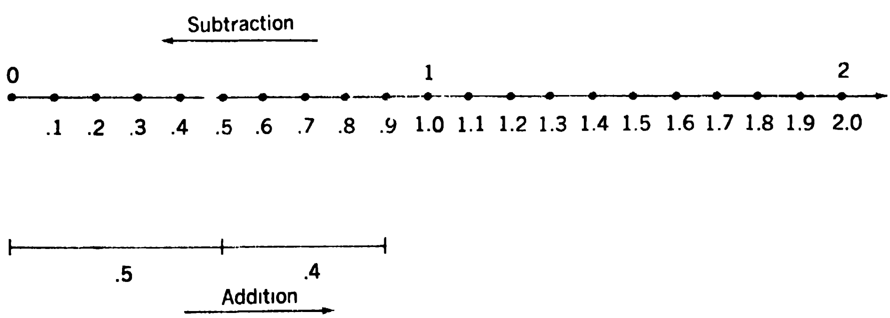


Figure 15.10

plies in addition of decimals by an example of the following type:

$$
\begin{aligned}
4 + .3 &= (4 \times \tfrac{1}{10}) + (3 \times \tfrac{1}{10}) \\
(4 \times \tfrac{1}{10}) + (3 \times \tfrac{1}{10}) &= \tfrac{1}{10} \times (4 + 3) \\
&= \tfrac{1}{10} \times 7 = \tfrac{7}{10}, \text{ or } 7
\end{aligned}
$$

In the same way, adding hundredths and hundredths illustrates the distributive property of multiplication over addition of rational numbers expressed as decimals.

After limited practice in adding rational numbers expressed as decimals, the pupil should make the following generalizations:

1. Decimals are added in a similar way as whole numbers.
2. If the addends are expressed as tenths, the sum will be tenths; if the addends are expressed as hundredths, the sum will be hundredths.
3. Adding numbers named by numerals in like places illustrates the distributive property of multiplication over addition.
4. Rational numbers are commutative and associative for addition.

It is easy for the pupil to understand the second generalization as it applies to the addition of two decimals of the type .3 + .4. He may not understand this principle as it applies to an example of the type .7 + .8. The sum of these two decimals is 15 tenths. When this number is written with decimal numerals, it becomes 1.5. The sum (15 tenths) is regrouped so that each digit has its positional value.

Frequently teachers stress "keeping the decimal points in a column" in addition and subtraction of decimals. A procedure of this kind stresses mechanics and not understandings. It is advisable, however, to emphasize that the sum of tenths and tenths is tenths, and the sum of hundredths and hundredths is hundredths.

Ragged decimals

The example .3 + .86 + .125 illustrates addition of *ragged decimals*. The numbers are expressed in different units. Usually both fractions and decimals represent units of measurement when these numbers are added or subtracted. It is not possible to have a social situation in which one measurement is expressed as tenths, another as hundredths, and another as thousandths if the measurements are combined as in the above example.

No measurement is ever exact. If a line segment is reported as 0.6 ft. long, the greatest possible error is .05 ft. To the nearest tenth, the length of a line segment .55 ft. is expressed as .6 ft. Similarly to the nearest tenth, the length of any line segment between .60 ft. and .65 ft. is expressed as .6 ft. Therefore, when the length of a line segment is expressed as .6 ft., the length may be from .55 ft. up to .65 ft. This fact may be written as .6 ±.05. When a measurement is expressed as hundredths, the greatest possible error is .005. Thus .62 ft. may be written as .62 ±.005. In the same way, .620 ft. may be written as .620 ±.0005. These illustrations show that the amount of error is different in measurements expressed to varying degrees of precision.

.725 in.
.454 in.
.621 in.
1.800 in.

The answer to the example on the right is 1.800 in. and not 1.8 in. Each of the numbers to be added is expressed as thousandths, hence the sum must be expressed as thousandths. If the sum is given as 1.8 in., the maximum error is 100 times as great as the maximum error in 1.800 in.

In addition of decimals involving measurements, the numbers must be expressed in the same unit. It is possi-

ble, however, to have illustrations of decimals to be added when these numbers may be classified as ragged decimals. Situations of this kind occur when dealing with numbers isolated from any measurement. The example $.3 \times 1.45$ may be written as $.3 \times (1 + .4 + .05)$ to illustrate the distributive law. The product is the sum of $.3 + .12 + .015$. The pupil finds the sum by adding tenths and tenths, hundredths and hundredths, and the like. The illustration shows that it is possible to devise a situation in which addition of ragged decimals will occur, but these situations are rare. The curriculum for most pupils in the elementary school should not include numbers of this kind.

Many teachers of elementary mathematics feel that it is necessary to instruct their classes in addition and subtraction of ragged decimals because some standardized tests in arithmetic include examples using decimals of this kind. There also are competitive examinations, such as civil service, which often include examples involving ragged decimals. As long as such practices persist, the teacher may feel justified in teaching addition and subtraction of ragged decimals.

To add in the example $.3 + .86 + .125$, the pupil should express each number as thousandths, as in (a). From a strictly mathematical viewpoint, the numbers should be rounded off to the number of least accuracy, or tenths, as in (b). Most pupils in elementary school, however, would not be able to understand the basis of this procedure. The teacher should therefore expect to follow the plan given in (a).

a.	.300	b.	.3
	.860		.9
	.125		.1
	1.285		1.3

MULTIPLICATION OF DECIMALS

Multiplying a decimal and a whole number

Multiplying a decimal and a whole number may include an example of the type $3 \times .4$ or of the type $.3 \times 4$. Since the order of multiplying two factors does not affect the product, these two examples are the same from a mathematical viewpoint.

The following activities should enable a pupil to discover how to find the value of n in an equation of the type $2 \times .3 = n$.

1. Use rectangular strips to model $2 \times .3$ by showing 2 sets of 3 strips. Each strip represents a tenth.
2. Find n from a number ray.
3. Solve by using fractions, as $2 \times \frac{3}{10} = \frac{6}{10}$.
4. Solve by using addition, as $.3 + .3 = .6$.
5. Solve with decimals, as $2 \times .3 = .6$.

The same activities may be used to find the value of n in an equation of the type $2 \times .7 = n$. The product is 14 tenths, which is regrouped or renamed as 1.4. The pattern described may be used to find the product of a number named by a mixed numeral and a whole number, as $2 \times 1.4 = n$. Different ways of finding the value of n include the following:

1. Use fractions, as $2 \times \frac{14}{10} = \frac{28}{10}$, or 2.8.
2. Add, as $1.4 + 1.4 = 2.8$.
3. Apply the distributive law. Rename 1.4 as $1 + .4$ or as $1 + 4 \times \frac{1}{10}$. The solutions will then be as follows:

a. $2 \times 1.4 = 2 \times (1 + .4) = 2 + .8$, or 2.8

b. $2 \times 1.4 = 2 \times (1 + 4 \times \frac{1}{10}) = 2 + \frac{8}{10}$, or 2.8

4. Multiply as with whole numbers. Use approximation to find the position of the decimal point in the product. The

number named by 1.4 is between 1 and 2, therefore the product of 2 and 1.4 must be between 2 and 4. The product 2.8 is between 2 and 4, therefore 2.8 is a sensible answer.

The class should solve equations of the following type. The solution of each equation illustrates a type of example in which one factor is a whole number and the other factor is a decimal.

$$\begin{array}{ll} 2 \times 3 = \square & \square \times 1.3 = 0 \\ \square \times .4 = 8 & 2 \times \square = 10 \\ 3 + 3 = 2 \times \square & 5 \times \square = 15 \\ .4 + .4 = \square \times .4 & \square \times 7 = 7 \end{array}$$

After solving equations in which it is necessary to find the product of ones and tenths, the class should make the following generalization: *The product of ones and tenths is tenths. There is one decimal place in the product.* Since the commutative principle applies to rational numbers, $4 \times .5 = .5 \times 4$.

The work with multiplication of a whole number and a decimal begins with tenths and should be extended to include hundredths. The teacher should use as many of the procedures described for multiplying ones and tenths as are needed for the class to discover how to find the product of ones and hundredths. Most pupils who understand how to deal with ones and tenths quickly discover how to deal with ones and hundredths or ones and any other decimal. The pupil should discover that the product of ones and hundredths is hundredths. Similarly, the product of ones and thousandths is thousandths. Thousandths represent three decimal places.

The class can use approximation to find the position of the decimal point in the product or apply the rule that the product of ones and tenths is tenths, ones and hundredths is hundredths, and so on. These are meaningful procedures, but they do not represent the stage of learning that may be described as meaningful habituation. After practice in multiplying a decimal and a whole number, a pupil should discover a short cut to find the position of the point in the product. He should see that the product contains as many decimal places as the decimal factor contains. Thus, the product of $7 \times .345$ will contain three decimal places because the decimal factor .345 contains three decimal places. It should be noted that the class discovers this rule for finding the position of the point in the product. The teacher does not give the rule and then have the class demonstrate the application of this mechanical way to find the position of the decimal point in the product. When the pupil uses this short cut to locate the point in the product, he should check the answer either by approximation to see if the answer is sensible or by use of place value. The superior pupil should be encouraged to use both methods.

Multiplying a decimal and a decimal

The teacher should introduce multiplication of two decimals with numbers expressed as tenths. The class should express each decimal as a fraction and then find the product. Examples (a–c) indicate the procedure.

The pupil should understand the work in each solution. If he understands the sequence of steps in (c), he knows that the product of .3 and .4 is .12.

The second step in (a) and (b) shows the basic structure for multiplying two factors expressed as tenths. It is important for the pupil to discover that *the product of tenths and tenths is hundredths.* The product contains two decimal places.

	Decimals	*Fractions*
a.	$.3 \times .4 =$	$\frac{3}{10} \times \frac{4}{10} = \frac{12}{100}$
	3 tenths $\times$ 4 tenths $=$ 12 hundredths	
	$.3 \times .4 = .12$	
b.	$.2 \times .3 =$	
	2 tenths $\times$ 3 tenths $=$ 6 hundredths	
	$.2 \times .3 = .06$	
c.	$.3 \times .4 = (3 \times \frac{1}{10})(4 \times \frac{1}{10})$	Renaming factors
	$(3 \times \frac{1}{10})(4 \times \frac{1}{10}) = (3 \times 4) \times (\frac{1}{10} \times \frac{1}{10})$	Rearranging factors (associative and commutative)
	$(3 \times 4)(\frac{1}{10} \times \frac{1}{10}) = 12 \times \frac{1}{100}$	Renaming numbers
	$12 \times \frac{1}{100} = \frac{12}{100}$, or .12	Renaming numbers

A few similar illustrations should enable the pupil to discover that the number of decimal places in the product is equal to the sum of the number of decimal places in the two factors. The procedure for multiplying two decimals is as follows:

1. Multiply as with whole numbers.
2. Find the sum of the number of decimal places in the two factors.
3. Counting from the right, mark off as many decimal places in the product as are found in step 2.

Often pupils do not understand why the product of two decimals, such as $.2 \times .4 = .08$, is less than either factor. The solution with fractional numbers will show clearly that the product is less than either factor. The class learned in dealing with fractional numbers how multiplying by a number less than 1 affects the product. Since each factor, .2 and .4, is less than 1, the product will be less than either factor.

The pupil uses the conventional pattern for multiplying two decimals of the type shown at the right. He should be able to write the example to illustrate the distributive law as follows:

$$\begin{array}{r} 4.5 \\ \times\ 1.5 \\ \hline 2\,2\,5 \\ 4\,5 \\ \hline 6.75 \end{array}$$

$$\begin{aligned} 1.5 \times 4.5 &= 1.5 \times (4 + .5) \\ &= (1.5 \times 4) + (1.5 \times .5) \\ &= 6.0 + .75 = 6.75 \end{aligned}$$

A challenge for the superior pupil consists in writing each factor as the sum of two numbers. The factors may then be written as follows:

$$\begin{aligned} 1.5 \times 4.5 &= (1 + .5) \times (4 + .5) \\ &= 1 \times (4 + .5) + .5 \times (4 + .5) \\ &= 4 + .5 + 2.0 + .25 = 6.75 \end{aligned}$$

Since multiplication is commutative, the order of the factors does not affect the product. Therefore, the order of the factors may be interchanged as follows:

$$\begin{aligned} &(1 + .5) \times (4 + .5) \\ &= 4 \times (1 + .5) + .5\,(1 + .5) \end{aligned}$$

Multiplying by 10 or a power of 10

It is not unusual to find college students who multiply by a power of 10, as shown.

$$\begin{array}{r} 4.25 \\ \times\ 100 \\ \hline 425\,00 \end{array}$$

A student at the junior high school level should learn to multiply by 10 or a power of 10 without writing the work. He needs to know how to multiply by these numbers in order to learn how to divide by a decimal.

The class should be able to discover how to multiply by a power of 10 by

applying the distributive property of multiplication, as illustrated by the following:

$$\begin{aligned} 10 \times 4.3 &= 10 \times (4 + .3) \\ &= 10 \times 4 + 10 \times \tfrac{3}{10} \\ &= 40 + 3, \text{ or } 43 \end{aligned}$$

$$\begin{aligned} 10 \times .24 &= 10 \times (.2 + .04) \\ &= 10 \times .2 + 10 \times \tfrac{4}{100} \\ &= \ 2 + \tfrac{4}{10}, \text{ or } 2.4 \end{aligned}$$

$$\begin{aligned} 100 \times 2.7 &= 100 \times (2 + .7) \\ &= 100 \times 2 + 100 \times \tfrac{7}{10} \\ &= 200 + 70, \text{ or } 270 \end{aligned}$$

After a few illustrations of the type shown, the class should formulate a rule for multiplying by a power of 10. The generalization is as follows: *To multiply by a power of 10, move the point in the numeral as many places to the right as the power of 10.* The indicated power of 10 is the same as the number of zeros in the standard numeral for that number, as $10^3 = 1000$.

In order for a pupil to understand how to divide by a decimal, he must know how to multiply by a power of 10. He should solve equations of the following type until he discovers the pattern for multiplying and dividing by a power of 10.

$10 \times 5.4 = \square$	$75 = \square \times 75$
$\square \times 5.4 = 54$	$\square \times 04 = 4$
$10 \times \square = 38$	$3 = \square \times 03$

When the pupil is able to solve equations of the type given in the exercise above, he has the mathematical readiness for understanding how to divide by a decimal divisor.

DIVISION OF DECIMALS

Kinds of examples in division of decimals

There are four major types of examples in division of decimals with respect to divisor, dividend, and quotient. No differentiation is made between a number named by a decimal numeral and a mixed numeral, as .3 and 2.5. The types may be represented as follows:

1. $2\overline{)6.2}$: Dividing a decimal by a whole number
2. $4\overline{)3}$: Dividing two whole numbers with a decimal in the quotient
3. $.2\overline{)6}$: Dividing a whole number by a decimal
4. $.2\overline{).14}$: Dividing a decimal by a decimal

If a pupil understands all phases of division of decimals, very probably no one type of example is more difficult than any other. On the other hand, for pupils who do not have a complete understanding of the procedure, finding the quotient in the third type of example results in more errors than in any of the other three. It is possible for the pupil to solve examples of these three types with a high degree of skill and yet have a limited understanding of the work. This is not true of examples of the type $3\overline{).5}$.[5]

The four types of examples in division of decimals may be reduced to two types if the divisor is changed to a whole number by multiplying by a power of 10. If the divisor is a whole number, a decimal may occur only in the quotient or in both the dividend and the quotient, as follows:

a. $\begin{array}{r} .25 \\ 4\overline{)1} \end{array}$ Decimal in the quotient only

b. $\begin{array}{r} .3 \\ 2\overline{).6} \end{array}$ Decimal in dividend and quotient

In reality these two types may be considered one type. In order to complete the division in example (a), the dividend 1 must be renamed as 1.00.

[5]See F. E. Grossnickle, "Types of Errors in Division of Decimals," *Elementary School Journal*, November 1941, 42:184–194.

Then example (a) represents the same pattern as given in example (b).

A decimal divisor may be changed to a whole number by multiplying the divisor by a power of 10. If the divisor is multiplied by a power of 10, the dividend must be multiplied by the same power of 10 to keep the quotient constant. The example $.4\overline{).1}$ or $.1 \div .4$ may be written in fractional form as $\frac{.1}{.4}$. By applying the identity element for multiplication, the number named by $\frac{.1}{.4}$ may be multiplied by 1 expressed as $\frac{10}{10}$. Then the number named by $\frac{.1}{.4}$ is renamed as follows:

$$\frac{.1}{.4} = 1 \times \frac{.1}{.4} = \frac{10}{10} \times \frac{.1}{.4} = \frac{1}{4}$$

The fractional numeral $\frac{1}{4}$ may be written as $4\overline{)1}$, which illustrates example (a) above. If the dividend contains fewer decimal places than the divisor, the transformed example will illustrate type (a) on page 282. If the dividend contains more decimal places than the divisor, the transformed example will illustrate example (b).

Dividing a decimal by a whole number

The pupil should use his kit material to discover how to divide a decimal by a whole number. To divide in the example $2\overline{)2.6}$, the pupil represents 2.6 as shown in Figure 15.11 and separates

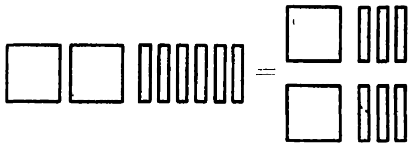


Figure 15.11

the markers into two equal groups. He then makes a symbolic representation of the experience on the chalkboard. He divides decimals as he divides whole numbers. Since tenths are divided by ones, the quotient is tenths.

A class has little difficulty in dividing in examples in which no regrouping is needed. When the number divided has to be regrouped, as in the examples $2\overline{)1.4}$ or $2\overline{).14}$, the work becomes more difficult to understand. To have the class understand how to divide in an example of the type $2\overline{).14}$, the teacher should have the pupils perform the following activities:

1. Use kit material to find the answer. Emphasize the regrouping necessary to make the representation.

2. Give a visual representation at the chalkboard with pocket charts. Have the class explain the steps in the model shown in Figure 15.12. In the figure (A) shows .14 expressed as .1 and .04, (B) shows .1 regrouped as .10 to form 14 hundredths, and (C) shows .14 separated into two groups of .07.

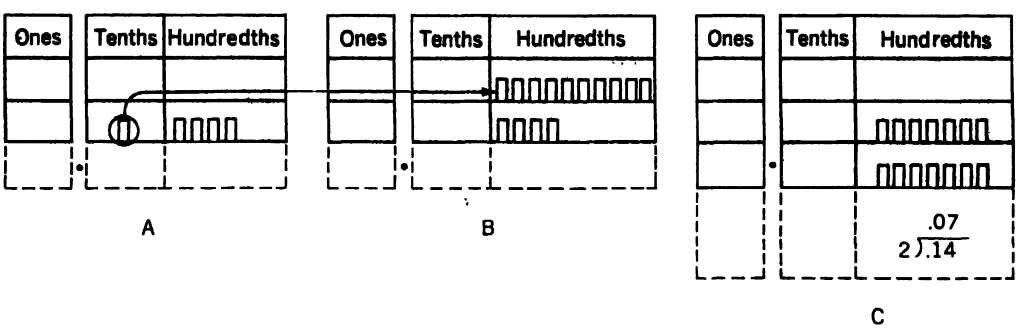


Figure 15.12

3. Use the identity element of 1 to show why the quotient is .07. Multiply $\frac{.14}{2}$ by 1 expressed as $100 \times \frac{1}{100}$.

Identity element of 1

$$\frac{.14}{2} \times 1 = \frac{.14}{2} \times \left(100 \times \frac{1}{100}\right)$$

Associative property

$$\frac{.14}{2} \times \left(100 \times \frac{1}{100}\right) = \left(\frac{.14}{2} \times 100\right) \times \frac{1}{100}$$

Renaming numbers

$$\left(\frac{.14}{2} \times 100\right) \times \frac{1}{100} = \frac{14}{2} \times \frac{1}{100}$$

Renaming numbers

$$\frac{14}{2} \times \frac{1}{100} = 7 \times \frac{1}{100} = \frac{7}{100} \text{ or } .07$$

4. Show the relationship between multiplication and division for the given example. Since $2 \times .07 = .14$, then

$$\frac{.14}{2} = .07, \text{ or } 2\overline{).14} \text{ with quotient } .07$$

5. Tell why the quotient must be less than one tenth.

6. Determine the position of the point in the quotient by a knowledge of place value. Since hundredths are divided by ones, the quotient must be hundredths. Hundredths imply two decimal places.

The distributive property also applies to division of decimals. .14 can be renamed as $.10 + .04$, or as $14 \times \frac{1}{100}$. The quotient of the example $2\overline{).14}$ can then be found by applying the distributive property, as in (a).

a $\frac{.14}{2} = \frac{.10}{2} + \frac{.04}{2} = .05 + .02$, or .07

b $\frac{.14}{2} = \frac{1}{2} \times \left(14 \times \frac{1}{100}\right) = \left(\frac{1}{2} \times 14\right) \times \frac{1}{100}$

$= 7 \times \frac{1}{100} = \frac{7}{100}$, or .07

Example (b) illustrates the use of the associative property and the renaming of numbers.

Expressing rational numbers with decimal numerals

Every rational number expressed with fractional numerals can be designated with decimal numerals, as $\frac{3}{4} = .75$ and $\frac{2}{3} = .666\cdots$. The first illustration represents a *terminating decimal* and the second, a *repeating decimal*, which indicates that one or more digits of the quotient repeat in sequence.

It is easy to determine if a fraction can be expressed as a terminating decimal provided the prime factors of the denominator are known. The only prime factors of the base of our number system are 2 and 5. The denominators of a fraction to be expressed as a decimal must therefore have prime factors of either 2 or 5. Such fractions as $\frac{1}{8}$, $\frac{7}{32}$, $\frac{4}{25}$, and $\frac{11}{160}$ can be expressed as terminating decimals because the prime factors of the denominators are 2 or 5 or both. The denominators of such fractions as $\frac{5}{12}$, $\frac{7}{36}$, and $\frac{11}{150}$ contain the prime factor 3; when expressed as decimals, therefore, these fractions will be repeating decimals.

The teacher has the class rename fractions as decimals, beginning with the familiar fraction $\frac{1}{2}$. The activities should include the following:

1. Draw a number ray to show that $\frac{1}{2}$ and .5 name the same number.

2. Demonstrate with a pocket chart or make a drawing on the chalkboard (see Fig. 15.13) to show the regrouping or dividing 1 by 2. In Figure 15.13 (A) shows 1 one, (B) shows 1 renamed as 10 tenths, and (C) shows 1.0 divided into two groups of .5.

3. Use the identity element of 1 expressed as $10 \times \frac{1}{10}$ and then multiply.

$$\frac{1}{2} = \frac{1}{2} \times 1 = \frac{1}{2} \times \left(10 \times \frac{1}{10}\right)$$

$$\left(\frac{1}{2} \times 10\right) \times \frac{1}{10} = 5 \times \frac{1}{10} = \frac{5}{10}, \text{ or } .5$$

4. Explain the short way to change to the decimal form, as indicated. Have the pupil tell the sequence of the steps in the solution.

$$\frac{1}{2} = 2\overline{)1.0} \text{ with quotient } .5$$

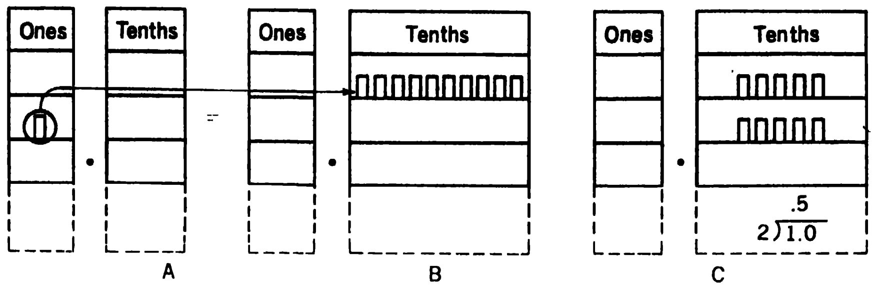


Figure 15.13

After expressing fractions having a denominator of 5 as decimals, the teacher has the class rename as decimals such fractions as $\frac{3}{4}$.

$$\frac{3}{4} \times 1 = \frac{3}{4} \times 10 \times \frac{1}{10} = \frac{30}{4} \times \frac{1}{10}$$

The pupil learned that a number is divisible by 4 when the last two digits of a numeral name a number that is divisible by 4. Since 30 is not divisible by 4, the 30 must be changed to 300. The solution is as follows:

$$\frac{3}{4} \times 1 = \frac{3}{4} \times 100 \times \frac{1}{100}$$
$$= \frac{300}{4} \times \frac{1}{100} = 75 \times \frac{1}{100} = .75$$

Eventually the pupil will learn the short cut procedure, illustrated at the right. He annexes as many zeros as needed to the dividend to make it a multiple of the divisor. The pupil understands the basis of renaming the dividend.

$$\begin{array}{r} 75 \\ 4\overline{)3\,00} \\ \underline{2\,8} \\ 20 \\ \underline{20} \end{array}$$

If the denominator of a fraction contains a prime factor that is neither 2 nor 5, that fraction can never be expressed as a terminating decimal. The digits in the quotient will repeat. The repeating decimal may consist of one digit, as for the fraction $\frac{1}{3} = .\overline{3}$, or two or more digits, as for the fraction $\frac{1}{7} = .\overline{142857}$. The bar above one or more digits in a decimal numeral indicates the digits that repeat. The set of repeating digits is called the *repetend.*

The example at the right shows the repetend for the fraction $\frac{1}{7}$. If the divisor is 7, the remainder from subtracting a partial product from a partial dividend may be 1, 2, 3, 4, 5, or 6. Each of these remainders is circled and occurs in the illustration; there are thus six digits in the repetend. If the remainder were 0, the decimal would be terminating. The sample illustrates the fact that the *greatest* number of places in the repetend is one less than the denominator of the fraction (divisor). The reader may test the validity of this generalization by showing that the repetend for the fraction $\frac{1}{13}$ contains 12 decimal places.

$$\begin{array}{r} 142857 \\ 7\overline{)1\,000000} \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 1 \end{array}$$

There are many properties of repeating decimals that a class may explore. These properties usually are not treated below the level of the junior high school.

The conventional form of expressing the decimal value of $\frac{1}{3}$ is $.33\frac{1}{3}$. However most fractions whose exact decimal value cannot be found are rounded off. Often teachers have the pupil round off the quotient to the nearest hundredth. No fixed number of decimal places to be retained can be decided arbitrarily. The number of decimal places kept in the quotient depends upon the problem. During initial instruction in

rounding off the quotient in decimals, it is satisfactory to express the quotient to the nearest hundredth.

The quotient should be expressed to one more decimal place than will be retained in the answer. Then the same plan for rounding off whole numbers applies to decimals. If the digit in the place to be dropped is less than 5, drop this digit; if this digit is 5 or more, increase by one the next digit to the left. Thus to the nearest hundredth, .473 is .47 but .475 is .48. The expression "to the nearest hundredth" may be symbolized as 0.01 and to the nearest tenth as 0.1.

Dividing a whole number by a decimal

Different methods may be used to find the position of the decimal point in the quotient in a program that emphasizes meaningful learning. In programs that did not stress meanings, the pupil learned to shift the point mechanically in divisor and dividend, as illustrated at the right. The pupil had limited understanding of the procedure. We shall give no consideration to this method. Four methods of finding the position of the point in the quotient may be characterized as follows:

$$\begin{array}{r} 9 \\ .2_{\wedge}\overline{)1_{\wedge}8} \end{array}$$

1. Use of the identity element to rename both divisor and dividend so that the divisor is a whole number
2. The subtractive method
3. Use of approximation
4. Use of place value.

The dividend in a division example corresponds to the product in a multiplication example. The product of two factors contains as many decimal places as are expressed in the sum of the number of decimal places of the factors. Therefore, the sum of the number of decimal places in divisor and quotient must be equal to the number of decimal places in the dividend. Hence the number of decimal places in the dividend less the number of decimal places in the divisor equals the number of decimal places in the quotient. This principle is the basis of the subtractive method of locating the decimal point in the quotient.

According to the fourth method, the pupil uses his knowledge of place value to find the position of the point in the quotient. In the example $.2\overline{)5}$, the thought pattern would be as follows: "Ones divided by tenths is tens."

The teacher must select one of the four methods given or improvise a method to introduce dividing a whole number by a decimal. It is important to understand that *the method to use in dealing with a decimal divisor is of great importance only in introductory learning of the topic.* The teacher may eventually present all methods, especially for enrichment for the more able pupil, but only one method will be given during introductory work.

The writers strongly recommend the method that applies the identity element to rename both divisor and dividend so that the divisor will be a whole number for the following reasons:

1. If the divisor is changed to a whole number, the types of examples in division of decimals is reduced from four to two (see p. 282).

2. The pupil is almost certain to understand the work because he had so many applications of the identity element.

3. The teacher finds it easy to introduce the work.

Making the divisor a whole number

The class should discover ways to find the answer to a problem involving

dividing a whole number by a decimal. A problem of the following type may be used to introduce the topic: How many pieces of ribbon 1.5 yards long can be cut from a ribbon 6 yards long?

The teacher writes the number sentence for the problem: $n \times 1.5 = 6$. The class then suggests different ways to find the value of n, for example:

1. Cut 1.5-yd. pieces from a string 6 yds. long.
2. Find n from a number ray (see Fig. 15.14).

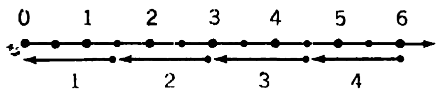


Figure 15.14

3. Divide 6 by $1\frac{1}{2}$.
4. Think, "In 3 yards there are 2 pieces, then there will be 4 pieces in 6 yards."
5. Solve equations of the following type:

$$\frac{6}{1.5} \times \frac{10}{10} = [illegible]$$
$$\frac{6}{1.2} \times \frac{10}{10} = \frac{60}{[illegible]}$$
$$\frac{10}{12.5} \times [illegible] = \frac{100}{125}$$
$$\frac{15}{[illegible]} \times \frac{10}{10} = 25$$

6. Subtract 1.5 from 6 as many times as possible.

After the class discovers that n is 4, the teacher demonstrates at the chalkboard how to find n by division. Since every fraction is an indicated division, every division is an indicated fraction. Thus $1.5\overline{)6}$ may be expressed as $\frac{6}{1.5}$. The denominator of the fraction can be changed to a whole number by multiplying both terms of the fraction by 1, expressed as $\frac{10}{10}$.

$$1 \times \frac{6}{1.5} = \frac{10}{10} \times \frac{6}{1.5} = \frac{60}{15} = \begin{array}{r} 4 \\ 15\overline{)60} \end{array}$$

The example illustrates the use of the identity element for multiplication. The identity element applies equally well when the conventional notation for division is used. Multiplying both divisor and dividend by 10, as $1.5\overline{)6} = 15\overline{)60}$, is the equivalent of multiplying the example by 1.

The steps in initial learning to divide a whole number by a decimal are as follows:

1. Write the division example as a fraction.
2. Multiply both terms of the fraction by 10 or 100 to make the divisor a whole number. The initial work should be restricted to divisors expressed as tenths or hundredths.
3. Divide the two whole numbers.
4. Check the answer by one or more of the following methods:
   a. Use fractions.
   b. Multiply the decimal divisor and the quotient to see if the product is equal to the number divided.
   c. Approximate the answer to see if it is sensible.

The pupil follows the plan given until he discovers a shorter procedure to use in dividing. He may discover that it is not necessary to transform the example so that the divisor will be a whole number. He should know that the decimal point in a numeral does not affect the operation of division with that number. In the example $.3\overline{)12}$, the pupil should know that multiplying both divisor and dividend by 10 shifts the decimal point one place to the right in each numeral. He may indicate the change in the position of the decimal point as shown. A pupil who discovers this short-cut procedure understands the work and should be permitted to use this means of

$.3\overline{)12.0}$

identifying the position of the point in the quotient. Other pupils should continue to use the method of making the divisor a whole number by applying the identity element of 1.

There are two points to emphasize in work dealing with a decimal divisor. First, the pupil should understand the mathematical principles involved in different steps in a solution. Second, he should be able to approximate the quotient. In the example $.3\overline{)12}$, the pupil should be able to approximate the answer to determine if it is sensible by a thought pattern of the following type.

"The divisor is approximately $\frac{1}{3}$, hence the quotient should be about 3×12, or 36."

"The divisor is less than $\frac{1}{2}$, hence the quotient should be more than 2×12, or 24."

It is very important for the more able pupil to be able to verify an answer by approximation.

Dividing a decimal by a decimal

Once the pupil understands how to divide a whole number by a decimal, he encounters no particular difficulty in understanding how to divide a decimal by a decimal, as in the example, $.4\overline{).36}$. In each case he transforms the example so that the divisor is a whole number. It is readily seen that the pupil must know by what power of 10 both divisor and dividend must be multiplied to affect this transformation. This power of 10 is indicated by the number of decimal places in the divisor.

The chief difficulties a pupil confronts in division of decimals occur in the division operation, the placement of the first quotient figure, and the location of the position of the decimal point in the quotient. The ability of the pupil to deal with the decimal point is the chief factor affecting success in this phase of division. Therefore a teacher should use a test consisting of easy examples from the standpoint of division in order to measure a pupil's ability in division of decimals.

Since the main problem in division of decimals is the placement of the point in the quotient and not the division operation as such, a very effective learning exercise consists in having the pupil place the point in the quotient when the quotient digits are given. Similarly, a multiplication example should have the product expressed with the decimal point missing. Examples of the following type are adapted for exercises in learning to locate the position of the point in the answer in multiplication and division of decimals.

Supply the decimal point in each product or quotient:

```
  3 · 8   24
 25 · 25 – 625
     65
 2)13
     7
 6)42
```

Summary

1. Every rational number can be expressed as a fraction or as a decimal.

2. The decimal point identifies ones' place and this place is immediately to the left of the point.

3. Every rational number represented by a fraction that has a denominator containing only the prime factors 2 or 5 can be expressed as a terminating decimal. If the denominator contains any prime factor other than 2 or 5, the fraction can be expressed by a repeating decimal.

4. Moving the decimal point to the right in a numeral multiplies the number named by a power of 10; moving the

point to the left divides the number by a power of 10. The power is the same as the number of places the point is moved in the numeral.

5. Add or subtract tenths and tenths, hundredths and hundredths, and the like.

6. The number of decimal places in the product is the sum of the number of decimal places in the factors.

7. A decimal divisor can always be made a whole number by multiplying both divisor and dividend by one expressed as $(10)^n/(10)^n$.

EXERCISES

1. Computation is more easily done with decimals than with fractions. For this reason some teachers introduce decimals before fractions. Evaluate this procedure.
2. Some teachers use only a number ray to model decimals. Evaluate this plan.
3. A manufacturer advertised that the cutting edge of a razor blade is 17 millionth inch in thickness. Write the standard numeral for this number.
4. Use the exponential form to write the following:
   a. .075
   b. 42.15
   c. .0015
   d. 100.05
5. Write the standard numeral for the following:
   a. $9 \times .1^2 + 4 \times .1^3$
   b. $5 \times .1^4$
   c. $3 \times .1 + 5 \times .1^4$
6. Give the total value of the 3 compared with the total value of the 6 in the following:
   a. 32.16 c. .3256
   b. 16.03 d. 6.023
7. A teacher introduced division by a decimal divisor by the use of a carat to shift the decimal point, as shown at the right. Give an appraisal of this technique. $.3_{\wedge})\overline{.1_{\wedge}8}$
8. Show how to find n in the equation $34 \div 2 = n$ by using the distributive law.
9. What is the greatest number of digits in the repetend of a repeating decimal for a fraction of the type $\frac{1}{n}$? Test the accuracy of your answer by finding the number of decimal places in the repetend of the fraction $\frac{1}{17}$.
10. Enumerate at least six different ways to find the value of n in the equation $2 \div .5 = n$.
11. Write the four number sentences that may be formed with the elements of each of the following sets:
    A: {.3, .5, .8} C: {12, .5, 24}
    B: {.2, 8, 1.6} D: {.03, 1.2, .216}
12. Solve each of the following equations:
    a. $\frac{1}{2} = \square \times \frac{10}{10}$ c. $\frac{1.5}{\square} \times \frac{\triangle}{\triangle} = \frac{15}{25}$
    b. $\frac{.1}{.4} \times \square = \frac{1}{4}$ d. $\frac{.7}{1.8} \times \frac{\triangle}{10} = \frac{70}{\square}$

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RATIOS AND PER CENTS

"Mike is a 300-hitter or batter" is a typical expression used by a baseball fan to describe a good hitter. A batter's rating is determined by finding the ratio of the number of hits the player made and the number of times he was at bat. On the average, a 300-hitter gets 3 hits out of 10 times at bat, or 30 hits out of 100 times at bat. This ratio may be expressed by a fraction, as $\frac{3}{10}$, by a decimal, as .3, or by a *per cent*, as 30%. The numerals $\frac{3}{10}$, .3, and 30% are different names representing the same rational number.

It was indicated earlier that one of the two uses of a fraction is to express the ratio of two numbers. If the measures of the line segments m and n are 3 inches and 4 inches, respectively, the ratio of the measure of m to the measure of n is $\frac{3}{4}$, .75, or 75%. The word "per cent" suggests a ratio in which a number is compared with 100; thus, 3% means the ratio of 3 to 100. Per cent is therefore another way of expressing hundredths.

This chapter deals with the following topics: teaching the meaning of per cent; finding a per cent of a number; finding what per cent one number is of another number; finding a number when a per cent of the number is given.

TEACHING THE MEANING OF PER CENT

A team won 7 out of the 10 games it played. The teacher writes the record of this team on the chalkboard as follows: A team won 7 out of 10 games played. The teacher then asks the class how to express this fact with numerals. This may be done in three ways:

Verbal. 7 out of 10
Fractional numeral $\frac{7}{10}$
Decimal 7

The verbal statement is the long form or the long way of expressing the fact. The fractional and decimal numerals represent short ways to express the fact. Each notation should be read as the verbal statement indicates. Thus, $\frac{7}{10}$ means 7 out of 10.

Next, the teacher has the class find the answer to the following problem: If the team should play more than 10 games and win at the same rate, how many games would the team win out of 20 games played? 30 games played? 40 games played? 50 games played? 100 games played? The teacher writes the number of wins out of 100 games played on the chalkboard as follows:

Long way 70 out of 100
Fractional numeral $\frac{70}{100}$
Decimal 70
Per cent 70%

The three numerals express the same number. The phrase "70 out of 100" is used in a variety of situations. The expression is in reality another numeral for the number commonly named as .70, $\frac{70}{100}$, or 70%. The different ways to represent a number of the kind shown may be described as the long way, as a fractional numeral, as a decimal, or as a per cent.

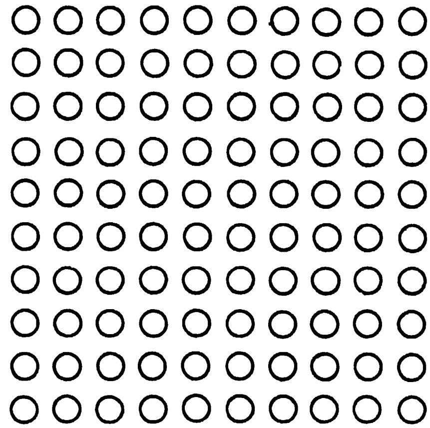

Figure 16.1

A hundred board

A hundred board is a most effective instructional aid for assisting the pupil in discovering the meaning of per cent (Fig. 16.1). The teacher has the class identify the number of disks on the hundred board by counting the number of disks in one row and the number of rows. The teacher then picks up a disk from the board and has the class describe this disk with reference to the total number of disks. The written records of the descriptions are as follows:

Long way 1 out of 100
Fraction $\frac{1}{100}$
Decimal 01
Per cent ·1%

The same procedure is repeated for other representations, such as 2 disks, 3 disks, 7 disks, and 10 disks. The class should give five numerals that represent the 10 disks in a row. These numerals are .1, .10, $\frac{1}{10}$, $\frac{10}{100}$, and 10%. The teacher encourages the class to give as many different numerals as possible to express a number. For each of the 10 rows of disks on the hundred board there are at least five different numerals to express each number when the fractional de-

nominator does not exceed 100. The row that contains 50 disks can be expressed with the numerals $\frac{1}{2}$, $\frac{5}{10}$, $\frac{50}{100}$, .5, .50, and 50%. A group of 25 disks can be represented as $\frac{1}{4}$, $\frac{25}{100}$, .25, and 25%. All the disks on a hundred board can be represented as 1, 1.0, 1.00, $\frac{10}{10}$, $\frac{100}{100}$, and 100%.

For such numerals as 3%, 15%, 25%, or 40%, the pupil expresses the same number as represented by each given numeral with decimal and fractional numerals. The same plan is followed for both fractional and decimal numerals. The pupil gives at least one numeral in each of the two remaining ways to represent the given number.

Renaming numbers

The numeral 3% names the same number as $\frac{3}{100}$ and .03. Therefore the numerals are interchangeable and the pupil should know how to express a per cent as a fraction or as a decimal, and vice versa. Expressing a fractional numeral or a decimal as a per cent involves the identity element of 1.

a. Express .03 as a per cent

Multiply .03 by 1 expressed as $\frac{100}{100}$

$$.03 = .03 \times 1 = .03 \times \frac{100}{100} = \frac{3}{100} = 3\%$$

b. Express $\frac{2}{5}$ as a per cent

Multiply $\frac{2}{5}$ by 1 expressed as $100 \times \frac{1}{100}$

$$\frac{2}{5} = \frac{2}{5} \times \left(100 \times \frac{1}{100}\right) = \left(\frac{2}{5} \times 100\right) \times \frac{1}{100}$$

$$\left(\frac{2}{5} \times 100\right) \times \frac{1}{100} = 40 \times \frac{1}{100} = \frac{40}{100} = 40\%$$

The pupil should identify the applications of the associative property of multiplication.

c. Express $\frac{2}{3}$ as a per cent

$$\frac{2}{3} = \frac{2}{3} \times \left(100 \times \frac{1}{100}\right) = \left(\frac{2}{3} \times 100\right) \times \frac{1}{100}$$

$$= \frac{66\frac{2}{3}}{100} = 66\frac{2}{3}\%$$

d. Express 75% as a fraction

$$75\% = 75 \times \frac{1}{100} = \frac{75}{100} = \frac{3}{4}$$

as a decimal

$$75\% = 75 \times \frac{1}{100} = \frac{75}{100} = .75$$

Examples (a–c) show that expressing a fraction or a decimal as a per cent involves the identity element of multiplication. The fourth illustration shows that expressing a per cent either as a fraction or as a decimal involves multiplying by $\frac{1}{100}$. This is true because of the definition of a per cent.

The pupil should be able to express any fraction as a per cent. This procedure involves expressing the ratio of two whole numbers as hundredths. There are certain fractions that are used frequently in the social applications of per cent. The student at the junior high school level should be familiar with the following fractions and their per cent equivalents:

$\frac{1}{2} = 50\%$	$\frac{7}{8} = 87\frac{1}{2}\%$
$\frac{3}{4} = 75\%$	$\frac{1}{3} = 33\frac{1}{3}\%$
$\frac{1}{8} = 12\frac{1}{2}\%$	$\frac{2}{3} = 66\frac{2}{3}\%$
$\frac{3}{8} = 37\frac{1}{2}\%$	$\frac{1}{6} = 16\frac{2}{3}\%$
$\frac{5}{8} = 62\frac{1}{2}\%$	$\frac{5}{6} = 83\frac{1}{3}\%$

Since different numerals may name the same number, the teacher should have the class practice identifying and showing the equivalence of different numerals. The following exercise is suggestive of the type of work to provide. The pupil makes each number statement true by inserting the correct symbol, =, >, or <, in the circle.

3%	○	3	4	○	75%
.05	○	5%	2%	○	2
[illegible]	○	.40	15%	○	$\frac{3}{5}$
1%	○	$\frac{1}{10}$	1	○	100%
90%	○	.09	1	○	10%

The three variables in dealing with per cents

It was stated on page 266 that there are three types of problems dealing with rational numbers. They are: (1) finding a fractional part of a number; (2) finding the ratio of two numbers; and (3) finding a number when a fractional part of it is given. Since a per cent is another way of naming a rational number, there are three types of problems in per cent. Equations for each of the three uses of per cent follow:

a Finding a per cent of a number: $\frac{3}{100} = \frac{n}{50}$

b Finding what per cent one number is of another number: $\frac{2}{5} = \frac{n}{100}$

c Finding a number when a per cent of it is given: $\frac{3}{100} = \frac{6}{n}$

In the equation for example (a), the *rate* is 3% ($\frac{3}{100}$), the *base* is 50, and the *percentage* is n. In the equation for example (b), the percentage is 2, the base is 5, and the rate is n. In the equation for example (c), the percentage is 6, the rate is 3%, and the base is n. We may write an equation, or a *formula*, to express the relationship among the three quantities, rate, base, and percentage, as $r \times b = p$, or $p = br$. The formula $p = br$ contains three variables. It is possible to find the value of any variable in this equation if the values of the other two variables are known.

The equations in examples (a), (b), and (c) have been designated as equations to represent the three cases of percent. Often a teacher erroneously deals with per cents by presenting each of the three cases as a separate topic. Each case represents a different variable in the equation $r \times b = p$.

FINDING A PER CENT OF A NUMBER

Oral work

The initial work involving finding a per cent of a number should be oral. There are four stages in presenting the work in finding a per cent of a number. First, the pupil should be able to express orally a per cent of 100, as 7% of 100, 45% of 100, and 100% of 100. After the class can find the percentages orally, the teacher should have the pupils write an equation for finding the percentages. The numerals 3% and $\frac{3}{100}$ name the same number. The ratio of some number n to 100 may be expressed as $\frac{n}{100}$. The two ratios form the following equations:

$$\frac{3}{100} = \frac{n}{100}$$
$$n = 3$$
$$3\% \text{ of } 100 = 3$$

The teacher must be sure that the pupil discovers the pattern for solving an equation of the type $\frac{4}{5} = \frac{\square}{5}$. The equal fractions have equal denominators, therefore the numerators must be equal.

Second, the pupil should be able to find the percentages in an example of the type 3% of 200 without the use of paper and pencil. The base for applying the per cent is a multiple of 100. His thought pattern would be as follows: "3% of 100 = 3; then 3% of 200 will be 2 × 3, or 6."

The teacher should then have the class write the equation for the same problem, as

$$\frac{3}{100} = \frac{n}{200}$$

The two equal ratios can be expressed with fractions having equal denominators by applying the identity element of 1. Multiply $\frac{3}{100}$ by 1 expressed as $\frac{2}{2}$

and then the fractions will have equal denominators.

$$\frac{3}{100} \times \frac{2}{2} = \frac{n}{200}$$

$$\frac{6}{200} = \frac{n}{200}$$

$$6 = n$$

$\therefore$ 3% of 200 = 6

Check: $\frac{3}{100} = \frac{6}{200}$ Replace n by 6

Third, the pupil should find the percentage in an example of the type 7% of 25. The denominators of the two fractions in the equation formed are unlike, but one denominator is a factor of the other. The equation and its solution are as follows:

$\frac{7}{100} = \frac{n}{25}$ The equation

$\frac{7}{100} = \frac{4}{4} \cdot \frac{n}{25}$ Multiply $\frac{n}{25}$ by 1 expressed as $\frac{4}{4}$

$\frac{7}{100} = \frac{4 \cdot n}{100}$

$n = \frac{7}{4}$, or 1.75

$\therefore$ 7% of 25 = 1.75, or $1\frac{3}{4}$

Fourth, the next type of example consists in solving an equation in which neither denominator of the fractional ratio is a common denominator, as illustrated in the equation for finding 5% of 36:

$$5\% = \frac{5}{100} = \frac{1}{20}$$

$\frac{1}{20} = \frac{n}{36}$ The equation

The pupil may find the prime factors of both 20 and 36 and then find the lowest common denominator to be 180. He renames each fraction as a fraction that has a denominator of 180.

$\frac{9}{9} \times \frac{1}{20} = \frac{5}{5} \times \frac{n}{36}$ Multiply $\frac{1}{20}$ by 1 expressed as $\frac{9}{9}$, multiply $\frac{n}{36}$ by 1 expressed as $\frac{5}{5}$

$9 \times 1 = 5 \times n$

$n = \frac{9}{5}$, or 1.8

$\therefore$ 5% of 36 = 1.8

The solution in the last equation is difficult for many pupils at the level of grade 6. There are two alternatives for meeting this difficulty. First, defer examples of this type to the next grade level. Second, provide a different solution for equations of this type. The second alternative is preferable (see p. 299 for an alternate solution).

Pupils frequently fail to understand the difference between a *per cent* and a *percentage*. The two terms are not synonymous. A percentage is the number that results from finding a per cent of a number. Thus, 5% of 80 is 4. The 4 is a percentage. The rate 5% is a per cent, as indicated by the per cent symbol. Therefore in all equations of the type 3% of 25 = n, the problem calls for finding the percentage.

Ways to solve an equation with equal ratios

There are many different ways to solve an equation of the type $\frac{3}{100} = \frac{n}{15}$. Perhaps the easiest method is to apply the multiplication axiom, in which both members of the equation are multiplied by the same number without changing the values of the equation. Usually the pupil does not learn this axiom in algebra at the grade 6 level. On the other hand, there are different ways to solve the given equation that embody principles that the pupil should have learned before per cent is introduced. These procedures are as follows:

1. Express both ratios with fractions having equal denominators. This is the plan used in previous illustrations.

2. Solve the equations by applying the principle that *dividend = divisor × quotient*. In the equation $\frac{3}{100} = \frac{n}{40}$, the dividend is n, the divisor is 40, and the quotient is $\frac{3}{100}$. The solution is as follows:

$$n = 40 \times \frac{3}{100} = \frac{120}{100}, \text{ or } 1.2$$

3. Apply the principle that if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. The pupil learned to compare fractions by applying this principle in number sentences of the type $\frac{a}{b} \bigcirc \frac{c}{d}$. If the product $ad = bc$, the two fractions are equal, as $\frac{2}{3} = \frac{10}{15}$, in which $ad = 30$ and $bc = 30$.

In beginning work involving how to find a percentage, a rate, or a base, the pupil should write an equation showing the two ratios, as $\frac{3}{100} = \frac{n}{50}$, $\frac{3}{4} = \frac{n}{110}$ or $\frac{3}{100} = \frac{6}{n}$. The equation for each use of per cent follows the same pattern. The teacher has the pupil solve the equation by one of the three ways described. For introductory work the pupil should use the first plan, in which fractions having unlike denominators are renamed as fractions having like denominators. This is especially true for fractions in which one of the denominators is a multiple of the other denominator, as $\frac{3}{10} = \frac{n}{60}$. When neither denominator is a multiple of the other denominator, the work can be simplified by applying the principle that in an equation of the type $\frac{a}{b} = \frac{c}{d}$, $ad = bc$. The teacher must be certain, however, that the pupil understands this procedure. Often a pupil "cross multiplies" and performs a mechanical operation that is meaningless.

FINDING WHAT PER CENT ONE NUMBER IS OF ANOTHER NUMBER

To find what per cent one number is of another number, it is necessary to express the ratio between these numbers. In Figure 16.2 the ratio of set A to set B in (A) is $\frac{1}{2}$; in (B), 1; in (C), $\frac{4}{5}$. These ratios expressed in per cents are 50%, 100%, and 80%, respectively.

If the ratio between two numbers is expressed as a fraction, the problem the pupil confronts is to determine which number is to be named in the numerator and which number is to be named in the denominator.

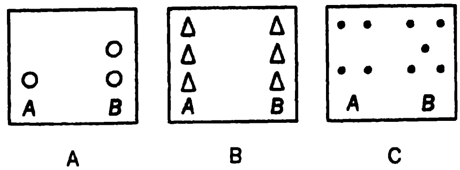

Figure 16.2

Let a and b represent any two numbers that are to be compared. A pupil who is not certain whether the ratio of the two numbers should be $\frac{a}{b}$ or the reciprocal $\frac{b}{a}$ should try to answer the following questions:

1. What number is to be compared?
2. With what number is it to be compared?
3. Could the answer be more than 100 per cent?
4. Must the answer be less than 100 per cent?

The number to be compared is the numerator of the fraction expressing the ratio of the two numbers. The number with which the comparison is made is the denominator of the fraction.

The following problem may be used effectively to introduce this application of per cent:

A basketball player made 9 baskets out of 25 shots at the basket. What per cent of his shots were successful? The equation for this problem is $\frac{9}{25} = \frac{n}{100}$. The pupil should solve the equation by the method he learned before. If he expresses the fractions with equal denominators, the equation will be as follows:

$\frac{4}{4} \times \frac{9}{25} = \frac{n}{100}$ Multiply $\frac{9}{25}$ by 1 expressed as $\frac{4}{4}$

If he uses the principle that in the two equal ratios $\frac{a}{b} = \frac{c}{d}$, $ad = bc$, the solution will be as follows:

$$\frac{9}{25} = \frac{n}{100}$$
$$9 \times 100 = n \times 25$$
$$n = \frac{9 \times 100}{25} = 9 \times 4 \times \frac{25}{25} = 9 \times 4 \times 1 = 36$$
$$\therefore 9 = 36\% \text{ of } 25$$

36 per cent of the shots were successful.

The equation $\frac{9}{15} = \frac{n}{100}$ may be written as $\frac{n}{100} = \frac{9}{25}$. If $a = b$, then $b = a$. It is important for the pupil to discover that $\frac{a}{b}$ and $\frac{c}{d}$ may be interchanged in an equation of the type $\frac{a}{b} = \frac{c}{d}$.

Per cents greater than 100 per cent

Sometimes the pupil finds it difficult to interpret a per cent greater than 100 per cent. A per cent greater than 100 per cent can occur only when a number is compared with a smaller number. The teacher must be certain that the pupil does not derive the generalization that the smaller number always is compared with the larger number when one number is expressed as a per cent of another number. Disks such as used on a hundred board, are effective instructional aids for showing a per cent greater than 100 per cent. Each pupil should have disks with which to form two stacks or piles. Then he should express the ratio of the two numbers as a per cent. In the example below, the ratio of (a) to (b) is $\frac{4}{5}$, or 80%:

a b

the ratio of (b) to (a) is $\frac{5}{4}$, or 125%. In a similar manner the pupil should form other piles of disks to express the ratios of any two piles as per cents. From these activities, he should be able to make the following generalizations:

1. If a number is compared with a larger number, the smaller number is less than 100% of the larger number.
2. If a number is compared with a smaller number, the larger number is more than 100 per cent of the smaller number.
3. If two equal numbers are compared, one number is 100 per cent of the other number.

An exercise of the following kind is an effective means of giving the class practice in representing the ratio of two numbers. The teacher writes a set of numerals {1, 2, 5, 8, 10, 14, 20} on the chalkboard and selects a numeral, as 8, for the class to use to form a ratio. The class writes the subsets in which the ratio of the numbers named in the set to 8 is less than 100 per cent, as {1, 2, 5}; greater than 100 per cent, as {10, 14, 20}; and equal to 100 per cent, as {8}. Then the class gives the equivalent per cent or the approximate per cent of the ratio. If the dictated number is not an element of the set of numbers named on the chalkboard, as 9 instead of 8, the subset equal to 100 per cent will be the empty set.

FINDING A NUMBER WHEN A PER CENT OF THE NUMBER IS GIVEN

Finding a number when a per cent of the number is given represents the third usage of per cent. Guiler found from a survey test given to 936 students in the ninth grade that 90 per cent of the students had difficulty in that part of the test dealing with finding a number when a per cent of the number is given.[1] The topic is difficult and seldom used in our daily affairs, but these are not adequate reasons for not teaching it. The topic must be taught in order for the pupil to have a complete understanding of per cent. It is not debatable

[1] Walter S. Guiler, "Difficulties in Percentage Encountered by Ninth-grade Pupils," *Elementary School Journal*, June 1946, 46:563.

whether the topic should be taught. The vital problem pertains to the manner in which the topic should be taught and when it should be introduced in the curriculum.

The first and second uses of per cent can readily be introduced in the elementary school. Because of the difficulty of the topic, it seems advisable to introduce the third use at the junior high school level. At least only part of the third use of per cent should be attempted at the elementary school level. The third use of per cent involves three types of problems.

Three types of problems in one use

The three types of problems that represent the third use of per cent are as follows:

1. Finding a number when the number given represents the per cent given, as follows: A book sold for \$4 at 80% of the regular price. What was the regular price?

$$\$4 = 400 \text{ cents}$$

$$\frac{8}{10} = \frac{400}{n} \quad \text{The equation}$$

$$n = \frac{10 \times 400}{8}$$

$$= \frac{10 \times 50 \cdot 8}{8} = 500$$

The regular price was \$5.

2. Finding a number when the per cent given must be subtracted from 100% to represent the number given, as follows: A dealer sold 20% fewer cars in June this year than he sold in June the previous year. If he sold 40 cars in June this year, how many cars did he sell in June the previous year?

$$100\% - 20\% = 80\%$$

$$\frac{8}{10} = \frac{40}{n} \quad \text{The equation}$$

$$n = 40 \times \frac{10}{8} = 50$$

He sold 50 cars in June the previous year.

3. The entering class in a high school had 20% more students this year than the entering class had the previous year. If this year's class had 180 students, how many were in the class the previous year?

$$100\% + 20\% = 120\%$$

$$\frac{12}{10} = \frac{180}{n} \quad \text{The equation}$$

$$n = \frac{10 \times 180}{12} = \frac{10 \times 15 \times 12}{12} = 150$$

Last year's class had 150 students.

The second and third problems in this group include an extra step that greatly complicates the difficulty in solving these problems. Many students erroneously solve the third problem by finding 20% of 180, which is 36, and subtract this percentage from 180. The enrollment would then be 144 instead of 150. Although the answer 144 is reasonable, it is incorrect. The base is 180. The identification of the base is the key step in the correct analysis of a problem involving per cents. Because of the difficulty in identifying the base in problems of types (2) and (3) of the third use of per cent, these problems should be deferred until the junior high school.

Teaching the third use of per cent

The teacher of a superior grade 6 group may decide that the class is ready for problems involving the third use of per cent. The work in this area should typify the method used in the first application. The given number then represents the given per cent. The teacher may introduce this use by a problem of the following type: A team won 60 per cent of the games it played. If the team won 15 games, how many games did it play?

The equation and its solution are as follows:

$$\frac{6}{10} = \frac{15}{n} \qquad \text{The equation}$$

$$6 \times n = 10 \times 15$$

$$n = 10 \times \frac{15}{6} = 25$$

The team played 25 games.

In the other two uses of per cent the variable as represented by n is the numerator of one of the fractions. In the third use, the variable is the denominator of one of the fractions. This change in pattern often results in the formulation of the wrong equation. The pupil may write the reciprocal of the correct ratio. In the given problem the equation then would be $\frac{6}{10} = \frac{n}{15}$, and n would be 9. The pupil can readily discover that this answer is absurd, since the team won 15 games. The pupil should check the answer by approximation to see if it is sensible. In the given problem, the thought pattern would be as follows: "60% is a little more than $\frac{1}{2}$. If the team had won only half of the games played, the number would be 30. Therefore the team played slightly fewer than 30 games, so 25 is a sensible answer."

Alternate plan for writing equations in per cent

An equation for a problem involving per cent may be written in different forms. According to the plan proposed, the pupil should write the equations as the equality of two ratios. To find 3% of 20, the equation is $\frac{3}{100} = \frac{n}{20}$. In equations expressing the other two uses of per cent, the variable may occupy the place held by either the 3 or the 20 in the given equation.

Some teachers prefer to have the pupil write the equation for finding 3% of 20 as follows:

$$3\% \text{ of } 20 = n, \text{ or } .03 \times 20 = n$$

In this equation n is the product of two given factors.

To find what per cent 5 is of 20, the equation by the second method is as follows:

$$5 = n\% \text{ of } 20, \text{ or } 5 = n \times 20$$

In this equation 5 is the product of the two factors n and 20. To find the variable, divide the product by the given factor and express the quotient as a per cent.

To find a number when a per cent of it is given, as 30 is 6% of some number, the equation is as follows:

$$6\% \text{ of } n = 30, \text{ or } .06 \times n = 30$$

The factors are .06 and n and the product is 30. Divide 30 by .06 to find the factor n.

Either plan is acceptable for writing equations involving per cents. For introductory work, the authors prefer the method of equal ratios for two reasons. First, this method emphasizes the standard way of comparing two numbers as hundredths, which is basic in an understanding of per cent. Second, the pattern for writing an equation having two equal ratios applies to a wide range of problems. Two equal ratios form a *proportion*, as $\frac{a}{b} = \frac{c}{d}$. All problems of the following type may be solved when the equation is written as a proportion. If 3 oranges cost 20 cents, what will be the cost of a dozen oranges? The equation is $\frac{3}{20} = \frac{12}{n}$ or $\frac{20}{3} = \frac{n}{12}$. After the pupil becomes familiar with per cent, a shorter procedure, such as used in the alternate plan, may be preferable to the method of equal ratios.

Reviewing per cents at different grade levels

A major topic that is introduced in one grade in arithmetic is reviewed in the next higher grade. This review may

be a repetition of the work of the preceding grade or it may be a *new view* of the topic. An important principle of learning indicates that a variety of experiences in dealing with a topic is more favorable for learning the topic than repetition of the same experience.

Let us assume that per cent is introduced in grade 6. The treatment of the topic in grade 7 should be different from the presentation in the previous grade. In grade 7 the work should involve the percentage formula, $p = br$ (see p. 294). The student would deal with all problems in per cent by replacing two of the variables by the given numbers and then solve the resulting equation. We may illustrate the procedure by solving the following problem: A screen for home movies sold for \$32 at 80% of the regular price. What was the regular price?

$r = 80\% = .8$	$p = br$, the formula
$p = 32$	$32 = .8 \times b$
$b = ?$	$b = 32 \div .8$, or 40

The regular price was \$40.

The equation $.8 \times b = 32$ is familiar to the student. In this equation the product of one factor and two factors is given. The missing factor is found by dividing the product by the given factor.

Often the student in the upper grades learned to solve problems involving the three uses of per cent by three different formulas. A program of this kind should never be perpetuated. The learner never discovered that the three formulas are different forms of the same formula. The relationship among the variables in the formula $p = br$ is the same as the relationship among factors and product in an equation of the type $3 \times 5 = 15$. Then $15 \div 3 = 5$ and $15 \div 5 = 3$. Similarly, in the percentage formula, the three related equations are: (1) $p = br$; (2) $p \div b = r$; and (3) $p \div r = b$.

When the student begins a systematic study of equations in algebra, usually in grade 7 or 8, he should review the topics of per cent from a different viewpoint. He would then apply a new set of principles in solving equations. The student applies the multiplication axiom to solve an equation of the type $\frac{3}{100} = \frac{n}{60}$. According to this axiom, both members of an equation are multiplied by the same number without changing the value of the equation. If both members are multiplied by 300, the resulting equation is $9 = 5n$, hence $n = 1.8$.

The equation $\frac{3}{100} = \frac{n}{60}$ represents a problem in which it is necessary to find 3% of 60. The chief function of learning to solve an equation of the type given is not primarily to find 3% of 60, but instead to learn how to solve a fractional equation. A per cent usage can represent an application of a fractional equation. The benefit derived from using per cents to form equations are twofold. First, the student learns how to solve a fractional equation. Second, he is helped to review and enrich his understanding of per cent by formulating the equations.

EXERCISES

1. What, for many pupils, is the most difficult part of the solution of finding what per cent one number is of another number? What are some things the teacher may do to help the pupil overcome this difficulty?
2. Write problems to illustrate each use of a fraction, a decimal, or a per cent.

3. Show why you would or would not introduce each use of per cent in grade 6.
4. What is meant by spacing a topic in the curriculum? How should the topic of per cent be spaced in the curriculum?
5. A book was marked to sell for \$9 at 20% above cost. What was the cost of the book? A student gave the following solution: 20% of \$9 = \$1.80; \$9.00 − \$1.80 = \$7.20, the cost. As a teacher, how would you have the pupil discover why his solution is faulty?
6. Illustrate different ways of solving an equation involving per cent. Give the principles that are involved in each solution.
7. If two numbers are equal, their reciprocals are equal. Verify this generalization. Show how this generalization applies to the equation $\frac{3}{4} = \frac{12}{n}$ and $\frac{4}{3} = \frac{n}{12}$.

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MATHEMATICAL SENTENCES AND PROBLEM SOLVING

Problem solving is the highest form of reflective thinking. A mathematics program, such as the one presented in this book, that emphasizes meaning and understanding is based largely on problem solving. Quantitative thinking is the basis of effective problem solving. When the classroom is regarded as a learning laboratory, instruction is conducted in such a way that basic relationships are discussed and formulated through experimentation with things and with quantitative aspects of things. The learner is challenged to make discoveries of mathematical relationships, which lead him, in turn, to analyze and interpret his experiences, to test his understanding of mathematical relationships, to organize his learnings, and to make generalizations that he can subsequently apply in new situations. Real interest in mathematics grows out of problem-solving experiences that deal with matters that are of concern to the learner.

When the teacher gives the young child the opportunity to use exploratory materials such as blocks to find the answer to the question, How many blocks are 2 blocks and 3 blocks? the pupil is required to do a simple type of quantitative thinking. An analysis of the way in which the child goes about finding the answer to the question af-

fords a method of evaluating his ability to do quantitative thinking. The goal of instruction in mathematics should be gradually to lead the learner to use increasingly mature procedures of quantitative thinking.[1] While the use of immature methods is characteristic of initial learning, the pupil should be encouraged to proceed to higher levels of responding.

Quantitative thinking is also involved in conveying numerical ideas in oral and written speech, in working examples, and in reading many kinds of materials. When a child is called on to answer questions about a graph, table, or chart in social studies, he is called on to use special types of reading and study skills involving quantitative thinking that should be developed and practiced as a part of the total mathematics program.

The ability to do the kind of quantitative thinking that is involved in solving real problems is most likely to be developed if the learner has frequent, directed experience in dealing with genuine problems that arise in daily life. It is difficult for the school to provide direct participation in more than a limited number of lifelike experiences because of the large number of children in a class. Thus it becomes necessary for the school to utilize a variety of planned vicarious experiences, such as a textbook contains, that give the children the opportunity to do quantitative thinking that is as similar to the thinking done in actual situations as is possible.

In this chapter the following topics are discussed: development of the ability to solve problems; teaching pupils to solve problems; reading and problem solving; activities for improving problem solving.

DEVELOPMENT OF THE ABILITY TO SOLVE PROBLEMS

Elements of a real problem

Psychologically speaking, problems grow out of the needs of individuals. The learner encounters a mathematical problem when he confronts a quantitative situation he cannot deal with in a habitual manner. When the problem interests him, he is instigated to take steps to explore and to solve it. These steps involve reflective thinking and may include experimentation. The individual's background of experience in dealing with quantitative aspects of problematic situations and his mental capacity are important factors that determine his success in dealing with the problem.

Elements of a problematic situation are:

1. A desired goal is to be attained.
2. There is a blocking of the path to be taken to attain the goal.
3. Available habitual responses are not suitable or adequate to attain the goal.
4. Various possible solutions (hypotheses) are proposed and tested.
5. A tentative conclusion is reached.[2]

Structured and unstructured problems

Verbal problems such as are commonly found in mathematics textbooks are structured in such a way that their interpretation by the pupil is relatively

[1]H. E. Moser, "Levels of Learning," *The Arithmetic Teacher* December 1956, 3:220–225.

[2]Robert L. Thorndike, "How Children Learn the Principles and Techniques of Problem Solving," *Learning and Instruction*, Forty-ninth Yearbook of the National Society for the Study of Education, Part I (Chicago: University of Chicago Press, 1950), pp. 192–215.

easy. The facts needed to find the answer to the question given in the problem are usually stated in the problem, and details concerning the situation presented give some guidance in determining the pattern to be followed in arriving at the solution. Experience in solving a wide variety of such problems contributes to the development of the ability to deal with many of the problems that arise in daily life.

Problems that arise in both childhood and adult experiences are often unstructured. Here the individual must do all of the preliminary structuring himself. He must learn to identify the relevant aspects of the problem; he must make any necessary assumptions that will simplify and clarify the situation; he must gather and organize the basic data needed and then determine the relationships among them by making any necessary computations, and when appropriate, bring them out sharply by tables and graphs. He must employ mathematical concepts to make interpretations and proceed according to principles to find the solution.

Working with unstructured problems is the most desirable form of mathematical applications. The teacher should plan the learning experiences of children in such a way that the work with problems of this type parallels the work with structured problems throughout the K–12 curriculum in mathematics.

TEACHING PUPILS TO SOLVE PROBLEMS

In its most general sense, the task of finding out how to subtract 28 from 41 is a mathematical problem. Pupils in a modern program are presented with this problem before being guided to a solution. Learning mathematics on any level will involve many problems that seem purely mathematical. Problem solving on the elementary level, however, usually means the solution of verbal problems. The following discussion on problem solving is directed specifically toward the solution of verbal problems, an activity that is common to both traditional and modern programs.

One of the major purposes of giving pupils verbal problems is to provide an opportunity to develop the thinking process. Rote methods for solving such problems reduce or eliminate thinking and therefore are self-defeating.

The range of problems considered suitable for the mathematics of the elementary school has varied widely. In the arithmetic programs of 50 years ago and more many of the problems had little relation to reality and social needs. About 20 years ago there was a tendency to confine such problems to those with demonstrated social applications. In current programs, more problems of a purely mathematical nature are common. There is little question that pupils perform at a higher level if they are working with material of interest to them, but interests vary widely and may be quite different in two adjacent classrooms on the same grade level. Teachers should make every effort to recognize class interests and to find problems related to these interests. However, the puzzle aspects of problems should not be overlooked. There are always some pupils who are willing to attack a problem on a puzzle basis without being concerned about its social value.

Guidelines for helping pupils solve problems must be very general or they become rote rules for special situations. The following are quite general:

1. Identify the problem question.
2. Determine what information is given and what is known.
3. Make a plan to determine what is wanted from what is known.

To identify the problem question the pupil must be able to read well enough to recognize what is wanted. One major cause of difficulty in dealing with verbal problems is a lack of basic reading skills. It must be recognized that the reading skills necessary for dealing with verbal problems in mathematics are much more specialized than those for general reading, but reading for problem solving does depend on basic reading ability. A pupil with low reading ability in the general sense will almost certainly find it difficult to handle verbal problems in mathematics.

Information that is given and information that is known are not necessarily the same. If a pupil is asked to change feet to inches, the problem may not mention that there are 12 inches in a foot, since the pupil is expected to know this fact. In analyzing a problem for what is given, the pupil should examine his basic fund of knowledge for information that may have bearing on the problem. Teachers may provide guidance in this matter by directing the pupil to sources of information, for example, the dictionary or collections of mathematical facts and formulas.

There is clearly no one simple plan that will guide one from the known information to the desired result called for in the problem. Plans for certain types of problems can be given in general terms. In problems involving per cent, it is a good idea first to identify the base in the problem, although no general rule for doing this may be given. As the pupil gains more experience in working with per cent problems, he will understand the problems better if in each case he identifies the base and recognizes its importance in the situation.

The most common plan for teaching problem solving has probably been to show pupils how to solve a few illustrative examples and then give them similar problems. As the process continues, problems from previous work are included with those of the type under discussion. This approach places a heavy burden on each pupil to form his own generalizations and procedures, verbal or nonverbal. Success in problem solving is largely dependent upon the pupil's ability to arrive at sound procedures based on experience as well as on his ability to read.

The current revolution in the teaching of elementary school mathematics has produced two new tools not generally available to teachers more than 10 years ago.

Writing an open sentence

The first involves the writing of an open sentence for the verbal problem. The basic plan for proceeding from the known information to the desired answer in the problem then requires the solution of the open sentence and the interpretation of the number obtained as the solution of the sentence. This plan is quite general and applies to all problems normally met on the elementary level. The plan is certainly not as simple as it sounds when stated in a single sentence. There is no easy way to teach a pupil how to write a correct open sentence for a verbal problem.

One of the advantages of writing an open sentence is that the teacher can almost immediately tell whether the pupil has correctly analyzed or interpreted the problem. If the problem is not ambiguous, an incorrect open sentence indicates that the pupil does not understand the problem. A correct open sentence is an indication that the pupil probably does understand the problem. Unfortunately, it is possible for the pupil to write a correct open sentence and not understand the problem. However,

a correct open sentence (equation) is probably a much better indication that the pupil has a grasp of the problem than the correct numerical answer, the most common criterion used in the past.

Identification of sets

The second tool is the identification of the sets in the problem. In most problems in the elementary school sets are combined, separated, or compared. The combining of sets may be described by addition or multiplication, while the separation and comparison of sets may be described by subtraction or division. Therefore, when a pupil recognizes that two sets in a problem are combined to get a third set, he can write the sentence $2 + 3 = \square$ provided he has correctly identified the numbers of the first two sets as 2 and 3 and has assigned the frame (variable) $\square$ to the set obtained by combining the first two sets. This procedure requires the ability to read the problem and recognize what happens to the sets involved. This approach does provide a mathematically sound procedure which is helpful to some pupils. A pupil may be able to write the correct open sentence for a problem without being able to give a verbal analysis of the set situation involved. The teacher again must recognize individual differences among the pupils and know when a discussion of the sets involved and how they are related will be of value to the class as a whole or to individual pupils. It is because a teacher must frequently make judgments of this nature that teaching is more correctly described as an art rather than a science.

With the tools just described, the overall plan for teaching pupils to solve verbal problems may be written as follows:

1. Read the problem to identify the problem question.

2. Identify the sets in the problem and the number associated with each set. If the number of a set is not known, assign it a number in the form of a letter, such as n, or a frame, such as $\square$. Determine how these sets are related.[3]

3. Write an open sentence that tells what happens to the sets as described in the problem. Identifying numbers in the problem as addends and sums (or factors and products) is an accepted guide in the writing of equations. Such identification may be a subverbal recognition of what happens to the sets in the problem.

4. Solve the open sentence to determine the number indicated by the letter or frame. The open sentence, when understood, indicates what operation (or operations) must be performed on the given numbers to obtain the number desired.

5. Interpret the number obtained in step 4 to indicate the answer to the problem.

The above plan applies to the standard "one-step" problems that have been part of arithmetic for years. A number of problems that traditionally involve more than one step may be solved by the above plan with a single open sentence. When the problems become too complicated for a single open sentence, they must be broken down into a sequence of "simple" problems.

Illustrating the steps in problem solving

The manner in which the five steps under discussion can be applied is demonstrated in the following examples:

[3]Some teachers prefer to break step 2 into two parts: (1) identify the sets and their numbers, and (2) determine how the sets are related. The above plan would then have six steps rather than five.

Problem 1 Jim has 15 cents and is given 45 cents. How much does he now have?

Step 1. The pupil should recognize that the problem is to determine the amount of money that Jim has after he is given 45 cents.

Step 2. The first set contains 15 cents; the second set contains 45 cents; the third set, obtained by combining the first and second sets, contains □ cents.

Step 3. The open sentence is $15 + 45 = \square$. This sentence indicates that 15 is to be added to 45.

Step 4. The number that makes the open sentence in (3) true is 60.

Step 5. Jim has a total of 60 cents after being given 45 cents.

Problem 2 Jane is given 30 cents and places it in her purse. She later discovers that she has 50 cents. How much money was in her purse at first?

Step 1. The problem is to determine the original amount of money in the purse.

Step 2. The first set is the set of □ cents originally in the purse; the second set is the set of 30 cents placed in the purse; the third set is the final amount of 50 cents and is obtained by combining the first two sets. It should also be noted that the first set may be obtained by removing the second set from the third, although the wording of the problem suggests combining the first two sets.

Step 3. Either of the following open sentences may be written as a result of the discussion in step 3:

$$\square + 30 = 50 \qquad \square = 50 - 30$$

Both of the above sentences are part of the same addition-subtraction pattern. Both are solved by subtracting 30 from 50.

Step 4. The required number is 20.

Step 5. The original amount in the purse was 20 cents.

The advantage of writing the first open sentence in (3) is that it stresses the relation between problem 1 and problem 2. In traditional arithmetic, problem 1 is usually classified as an addition problem and problem 2 as a subtraction problem. With a set analysis, the two problems can be classified as two different forms of the same problem.

Problem 3 How much must Sue pay for 8 five-cent stamps?

Step 1. The problem is to determine the cost of 8 stamps.

Step 2. The first set is a set of 8 stamps. A set of 5 cents is associated with each stamp. There are 8 such sets of 5 cents. The third set is the set of □ cents paid for the 8 stamps and is obtained by combining the 8 sets of 5 cents.

Step 3. The open sentence is $8 \times 5 = \square$.

Step 4. The required number is 40.

Step 5. 8 stamps cost 40 cents.

Problem 4 How many 5-cent stamps can Gene buy for 60 cents?

Step 1. The problem is to determine the number of stamps.

Step 2. The first set contains n stamps. A set of 5 cents is associated with each stamp. There are n such sets of 5 cents. The third set contains 60 cents, the total paid for the stamps, and is formed by combining n sets of 5 cents.

Step 3. The open sentence is $n \times 5 = 60$.

Step 4. The required number is 12.

Step 5. 12 stamps can be purchased for 60 cents.

Problem 4 can be visualized as requiring that the set of 60 cents be partitioned into equivalent subsets of 5

cents each. This leads to a comparison or measurement division, and the open sentence is:

$$n = 60 \div 5$$

The above sentence is again part of the same pattern as the sentence given in step (3) and both sentences indicate that 60 must be divided by 5 to obtain the desired number. The advantage of using the sentence given in step (3) is that problems 3 and 4 can then be more readily recognized as related problems.

Problem 5 Joe buys 5 stamps for 40 cents. How much does each stamp cost?

Step 1. The problem is to determine the cost of each stamp.

Step 2. The first set contains 5 stamps. A set of n cents is associated with each stamp. There are 5 such sets of n cents. The third set contains 40 cents, the total spent for stamps, and is obtained by combining 5 sets of n cents.

Step 3. The open sentence is $5n = 40$.

Step 4. The required number is 8.

Step 5. Each stamp costs 8 cents.

The above problem may be visualized as requiring that 40 cents be partitioned into 8 equivalent subsets. This suggests a partition division and the open sentence $n = 40 \div 5$. This sentence is part of the same pattern as that written in step (3). Both sentences indicate that the required number is obtained by dividing 40 by 5. The advantage of the sentence written in step (3) is that it indicates how problem 5 is related to both problems 3 and 4. A pupil who can recognize problems 3, 4, and 5 as different forms of the same problem should have a distinct advantage over the one who sees them as three unrelated problems.

It is important to recognize that all of the open sentences in the five illustrative problems above can be written as the result of combining two or more sets into a single set. In the last three problems, the sets that are combined are equivalent. If a pupil can recognize that 3 sets are combined to form a single set, he should recognize that the latter set can be partitioned into the 3 original sets. To talk about the cost of 8 stamps may suggest combining 8 sets, whether the cost of the stamps is known or not. However, if 40 cents is to be divided equally among 5 children, the separation of sets is clearly suggested. If the pupil can visualize putting the 5 sets back together, the sentence $5n = 40$ is suggested. Pupils will react differently to such situations and should not all be required to give the same solution. It is generally agreed that different methods of solving a problem should be encouraged and discussed. Children should be encouraged to do original thinking.

Problem 6 Joan has 50 cents. Sandy has 75 cents. How much more money does Sandy have?

Step 1. The problem is to determine how much more money Sandy has than Joan or to determine the difference in the two amounts of money.

Step 2. The first set contains 50 cents; the second set contains 75 cents; a third set may be constructed, but there should be no necessity to do so. The problem here is to compare the two sets, and this may be done by subtraction of the cardinal numbers of the sets.

Step 3. The open sentence is $n = 75 - 50$.

Step 4. The required number is 25.

Step 5. Sandy has 25 cents more than Joan.

Problem 7 Find the ratio of the amount that Sandy has to the amount that Joan has.

Step 1. The problem is to determine the ratio of two numbers.

Step 2. The sets are the same as in problem 6. In this case the sets are to be compared by division rather than by subtraction. A ratio is the comparison by division of two sets with similar elements.

Step 3. The open sentence is $n = 75 \div 50$.

Step 4. The required number is $\frac{3}{2}$, $1\frac{1}{2}$, or 1.5. All three numerals represent the same number and are correct solutions to the open sentence in step (3). Unless very specific instructions to the contrary are given, all answers should be considered correct.

Step 5. The ratio of Sandy's money to Joan's is $\frac{3}{2}$. The problem specifically indicates the ratio of Sandy's money to Joan's. If the problem merely asked for the ratio of the two amounts, either $\frac{3}{2}$ or $\frac{2}{3}$ would be correct.

Problem 8 Tom had a dollar (100 cents) and bought 2 candy bars at 10 cents each and a package of paper for 25 cents. How much money did he have left?

Step 1. Find how much Tom had left from one dollar.

Step 2. There is a set of 2 candy bars (some pupils may prefer to say there are 2 sets of 10 cents), where each bar is associated with a set of 10 cents. There is a set of 25 cents. There is a set of 100 cents (one dollar). There is a set of n cents remaining. The set of n cents combined with the 2 sets of 10 cents and the set of 25 cents form the set of 100 cents.

Step 3. The open sentence follows: $n + 2 \times 10 + 25 = 100$.

Step 4. The required number is 55.

Step 5. Tom has 55 cents left.

Problem 8 illustrates how a multistep traditional problem can be solved with a single open sentence. Some pupils may prefer to use several steps. Many such pupils will eventually prefer to use the one-step approach and should usually be allowed to proceed at their own pace.

Set language has been stressed in the analysis of the previous problems. Any teacher may accomplish similar results without emphasizing set language to this extent. While set language is now widely used in elementary school programs, it is not universally accepted. The teacher must make judgements as to the value of such language in the classroom. It will be difficult to make true evaluation without experimental evidence of its worth.[1]

In problem 1, step (2) may be simplified by saying that amounts of 15 cents and 45 cents are given. These numbers may then be identified as addends which will then lead to the same open sentence in step (3). Identifying the numbers as addends is essentially a subverbal way of recognizing that they are numbers associated with sets that are being combined.

The approach outlined in the illustrative problems above, with minor variations, is now almost universally accepted in modern programs. Emphasis on the set analysis varies from program to program, however. There is no substantial evidence to indicate that this approach, with or without the set analysis, is better than any other. Individual teachers report varying success, but many other factors are involved. Future research may provide more information on this matter.

Pupils should not write equations with labels on numbers, as $n + \$5 = \$10$. Numbers and not dollars are added or subtracted. One exception to this

[1]John W. Wilson, "The Role of Structure in Problem Solving," *The Arithmetic Teacher*, October 1967, 14:486–497.

rule is necessary because the notation \$1.25 for one dollar and 25 cents was introduced for social reasons long before decimal notation came into use. Before the introduction of decimal notation, 1.25 without the dollar sign is meaningless.

Verbal problems will always be difficult for a substantial number of pupils no matter what approach is used. The ability to read mathematical problems with discrimination usually comes only with much effort. Teachers should constantly be investigating ways in which this skill may be developed in pupils.

Solution of open sentences

The solution of open sentences is an integral part of the solution of verbal problems. When a sentence such as $\square + 3 = 5$ is solved in grade 1, the pupil need only know that $2 + 3 = 5$ is a true statement. Solution of such sentences provides additional practice in dealing with facts and valuable knowledge about the nature of an open sentence. The pupil solves an open sentence by intuition, by application of a mathematical principle, or by applying an axiom. For a description of these three procedures, see page 135.

A useful review of the ideas needed for understanding how to solve most of the open sentences occurring in the elementary program may be obtained with the following activities:

1. Ask the pupils to write all the open sentences they can with addends of 2 and 3 and a sum of n. There are 8 sentences:

$$\begin{array}{ll} n = 2 + 3 & n = 3 + 2 \\ 2 + 3 = n & 3 + 2 = n \\ n - 3 = 2 & 2 = n - 3 \\ n - 2 = 3 & 3 = n - 2 \end{array}$$

It is not usually necessary to write the right-hand set, but it is worth doing occasionally because of its value in stressing that if $a = b$ then $b = a$. This fact is not completely understood by many college students. The teacher should stress the fact that n represents the same number in all of the sentences above. The sentence $n = 2 + 3$ makes it clear that the number represented by n is obtained by adding 2 and 3 no matter which of the above equations is given.

2. Ask the pupils to write sentences in which the addends are 2 and n and the sum is 9. The following four equations illustrate the addition-subtraction pattern discussed on page 119. Again, n names the same number in all of the sentences shown below:

$$\begin{array}{ll} n + 2 = 9 & 2 + n = 9 \\ 9 - n = 2 & 9 - 2 = n \end{array}$$

The sentence $9 - 2 = n$ indicates that the number represented by n may be obtained by subtracting 2 from 9 and will be the same no matter which of the above sentences is given (as well as any of the other four discussed in (1) above.

3. Ask the pupils to write open sentences with factors of 3 and 4 and a product of n. Four such equations are:

$$\begin{array}{ll} 3 \times 4 = n & 4 \times 3 = n \\ n \div 3 = 4 & n \div 4 = 3 \end{array}$$

All of these equations indicate that the number n can be obtained by multiplying 3 by 4.

4. Ask the pupils to write open sentences in which the factors are 6 and n with a product of 30. Four such sentences are:

$$\begin{array}{ll} 6 \times n = 30 & n \cdot 6 = 30 \\ 6 = 30 \div n & n = 30 \div 6 \end{array}$$

All of the above equations indicate that the number (factor) n can be determined by dividing the product 30 by the known factor 6.

READING AND PROBLEM SOLVING

Problem solving cannot be taught as a skill, since the conditions in verbal problems dealing with social situations usually vary from problem to problem.[5] The learner's ability to solve problems depends on his intelligence, his reading ability, his understanding of number operations, and his background of experience. However, the teacher should give special help in vocabulary development, reading, and procedures for dealing with verbal problems such as are described in the following pages. Special work should be done to assist children to formulate mathematical sentences that provide the basis for solving equations.

The incidence of mathematical terms in reading is very large. Horn reported:

> Of the first 1069 words in the list compiled by Thorndike and Lorge, more than one in ten are reasonably specific arithmetical, geome'ı ıcal, or statistical terms. And if indefinite mathematical terms are included, the proportion is about one in four. A large number of these mathematical terms appear frequently in the texts and references in the content fields.[6]

Horn reported that such technical terms as the following were rated essential by both teachers of art and of mathematics: area, balance, breadth, circle, cube, depth, dimension, distance, horizontal, length, measure, parallel, perpendicular, rectangle, square, triangle, and unit.

[5]H. P. Spitzer and Frances Flournoy, "Developing Facility in Solving Verbal Problems," *The Arithmetic Teacher*, November 1956, 3:177–182.

[6]Ernest Horn, *The Teaching of Arithmetic*, Fiftieth Yearbook of the National Society for the Study of Education, Part 2 (Chicago: University of Chicago Press, 1951), p. 11.

Mathematical concepts frequently appear in combinations in reading, thus increasing the difficulty of dealing with them. The ability to understand such statements, in which mathematical terms have relations to other facts in a larger social setting, is heavily conditioned by the reader's grasp of the number system and requires functional quantitative thinking:

The difficulty of dealing with the mathematical concepts in reading is increased by the fact that they frequently appear in combination, as:

> almost two hundred years; through many centuries; millions of dollars; nearly two miles wide; ranges from twenty-five to one hundred twenty-five per square mile; nine hundred square miles; three and a half million; [the river] falls only four inches in a mile.[7]

It appears evident that teachers of mathematics should not limit the treatment of technical, quantitative, and mathematical terms to their strictly specialized meanings but should also help students to understand their interpretation in broader social situations, especially in the reading they do in all courses i.ı school. The significance of this poin ›r teachers of all curriculum areas cannot be too strongly stressed. Every t·acher in a sense is a teacher of mathematics. Similarly, every teacher of mathematics is a teacher of reading.

The ability to read plays an important role in quantitative thinking. Special kinds of reading skills are required in problem solving in addition to the ability to read and comprehend explanations of algo.:sms that are included in textbooks and workbooks.

[7]Horn, p. 11.

Research has shown that pupils who excel in problem solving are significantly superior to those who are poor in problem solving in the following fields:

1. Computational ability
2. Ability to apply the sequence of steps involved in problem solving
3. Ability to estimate answers to verbal problems
4. Range of information concerning social uses of arithmetic
5. Ability to read graphs, charts, and tables
6. Ability to see relations in number series
7. General and nonverbal reasoning ability
8. General reading level
9. Level of mental ability.

The good and poor problem solvers are not significantly different in the general reading skills used in literary reading, such as those included in the Gates Tests in General Reading, but they do differ significantly in the special reading skills required in mathematics, namely, ability to follow the steps in problem solving. It evidently is necessary to give special attention to teaching pupils inferior in problem solving the special reading skills peculiar to mathematics.

Well-constructed mathematics textbooks and workbooks often provide excellent reading exercises, which develop the reading skills required in problem solving and extend the vocabulary. Teachers should not hesitate to use suitable reading exercises in mathematics textbooks and workbooks, beginning with those that are somewhat below the level of problems the pupil can solve reasonably well and gradually progressing to exercises of greater difficulty found in textbooks for the higher grades. Some of the more valuable kinds of helps in problem solving are described in the following pages.

ACTIVITIES FOR IMPROVING PROBLEM SOLVING

The following activities are among those that many educators believe are helpful in aiding pupils to achieve a higher level of ability in problem solving:

1. Encourage pupils to submit different solutions to each problem. If the solution suggested is incorrect, show why it is wrong. If the solution is correct but inefficient, the pupil should be made aware of this fact. Stressing a variety of solutions helps to reduce the tendency of pupils to learn rote rules for solving verbal problems.

2. Encourage the pupil to estimate answers to problems. This is not an easy skill to develop and the teacher should constantly supply guidance for pupils. For example, if the correct answer is 915, an estimate of 1000 is usually considered acceptable, but a better practice would be to say that the correct answer is between 800 and 1000. The practice of estimation helps pupils decide which answers are sensible.

3. Give the pupils an open sentence and ask them to construct a verbal problem that will require the given open sentence for a solution. For example, if the sentence $3 \times n = 24$ is given, the following might be among the problems given by the pupils:

a. If the product is 24 and one factor is 3, find the other factor.

b. What is the cost of 1 bar if the cost of 3 bars is 24 cents?

c. Joe traveled 24 miles in 3 hours on his scooter. What was his average rate in miles per hour?

d. A test with 3 questions had a value of 24 points. What was the value of each question if all the questions had the same weight?

e. Each paper cost n cents. Three papers cost 24 cents. What is the cost of each paper?

If this activity is repeated with some regularity, it may provide one of the most useful aids to problem solving.

4. Use set sentences as an intermediate situation to help pupils learn to write verbal problems. The following examples give the set sentence and the corresponding sentence.

a. A set of 3 combined with a set of 5 is a set of n; $3 + 5 = n$.

b. A set of n combined with a set of 4 is a set of 10; $n + 4 = 10$.

c. A set of n combined with a set of m is a set of t; $n + m = t$.

d. 3 sets of 4 is a set of n; $3 \times 4 = n$.

e. 4 sets of n is a set of 20; $4n = 20$.

f. n sets of 7 is a set of 28; $7n = 28$.

g. n sets of t is a set of m, $nt = m$.

5. Write a set sentence as in (4), and ask the pupils to give verbal problems that can be described by the set sentence. For the set sentence in (4a), a pupil might write the following verbal problem: John earned $3 in the morning and $5 in the afternoon. How much did he earn in a day?

6. Write an open sentence, for example, $3n + 10 = 50$, and ask the pupil to write or state a set sentence that corresponds to it. In this case the answer is 3 sets of n combined with a set of 10 is a set of 50. The activities outlined in (4–6) should give the pupil specific help in learning how to read and analyze simple verbal problems.

7. Encourage able pupils to generalize. If a pupil solves the following sequence of problems he should be able to recognize that the cost per stamp is obtained by dividing the total cost, c, by the number of stamps, n. In conjunction with the previous exercises one can say that n sets of t cents is a set of c cents where t is the cost of one stamp. This leads to the open sentence $n \times t = c$, which may also be written as $t = c \div n$.

a. If 3 stamps cost 15 cents, what is the cost of 1 stamp?

b. If $\square$ stamps cost 15 cents, what is the cost of 1 stamp?

c. If n stamps cost 15 cents, what is the cost of 1 stamp?

d. If 3 stamps cost $\square$ cents, what is the cost of 1 stamp?

e. If 3 stamps cost c cents, what is the cost of 1 stamp?

f. If n stamps cost c cents, what is the cost of 1 stamp?

8. Use problems with multiple answers. In some problems there is no single answer. Consider the following problem:

Mr. Smith plans a 240-mile trip. He plans to drive at a speed of not less than 40 miles an hour or more than 60 miles an hour. How long will the trip take if he carries out his plan?

If he were to drive 40 miles an hour at all times, his time would be $240 \div 40$, or 6 hours. If he were to go 60 miles an hour at all times, his time would be $240 \div 60$, or 4 hours. So his time would undoubtedly be between 4 hours and 6 hours. We can express the number of hours as the answer to the problem mathematically as $t \leq 6$ and $t \geq 4$; that is, t is equal to or less than 6 hours and equal to or greater than 4 hours.

The following problem is one that yields multiple answers that the class should discuss:

Tom had $5.75. He wants to buy two 75-cent records and an album. How much could he pay for the album?

The answer obviously is that he can pay any amount less than \$5.75 − (2 × \$.75) for the album. The pupils should be led to discuss the fact that the answers to such problems cannot be given with a single number.

9. Use exercises specifically designed to develop vocabulary. Exercises similar to those applied in reading will broaden and sharpen the understanding of words used in mathematics and improve the work in problem solving. The important point for the teacher to bear in mind is that the ideas represented by a given vocabulary contribute to the reading difficulty of a statement. Thus the words in the following sentence are all in the first 2500 in the Thorndike word list, but the context in which they are used involves ideas that are unintelligible to grade 4 children: "The square of the sum of two numbers is equal to the square of the first number added to twice the product of the first and second numbers, added to the square of the second number."[8]

When children do not understand the situation a verbal problem presents, they either give up or venture guesses as to procedures to use, or merely juggle the numbers that are given and arrive at answers that are meaningless. The teacher must develop the meaning of a situation and the *background* necessary to understand it. Otherwise the children will not succeed in solving the problem. Vocabulary exercises[9] such as the following should be supplemented by lessons in the looking up of the specific meanings of difficult and unfamiliar words in problems, explanations, and discussions when the need arises:

a. Matching words with definitions, objects, pictures

b. Multiple-choice exercises requiring selection of correct word from several choices

c. Completion exercises in which missing words are supplied by the learner

d. Naming the unit of measure or the instrument used in measuring various items or aspects of things

e. Writing the correct words for abbreviations

f. Naming geometric figures or parts of drawings or drawing representations of expressions

g. Performing some action to show meaning

h. Restating expressions in other words

i. Correcting faulty statements

j. Preparing original lists of words arranged according to headings given

k. Naming units and instruments of measurement used by various workers, such as grocers, carpenters, and clerks

l. Writing lists of words that relate to given words, such as fraction, money, circle, time.

The left-hand column of Table 17.1 lists verbal statements while the right-hand column lists algebraic or symbolic equivalents. Pupils should be given practice in supplying the verbal equivalent to symbolic statements *and* in supplying the symbolic equivalents of verbal statements.

10. Watch for opportunities to make problems from major news developments of the day. If a new plane is being released that has a speed of 2000 miles per hour, ask the pupils how long it will take this plane to fly across the country; to London; around the world. Encourage pupils to bring in newspaper

[8] W. E. Young, "The Language Aspects of Arithmetic," *School Science and Mathematics*, March 1957, 57:172.

[9] Harry C. Johnson, "The Effect of Instruction in Mathematical Vocabulary upon Problem Solving in Arithmetic," *Journal of Educational Research*, October 1944, 38:97–110.

TABLE 17.1

Developing a Mathematical Vocabulary

Verbal Statement	*Symbolic Statement*
The sum of $\square$ and 3	$\square + 3$
The number $\triangle$ increased by 2	$\triangle + 2$
12 decreased by some number	$12 - n$
21 increased by some number	$21 + n$
The product of n and 4	$n \times 4$
The quotient of n and 2	$n \div 2$
Tom is n years old. Harry is 3 years older	$n + 3$
The cost in cents of n dolls at 40 cents each	$n \times 40$
The cost in cents of 4 apples at n cents each	$4 \times n$
The cost in cents of n apples if 1 apple costs 5 cents	$n \times 5$
The average weight of 4 chickens whose total weight is n pounds	$n \div 4$
The number n decreased by $\frac{1}{2}$	$n - \frac{1}{2}$
$\frac{1}{4}$ of a number	$\frac{n}{4}$, or $\frac{1}{4} \times n$
The area of a rectangle whose length is 10 inches and whose width is x inches	$10 \times x$
The distance an automobile goes in 1 hour if it goes 120 miles in x hours	$120 \div x$

clippings from which they have constructed problems.

11. Occasionally give a problem with some necessary information missing or one that contains some information not needed. Encourage class discussion of such problems and help pupils recognize what is missing or what information is not needed.

A general program for developing ability to solve verbal problems

The elements of an effective program for developing the ability to solve verbal problems may be stated as follows:

1. Discuss uses of number in social situations that arise or in pictures and illustrations in textbooks to develop vocabulary and background experiences.

2. Give the pupil many opportunities to write open sentences for simple situations. This activity should be started in grade 1 and continued on increasing levels of difficulty.

3. Give the pupil abundant experience in reading and solving easy, meaningful verbal problems in textbooks and workbooks, beginning at or slightly below his level of development. Emphasize the u[illegible] of mathematical sentences as the basis of problem solving. Keep the numbers small at the start. Gradually increase the difficulty of the vocabulary, computations, and situations.

4. Be sure to help the learner to understand the meaning of each number process by manipulative experience and by use of set situations.

5. Have pupils use manipulative materials to demonstrate and work out solutions of simple problems so as to help them to visualize the situations and the relations involved. Have pupils tell orally about the procedures they use in terms of sets.

6. Plan reading experiences that will develop the special reading skills in problem solving. Take one skill at a time on the sequence listed. This is very important. See workbooks for special helps.

7. Emphasize vocabulary development by suitable exercises similar to those used in reading.

8. Do special work on measures and their use in social situations. Help pupils to develop and then use tables of measure. Demonstrate or visualize the processes of conversion from large to small, and small to large, a basic source of difficulty in problem solving.

9. Develop basic rules, formulas, and procedures through real situations involving manipulation of representative materials, drawings, visualization, and thinking through the relations involved; for example, perimeter, area, costs, interest, percentage, and so forth.

10. Use problems without numbers to help the pupil learn to state in his own words how to find answers.

11. Emphasize the need for accuracy in all computations. Teach pupils to go over their work to check it.

12. Teach pupils above grade 5 procedures for approximating and estimating answers to see if their answers are sensible.

13. Use a consistent plan for writing solutions of verbal problems similar to that outlined on page 306.

EXERCISES

1. What is the difference between problem solving and quantitative thinking?
2. List steps in solving a verbal problem.
3. Give an example of problem solving growing out of a life situation.
4. Write a set sentence for $3n + 5 = 28$.
5. Illustrate the concepts underlying problem solving by applying them to a real situation.
6. What is the value of open sentences in problem solving?
7. What are the three activities with sets that are related to problem solving in the elementary school?
8. Prepare a verbal problem that can be solved in more than one way.
9. How would you develop the mathematical concepts underlying the cost-number of items-price relationship?
10. Show why the teaching of reading has an important place in arithmetic.
11. Describe several vocabulary-building exercises useful in elementary mathematics.
12. What specific methods can you recommend for teaching slow learners to solve problems?
13. Why is problem solving not a skill?

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NONMETRIC GEOMETRY

Sets of numbers are the building blocks of arithmetic and algebra. Sets of points are the building blocks of geometry. The concept of a point is undefined, as is the concept of a set.

Figure 18.1

On the elementary level a point is usually described as a position in space. A point is represented by a dot such as the period at the end of this sentence. Points are usually labeled by capital letters, as illustrated in Figure 18.1. A *geometric figure* is a set of points. The set of two points in Figure 18.1 is a geometric figure but has no common name other than a pair of points. *Space* is the set of all points. Some subsets of space have specific names, as lines, curves, planes, solids, and angles.

The numeral 3 is a collection of ink that represents an idea called a number. The number cannot be seen or touched, but it is easier to work with when it has a symbol or name. The two dots in Figure 18.1 are also symbols that represent the geometric idea called a point. A point is also an idea that cannot be seen or touched. The dot makes the point easier to visualize and work with, just as the numeral does for the number.

Points, lines, planes, and other sets of points are ideas that cannot be seen or touched, but many things in everyday life are good approximations of these ideas. Nature and the works of man are

full of things that suggest points, lines, and other common geometric figures. The distinction between the dot and the point that it represents should not be overemphasized but must be understood. The dot has measurable dimensions while the point does not. The geometric idea called a line has no width, while the physical representation of a line does. Confusion may result if the differences between the geometric idea and its physical representation are not understood.

Nonmetric geometry deals with properties of sets of points not involving measurements. Nonmetric properties include such ideas as shape, interior, exterior, and between.

The following topics will be discussed in this chapter: one-dimensional figures; two-dimensional figures; three-dimensional figures.

ONE-DIMENSIONAL FIGURES

Place a pencil on a dot representing point A. Move the pencil to a dot representing point B. The result is a physical representation of a part of a *curve* (curved line).

There are an infinite number of paths between two different points. The shortest path between two such points represents the *line segment* connecting these two points.

Points, curves, lines, and line segments cannot be defined on an elementary level and are considered as undefined terms on much higher levels. The pictorial aspect of geometry makes it possible to illustrate these ideas readily and to provide many activities that will contribute to an understanding of their properties.

The earliest activities in geometry should involve drawing dots as representations of points and connecting these dots with a variety of paths. Rulers should be introduced to aid in drawing paths that represent line segments. At this point the rulers are used as straight edges and not as measuring instruments. Some suggested activities are as follows:

1. Place two dots on a sheet of paper and draw three paths that do not represent (straight) line segments. It is almost universally accepted that the term "line segment" means a straight line segment.
2. Draw three different dots (representing points) on a sheet of paper. Draw as many paths representing line segments as possible connecting these three dots.

As a challenge, ask the pupils if there are any special situations. A special case exists when the three points lie on the same straight line. In all cases there are three line segments connecting the points A, B, and C. These are usually labeled AB, BC, and AC. When the three points lie on a line, AB and BC lie on AC (see Fig. 18.2). While the triangle in Figure 18.2 is two-dimensional, the emphasis here is on one-dimensional line segments. It would be too artificial to restrict these activities to one dimension.

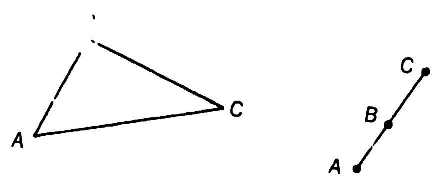


Figure 18.2

3. Draw four dots and label them A, B, C, and D. Connect these dots with paths representing line segments in as many ways as possible. The six possible line segments are AB, AC, AD, BC, BD, and CD.
4. Challenge superior students to discuss the special situations that may

occur in the previous activity (item 3). Three points may be on a straight line or all four points may lie on a line.

5. Identify the nine line segments in the drawing in Figure 18.3.

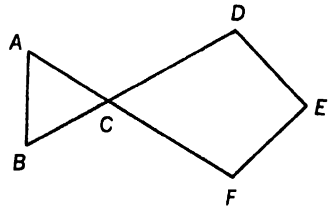


Figure 18.3

6. Use pictures in a variety of textbooks and identify figures that suggest points, portions of curved lines, and line segments.

7. Use the classroom to find representations of points and line segments as well as portions of curves. The corner of a room or a window suggests a point. The edges of books, walls, or windows suggest line segments. Maps and pictures provide representations of portions of curved lines.

8. Examine the playground and its vicinity for representations of points, line segments, and portions of curved lines.

9. Have the class draw designs using only line segments (see Fig. 18.4).

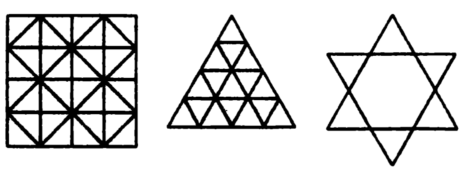


Figure 18.4

10. If the pupils are familiar with the use of a compass, have them draw designs using only portions of curved lines (see Fig. 18.5). A coin or some other circular object may also be used.

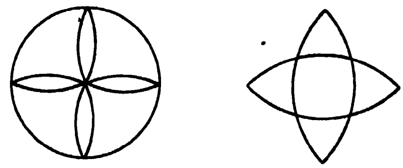


Figure 18.5

11. Have the class draw designs involving both line segments and portions of curved lines (see Fig. 18.6).

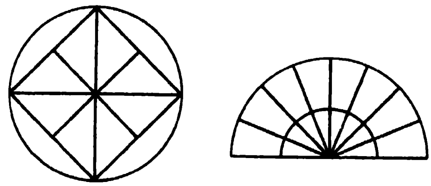


Figure 18.6

12. If reproductions of abstract art are available, attempt to find some that fall into the categories described in activities (8–10). Many paintings by Mondrian involve only rectangles.

These and similar activities should enable students to become familiar with the geometric ideas of points, line segments, and portions of curves (curved line segments).

Up to this point, care has been taken to distinguish between a point or a line and its representation. Complete accuracy demands the statement that the representation of a line be drawn. Ordinary usage involves the statement that the line segment be drawn. Although the latter statement is incorrect, its intent is clear and it is much less awkward than the correct statement. From this point on, the abbreviated but inaccurate statement will be used, as is common practice, except for an occasional reminder or in situations where precise language is necessary to avoid confusion. For example, in activity (3),

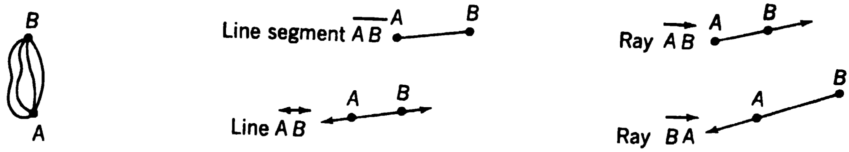


Figure 18.7

the second statement will now be written: Connect the three points with line segments in as many ways as possible. This statement is less awkward and is not likely to be misinterpreted.

Lines and rays

A line segment is a part of a *line*. In set language, the points of a line segment form a subset of the set of points forming the line. The concept of a line segment comes first. A *line* is obtained by extending a line segment indefinitely in both directions. A *ray* results when a line segment is extended indefinitely in only one direction. Figure 18.7 illustrates how line segments, lines, and rays are represented on paper. The notation shown in Figure 18.7 is now quite standard. Rather than writing *ray AB* or *line AB*, it is easier to write $\overrightarrow{AB}$ or $\overleftrightarrow{AB}$, respectively. When metric geometry is introduced, the symbol AB is used to represent the length or measure of the line segment $\overline{AB}$. The symbol $\overline{AB}$ represents a set of points while the symbol AB represents a number (the length or measure of the line segment).

Figure 18.7 also illustrates the sequence for a child to follow in learning fundamental geometric concepts.

Two points (dots) are first drawn and connected with a variety of paths. Two points are then connected with a line segment (a special path). The line segment is then extended indefinitely in both directions to obtain a line. The line segment is extended indefinitely in only one direction to obtain a ray.

This sequence introduces points, curved line segments, straight line segments, lines, and rays in that order. Some teachers may prefer to introduce the line after the ray.

The following activities illustrate how to introduce the concepts of a line and a ray:

1. Have the class connect two points with a line segment. Ask for the symbol that represents this set of points ($\overline{AB}$). Ask the class for suggestions for indicating that this segment is to be extended indefinitely in both directions. If no one in the class suggests the use of arrows, draw several different symbols, including $\overleftrightarrow{AB}$, and ask the class which one suggests an extension in both directions. Name this idea as a line and introduce its symbol, $\overleftrightarrow{AB}$.

2. Ask each pupil to draw pairs of lines in as many different relative positions as possible (see Fig. 18.8).

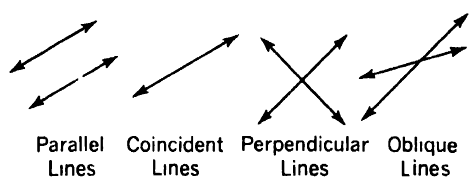


Figure 18.8

This type of activity provides readiness for parallel and perpendicular lines with no necessity to refer to these ideas by name. Two lines are parallel (with no common point) or intersect (with at least one point in common). If the lines intersect, they are either coincident

(with all points in common) or have exactly one point in common. If the lines intersect and are not coincident, they are either perpendicular or oblique. Figure 18.9 shows a set diagram illustrating how sets of pairs of lines are related.

Pairs of Lines

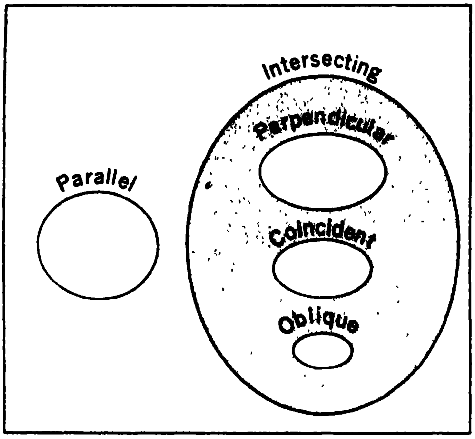


Figure 18.9

3. Have the class draw a line segment connecting the points A and B. Ask for suggestions for showing how to indicate that the segment is extended indefinitely in one direction. Ask in how many different ways this can be done. Introduce the notation $\overrightarrow{AB}$ and $\overrightarrow{BA}$ for the two rays that can be related to the pair of points A and B. Help pupils recognize that the first letter in the symbol for a ray always indicates the starting point of the ray.

4. Have the pupils draw pairs of rays in as many different relative positions as possible (see Fig. 18.10).

The concept of a line segment is simpler than that of a line or ray because representations of a line segment occur everywhere in the child's environment. The concept of a line and ray is introduced after that of a line segment. The ray is usually introduced as the basis

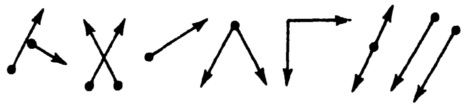


Figure 18.10

for understanding the angle concept. The angle will be discussed under two-dimensional figures.

Figure 18.11

A *curve* (curved line) that is not closed extends indefinitely in both directions. Figure 18.11 shows a curve that is not closed. Curves, lines, rays, line segments, and curved line segments are all one-dimensional figures because they have no width. These five ideas form the fundamental set of one-dimensional figures. Other one-dimensional figures may be formed by combining two or more of these ideas, as illustrated in Figure 18.12.

Figure 18.12

Mathematicians usually consider the set of curves to include the set of lines. In set language, the set of lines is a subset of the set of curves.

It is a worthwhile bulletin board project to post a list of everyday representations of points, line segments, lines, rays, curved line segments, and curves. A brief illustrative example of such a list is given below:

Points

End of a pin
End of a pencil
Corner of a room
Corner of a sheet of paper
Position on a map

Line segments

Edge of a sheet of paper
Road on a map
Boundary on a map
Edge of an athletic field
Graphs
Edge of a building
Goal line
Center stripe on a road

Lines

A straight line graph over an infinite period of time

Radio waves from the same station in opposite directions

A center stripe on a long portion of straight road

A railroad rail on a long section of straight track

The latter two are not lines because they do not extend indefinitely, although they appear to do so to the observer.

Rays

A line of sight starting from one's eye
A beam of light from a searchlight
A ray of light from the sun
A radio beam from a transmitter

Curved line segments

Boundaries on a map
Graphs
A river on a map
A wire from one pole to another
Veins in leaves
A clothes line

Curves

A curved line graph over an infinite period of time

The path of a comet not in closed orbit

The center stripe on a curved road

A railroad rail on a curved section of track

The latter two are not curves, since they do not continue indefinitely, although they appear to do so to the observer.

It may be helpful to keep such a list for an extended period of time and let the pupils add to it as they get new ideas.

Distinctive points

A line segment is an infinite set of points. In a line segment $\overline{AB}$, as illustrated in Figure 18.7, A and B are the *endpoints*. Each endpoint has points of the set on only one side of it. All the other points of the set have points on both sides. This property may be described in a different way. From an endpoint it is possible to travel in only one direction and remain in the set. From any other point of the line segment, it is possible to travel in two directions and remain in the set. The endpoints are sometimes called *distinctive points* because their properties are different from those of other points in the set. After the distinctive nature of the endpoints has been discussed and understood by the class, it may be interesting to examine other geometric figures for distinctive points.

A curved segment has two distinctive points, the endpoints.

A line has no distinctive points, since it has no endpoints.

A ray has one distinctive point, its initial point.

A curve has no distinctive points, since it has no endpoints.

Other activities that will enable pupils to recognize important properties of lines and curves are the following:

1. Have the pupils draw a line $\overleftrightarrow{AB}$ on a sheet of paper.

2. Have the pupils draw a curved line that intersects $\overleftrightarrow{AB}$ twice.

b. Have the pupils draw a curved line that intersects $\overleftrightarrow{AB}$ three times.

c. Have the pupils draw a line that meets $\overleftrightarrow{AB}$ only once.

d. Have the pupils draw a line that meets $\overleftrightarrow{AB}$ twice. The class should learn from this activity that two different lines can intersect in only one point, and if two lines have more than one point in common, they have all points in common.

e. Have the pupils draw a line that does not meet $\overleftrightarrow{AB}$ on the paper but such that it would if the paper were large enough.

f. Have the pupils attempt to draw a line that would not meet $\overleftrightarrow{AB}$ no matter how large the sheet of paper. Such lines can then be defined as *parallel*.

2. Have the pupils identify representations of line segments belonging to parallel lines, such as opposite edges of sheets of paper, windows, and rooms.

3. Have the pupil place two dots labeled A and B on a sheet of paper.

a. Ask the pupil to draw three lines passing through point A.

b. Discuss with the class how many lines can be drawn through a point, such as A. An infinite number of lines can pass through a point such as A. When physical representations of a line are drawn through point A they will tend to merge with a relatively small number and become indistinguishable. This example is a good illustration of the difference between the abstract line (the idea) and its physical representation on a sheet of paper.

c. Ask the pupils in the class to draw as many different lines as possible through points A and B. Pupils should discover in this manner that only one line can pass through two points.

4. Have each pupil place three dots on a sheet of paper. Ask that a line be drawn through the three points. The class should discover that it is not always possible to draw a line through three points. When three points are in the special position so that a line can pass through all three of them, the points are said to be *collinear*.

5. Have each pupil draw a line $\overleftrightarrow{AB}$ on a sheet of paper. Direct them to place a point C on the paper that does not lie on $\overleftrightarrow{AB}$.

a. Ask the pupils to draw a line through C that will not meet $\overleftrightarrow{AB}$ even if the paper extended indefinitely.

b. Ask the pupils to draw two different lines through point C that will not meet line $\overleftrightarrow{AB}$ no matter how far extended. The pupils should then discover that only one line can be drawn through point C parallel to $\overleftrightarrow{AB}$.

6. Have each pupil draw a line $\overleftrightarrow{AB}$ on a sheet of paper and place a point C on this line.

a. Have the pupils draw two lines through C that are different from $\overleftrightarrow{AB}$.

b. Discuss with the class if any particular line seems more special than any other line. Draw several different situations, including one in which the lines are perpendicular. Ask which of these is a special line. The class will probably identify the perpendicular line as a special situation. Perpendicular lines must be recognized on a purely nonverbal basis before the angle concept is introduced. The term "square corner" is sometimes used to describe perpendicular lines before the angle concept is presented. A corner of a card or sheet of paper may be used to determine if two lines are perpendicular or to draw perpendicular lines.

Drawing conclusions

The art of teaching involves the ability to choose and direct activities that are interesting and instructive. The activities are most successful when they

lead many pupils to discover important mathematical relationships. Drawing conclusions from a limited amount of evidence may be a dangerous procedure. It is most productive under the guidance of a skillful teacher who will help pupils to reject false conclusions as well as to accept correct ones.

It may be helpful occasionally to deliberately lead a class to make a conclusion that can then be shown false. Optical illusions, as illustrated in Fig. 18.13, are frequently used for this purpose.

Eventually a pupil must understand the role of mathematical proof as the

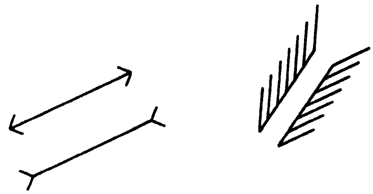


Figure 18.13

basis for judging conclusions. Construction of proofs is not usually considered an acceptable activity for the elementary school. In directing activities such as those discussed here, an alert teacher will find many opportunities for providing readiness for the concept of proof.

EXERCISES

1. Define space.
2. Name several proper subsets of space.
3. Describe a point.
4. What is the distinction between a point and a dot?
5. Define a geometric figure.
6. What is the simplest geometric figure?
7. Identify the 10 line segments in Figure 18.14.

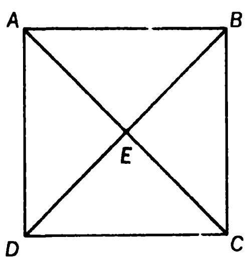


Figure 18.14

8. How many curved lines can be drawn from point A to point B?
9. How many line segments can be drawn from point A to point B?
10. How many lines can be drawn that pass through the points A and B?
11. How many line segments can be drawn connecting four different points?
12. Name an important subset of the set of curves.
13. Identify each of the geometric figures of Figure 18.15.

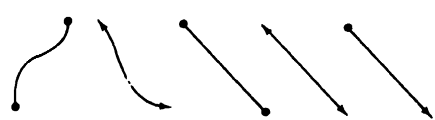


Figure 18.15

TWO-DIMENSIONAL FIGURES

A one-dimensional figure is a curve or a subset of a curve. A two-dimensional figure is a *surface* or a subset of a surface. Rolling farm land suggests a curved surface. A table top suggests a flat or *plane* surface. Almost all elementary work with two-dimensional figures is done with a plane and its subsets. A sur-

face has no thickness. A physical representation of a surface, such as a sheet of paper or the skin of a ball, does have thickness. A *region* is a part of a surface. A curve (extending indefinitely in both directions) separates a surface into two regions. The curve (which may be a straight line) acts as the boundary that must be crossed to travel from one region to the other.

Closed curves

A circle is the most common example of a closed curve. Figure 18.16 illustrates some closed curves.

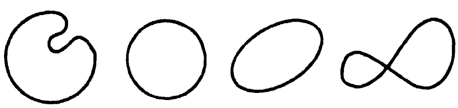
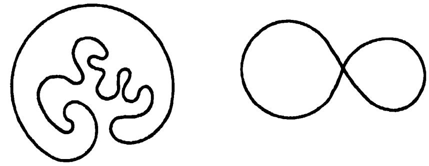


Figure 18.16

A closed curve may be drawn by placing a pencil on any point and making a path that ends when it returns to the starting point. Such a path may or may not cross itself. It is mathematically important to know whether a closed curve does or does not cross itself. A closed curve that does not cross itself is called a *simple* closed curve. The word "simple" in this case is used in a scientific sense and not in a descriptive one. A simple closed curve may appear to be more complex than a nonsimple closed curve, as illustrated in Figure 18.17. A simple closed curve separates the plane into two regions, an inside and an outside. It can also be said that a simple closed curve partitions the plane into three subsets, the inside, the outside, and the boundary (the curve itself).

An interesting activity is for the class to investigate distinctive points on a closed curve (see p. 323). A curve that is not closed has no endpoints because it extends indefinitely in both directions. A closed curve has no endpoints because it ends at the point at which it begins. Any point in a simple closed curve may act as the beginning and endpoint. A simple closed curve has no distinctive points. When any point is removed from a closed simple curve, the remaining curve is in one piece but is no longer closed. When the crossing point is removed from a nonsimple closed curve, as in Figure 18.17, the remaining curve is in two parts, neither of which is closed. From any point on a simple closed curve it is possible to travel in two directions and remain on the curve. From a crossing point on a nonsimple closed curve it is possible to travel in more than two directions and stay in the set (or on the curve).



A simple curve does not cross itself | A nonsimple curve crosses itself

Figure 18.17

Inside and outside

The meaning of "inside" and "outside" in the previous discussion should be clear from the everyday meaning of the words. It is now common practice to include exercises in the early elementary grades that require the pupil to recognize whether a point is inside, outside, or on the boundary of a simple closed curve.

Both the curve that is not closed (including the line) and the simple closed curve separate the plane into two regions, but only in the case of the closed curve may the two regions be readily labeled inside and outside. The two regions formed by a curve that is not closed may be identified as upper and

lower or left and right but not as inside and outside.

A line separates the plane into two regions while a line segment does not. Figure 18.18 illustrates how it is possible to get from point A to point B without crossing the line segment. In the case of the line and the closed curve, it is not possible to get from point A to point B (remaining on the surface) without crossing the boundary.

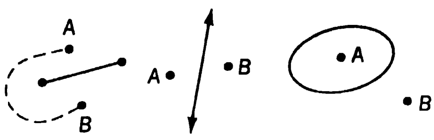


Figure 18.18

The following activities can be used to increase the understanding of inside and outside.

1. Refer to Figure 18.19. Have pupils identify points inside, outside, and on the square. Do the same for the circle and the triangle.

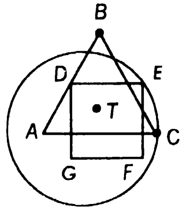


Figure 18.19

2. Have pupils draw four different points on a sheet of paper and label them A, B, C, and D. Ask the pupils to draw a simple closed curve so that A and B are inside and C and D are outside. These instructions may then be changed to include one or more of the points on the line, and so on.

3. Refer to Figure 18.20. Establish the fact that all points in this figure are outside the simple closed curve. Have the pupils count the intersections of each line segment with the curve and discover that there are an even number of crossings.

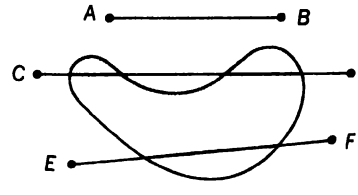


Figure 18.20

4. Refer to Figure 18.21. Establish the fact that the pairs of connected points are such that one is inside and one is outside. Help the pupils discover that the number of intersections is odd in this case.

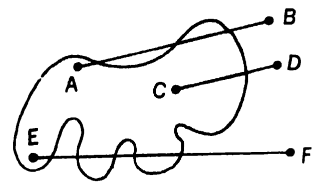


Figure 18.21

5. Refer to Figure 18.22. Establish the fact that all points are inside the simple closed curve. Again count the intersections between connecting line segments and the curve and help pupils discover that in each case the number of intersections is even.

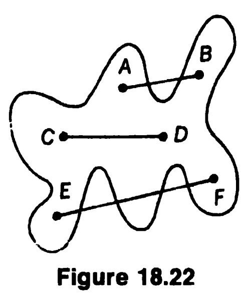


Figure 18.22

Items (3–5) should enable the pupils to discover that if points are on the same side (both inside or both outside) of a

simple closed curve, connecting line segments will intersect the curve an even number of times. If pairs of points are situated so that one point is inside and the other is outside the simple closed curve, connecting line segments will intersect the curve an odd number of times. Figure 18.23 illustrates that the even-odd generalizations do not apply to nonsimple curves. In this case, two interior points are connected with a segment that intersects the curve in only one point (the distinctive point).

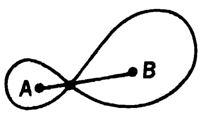


Figure 18.23

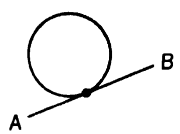


Figure 18.24

Figure 18.24 illustrates another situation that apparently contradicts the even-odd generalization. In this case, the connecting segment is tangent to the circle and again apparently intersects the circle in an odd number of points (one point). This apparent contradiction cannot be explained on an elementary level. When interpreted algebraically, it can be interpreted as a double point. It should be clear that such unusual cases are not to be introduced unless some very able pupil asks about them. The activities described in items (3–5) should be used with more able or smaller groups.

Polygons

A *polygon* is a closed curve. This is a correct statement and a good description but not an acceptable definition. A circle is a closed curve but is not a polygon. A polygon is a closed broken line segment. This statement is also correct but is not an acceptable definition. A coat hanger can be used to illustrate why the last statement is not an acceptable definition for a polygon. Think of the coat hanger as a crude triangle. Ignore the handle or cut it off. Bend the hanger, as illustrated in Figure 18.25. The result is a four-sided closed broken line segment that is not a polygon because it is not a plane figure. An almost complete definition follows: A polygon is a closed plane figure formed by line segments joined end to end.

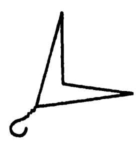


Figure 18.25

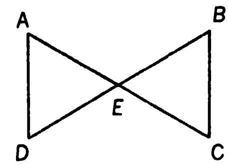


Figure 18.26

By the above definition, Figure 18.26 is a polygon formed by the six segments $\overline{DE}$, $\overline{EB}$, $\overline{BC}$, $\overline{CE}$, $\overline{EA}$, and $\overline{AD}$. However, DE and EB are not acceptable as adjacent sides, since they lie on the same line.

A correct definition is: A polygon is a closed plane figure formed by line segments joined end to end with no two adjacent segments on the same straight line.

By this definition, Figure 18.26 is a four-sided polygon formed by the segments $\overline{AC}$, $\overline{CB}$, $\overline{BD}$, and $\overline{DA}$. It should be clear that a complete definition need not be given at the elementary level. Some authors use the almost complete definition given above, but it is probably best to deal with polygons at the elementary level on a pattern basis. The use of precise language too soon may be more confusing than helpful.

Identifying figures

Triangles, squares, and rectangles are identified by pupils in the early elemen-

tary grades in many current programs but they are not usually recognized as polygons until the upper elementary grades.

Early identification of geometric figures must be entirely by shape on a nonverbal basis because identification takes place before concepts of equal segments and right angles are introduced. Squares should be recognized as special cases of rectangles in the earliest stages. The set of squares is a subset of the set of rectangles. Current texts provide numerous opportunities to identify geometric figures. In some instances it may be desirable to supplement the work in the text. Some of the following activities may be helpful:

1. Have the pupils place three points (not on the same line) on a sheet of paper and connect these points with line segments Ask for the name of the resulting figure.

2. Refer to a drawing such as that of Figure 18.27. Ask the pupils to identify the triangles (*ABD*, *BDC*, *BGC*, and *CDG*); the rectangles (*CBEF*, *AEFD*, and *ABCD*); the square (*ABCD*).

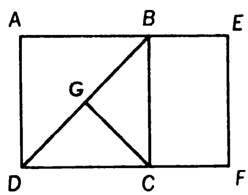


Figure 18.27

3. Ask the pupils to make a design with two or more triangles.

4. Have the pupils make a design with two squares; with three squares; with a circle and a square.

5. Examine the classroom and the immediate vicinity of the school for representations of squares, rectangles, triangles, and circles.

6. Ask the pupils to draw a set diagram to show how triangles, rectangles, squares, circles, and closed curves are related (see Fig. 18.28).

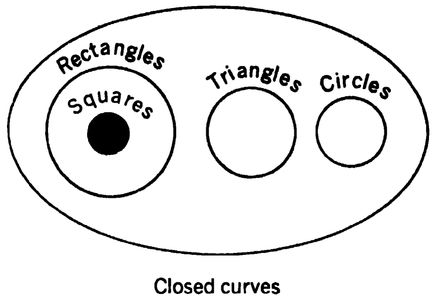


Figure 18.28

While early identification of geometric figures must be on shape alone, this means of recognition must gradually be broadened to include verbal descriptions of various features. A square and a circle may both be recognized as closed curves. As the scope of recognition broadens, the concept of a polygon may be introduced as a unifying idea. The following list includes activities designed to help pupils discover properties of polygons.

1. Have the pupils place four points on a sheet of paper with no three of the points on the same line.

a. Ask the pupils to connect these points in any way they wish.

b. Ask the pupils to connect the points so that the result is a closed curve. Some curved figures should result.

c. Ask the pupils to connect the points with line segments to obtain a closed curve.

d. Have a number of different drawings placed on the chalkboard and help the pupils identify which ones are polygons. Stress that the polygon must be a plane, closed figure formed by line segments joined end to end. Use a coat hanger to demonstrate a four-sided closed figure that is not a polygon (see Fig. 18.25).

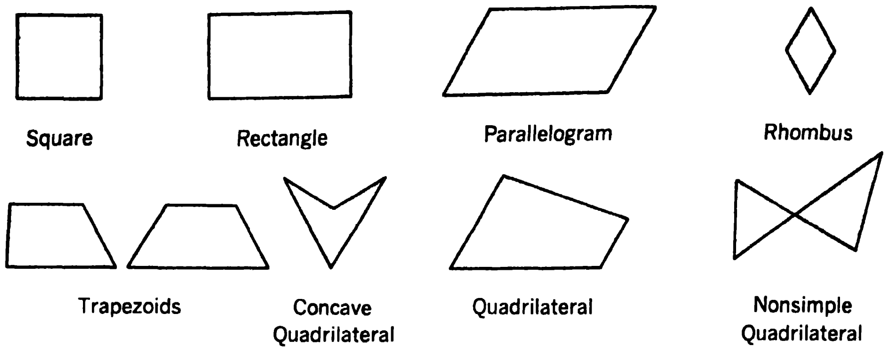


Figure 18.29

2. Repeat the above sequence with five points and help the pupils identify the polygons as well as the figures that are not polygons.

3. Ask the pupils to draw as many differently shaped four-sided polygons as they can (see Fig. 18.29).

Quadrilaterals

All the drawings in Figure 18.29 represent *quadrilaterals* (four-sided polygons). Every quadrilateral has four sides, four angles, and four vertexes. A *vertex* is the common endpoint of two sides. In a *concave* quadrilateral, it is possible to draw a line segment connecting two interior points that does not lie completely within the quadrilateral. In a *nonsimple* quadrilateral, two sides have a point in common other than the vertexes. A nonsimple polygon is a closed curve that is not simple because it crosses itself. Nonsimple polygons are usually reserved for more advanced levels and should probably not be introduced in the elementary school.

By the end of grade 6, pupils should be able to identify the special features of and the relations among the square, rectangle, parallelogram, trapezoid, and possibly the rhombus.

A square is a rectangle with four equal sides (four sides with equal measures). A rectangle is a parallelogram with four right angles (four angles with equal measures). A quadrilateral is any four-sided polygon. A parallelogram is a quadrilateral with two pairs of parallel sides. A trapezoid is a quadrilateral with exactly one pair of parallel sides. After more than 2000 years of geometry, the preceding definition is not completely standard, but it is used in the majority of traditional geometry textbooks. Some authors define a trapezoid as a quadrilateral with one pair of parallel sides. With the latter definition, a parallelogram is a trapezoid. With the former definition of a trapezoid, the set of trapezoids and the set of parallelograms are disjoint. A set diagram is helpful in visualizing the manner in which quadrilaterals are related (see Fig. 18.30). The set diagram in Figure 18.30

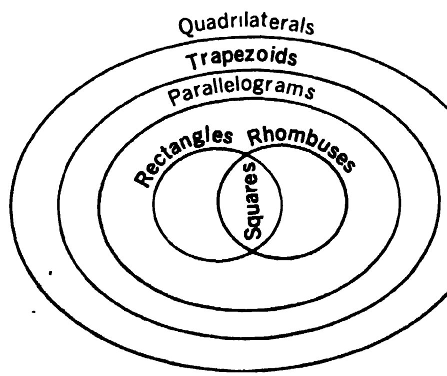


Figure 18.30

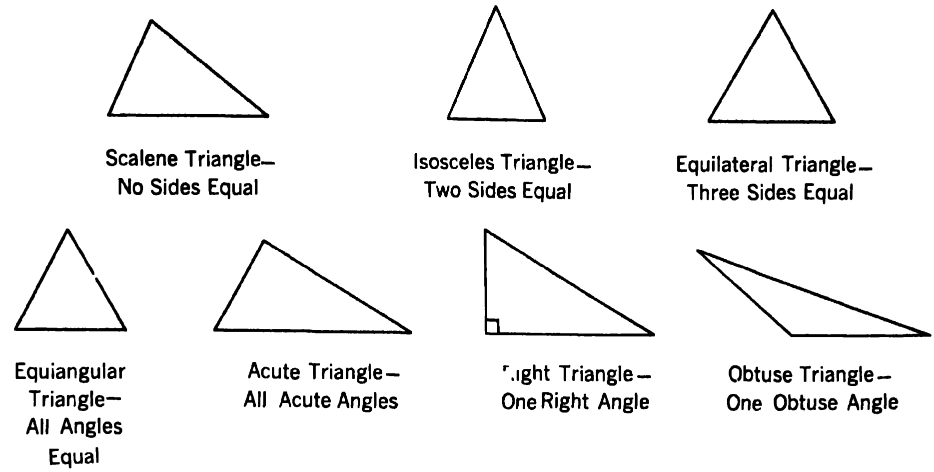


Figure 18.31

is based on the more modern definition of a trapezoid.[1] A rhombus is a parallelogram with four equal sides. A square is also a rhombus such that all four of its angles are equal (have equal measures). Some authors define a rhombus so that it cannot be a square.

Triangles and quadrilaterals are the polygons that the elementary pupil meets most frequently. Just as there are many kinds of quadrilaterals, there are many kinds of triangles, as illustrated in Figure 18.31. The differences between the types of triangles listed in Figure 18.31 depend upon metric ideas, such as the length of the sides or the size (measure) of the angles. Only a few of these types of triangles are usually identified by most elementary pupils. All elementary pupils should, however, be able to identify right triangles. This is possible even before pupils are introduced to the measurement of angles because of the "square corner."

For the elementary pupil, a triangle is a right triangle or it is not. Triangle that are not right triangles are oblique. Figure 18.32 shows how right, oblique,

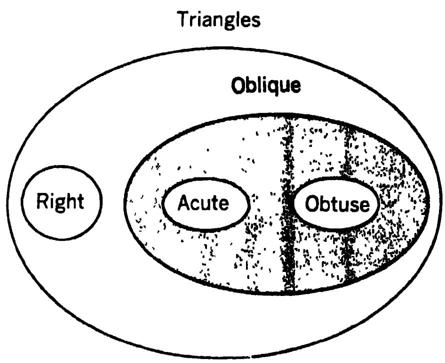


Figure 18.32

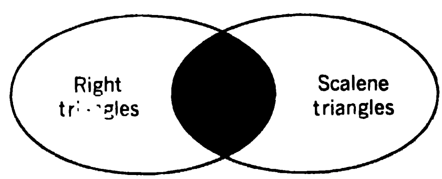


Figure 18.33

acute, and obtuse triangles are related, while Figure 18.33 shows how right triangles and scalene triangles are related (overlapping).

[1]David L. Dye, "What Is a Trapezoid?," *The Mathematics Teacher*, November 1967, 60:727–728.

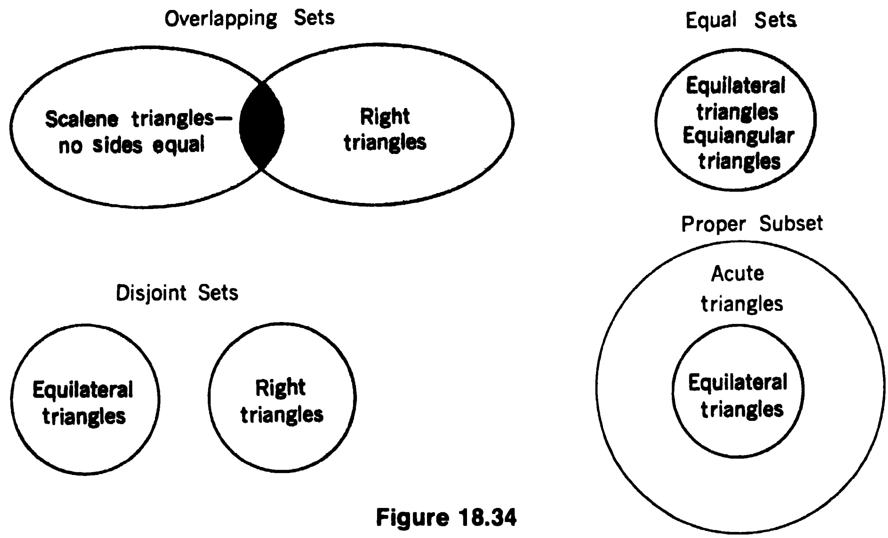


Figure 18.34

Set diagrams may be used to illustrate how any two types of triangles are related, as in Figure 18.31. Figure 18.34 shows four such diagrams.

Enrichment activities

The following activities describe advanced work with polygons suitable for very able classes in the upper grades:

1. Refer to Figure 18.35. Help the class identify the two figures as simple four-sided polygons (quadrilaterals). They should be able to recognize that both figures have four sides, four angles, and four vertexes. Ask if the point A is inside or outside the polygon. Help the pupil to recognize that the line segment $\overline{AB}$ connects two interior points of the polygon but contains many points, as C, which lie outside the polygon. Inform the class that such a polygon is called a *concave* polygon. The right-hand polygon is *convex* because every line segment connecting interior points lies entirely within the polygon.

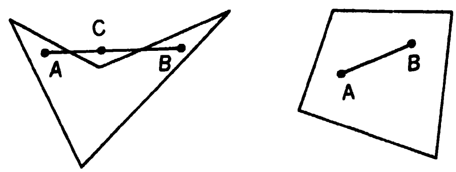


Figure 18.35

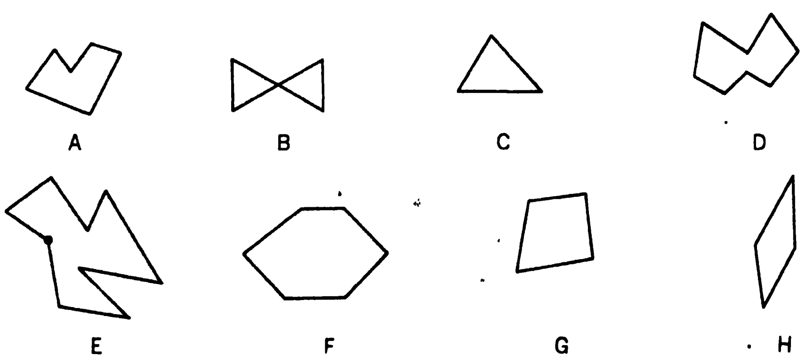


Figure 18.36

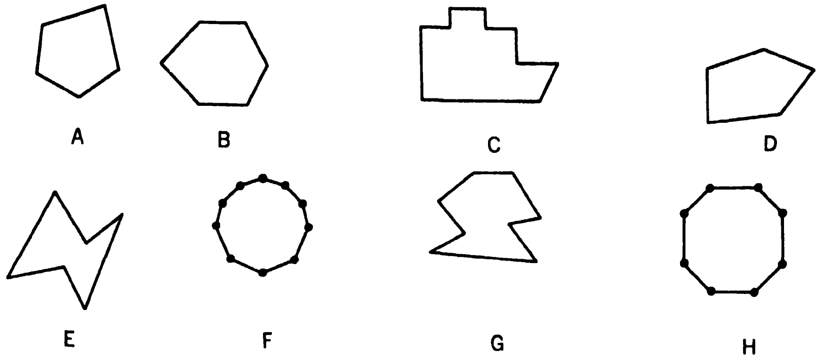


Figure 18.37

2. Refer to Figure 18.36. Identify the concave polygons (A, B, D, and E).

3. Write the following list on the board. If slides, filmstrips, or overhead projectors are available, they may be preferable.

Pentagon – five-sided polygon

Hexagon – six-sided polygon

Octagon – eight-sided polygon

Decagon – ten-sided figure

The above list includes the polygons of more than four sides that occur most frequently in subsequent mathematics. The dodecagon (12 sides), the septagon (7 sides), and the nonagon (9 sides) occur less frequently.

Place a set of figures on the board similar to those in Figure 18.37. Ask the pupils to identify the pentagons (A and D), hexagons (B and E) octagons (G and H), and decagons (C and F). Be certain that not all figures have all sides and angles equal (having equal measures). Some of the figures should have sides that are not equal (with unequal measure) and angles that are not equal. Polygons with all sides equal and with all angles having equal measure are called *regular* polygons. Pupils should see enough polygons (with more than four sides) that have unequal sides and angles to recognize that all pentagons, hexagons, octagons, and decagons are not regular.

4. Ask the pupils to create designs using polygons with more than four sides (regular or not).

5. It may be desirable for some very able grade 6 groups to distinguish between simple quadrilaterals and nonsimple ones (see Fig. 18.38). Identify the four vertexes in both figures (*A*, *B*, *C*, and *D*). Identify the four sides (*AB*, *BC*, *CD*, and *DA*). Ask the pupil how he can recognize that point *E* is not a vertex. One can move in only two directions from a vertex and remain within the set. If the vertex is removed, the remaining set is in one piece. If the point *E* is removed, the remaining set is in two parts. It is also possible to move in four different directions from point *E* and remain in the set. Point *E* cannot be a vertex, since *DE* and *EB* lie on the same line.

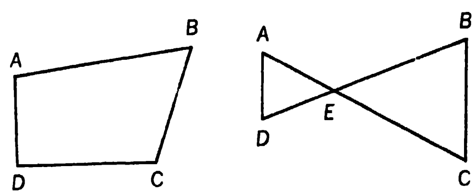


Figure 18.38

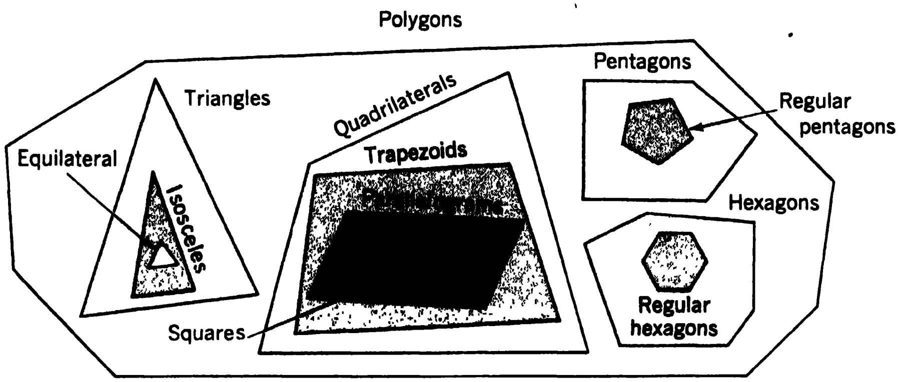


Figure 18.39

6. Have the pupils construct a set diagram showing how some polygons are related. Figure 18.39 shows how this may be done. Figure 18.28 also suggests using a polygon, which suggests the set involved rather than the usual circle or closed smooth curve.

Angles

Pupils can readily recognize that a quadrilateral has four sides and four angles before they can describe or define an angle. An angle cannot be introduced in its logical sequence (before triangles, squares, and rectangles) in the elementary school. The angle concept is difficult to define and must be dealt with on a pattern and nonverbal basis before it can be discussed in a sound mathematical manner.

Right angles are recognized in early work with squares, rectangles, and perpendicular lines and are sometimes described as "square corners."

In the upper grades, a child should be able to deal with an angle on a more precise basis. At such a time, the *angle* is then defined as two different rays with a common endpoint such that the rays do not lie on the same line. The common point is the *vertex* of the angle and each ray is a *side* of the angle.

It can be debated whether an angle is a one-dimensional or a two-dimensional figure. When an angle is viewed as two rays with a common endpoint, the rays have no width and the angle in this sense is one-dimensional. However, every angle has an interior region. Recognition of the interior and exterior is important in a modern approach to geometry. An angle with its interior is a two-dimensional figure.

Two differences between the current interpretation of an angle and the traditional approach involve the "zero" and "straight" angles. The definition of an angle given above excludes the zero angle by demanding two different rays. The straight angle is excluded by the statement that the two rays cannot lie on the same line. A zero angle and a straight angle have no interior or exterior. For this reason, most high school geometry textbooks define an angle in a manner that excludes zero and straight angles. While early work with angles need not be concerned with the interior and exterior of an angle, in the late elementary grades many textbooks now deal with the concept of the interior and exterior. The following activities may be helpful in enabling the pupils to understand the concept of the interior of an angle:

1. Direct each pupil to draw a line on a sheet of paper and to shade one of the two regions formed. The two regions thus formed by the line may be referred to as the shaded and unshaded regions.

2. Have the pupils draw a ray on a sheet of paper. Ask if the ray separates the paper into two regions. Explain that the line associated with the ray divides the paper (plane) into two regions. The two regions associated with a ray are those formed by the line of which the ray is a part.

3. Have the pupils draw the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ on the same sheet of paper. Identify the two regions associated with the ray $\overrightarrow{AB}$. Shade the region containing the point C with horizontal shading. Identify the two regions associated with the ray $\overrightarrow{AC}$ and shade the region containing point B with vertical shading. The intersection of the set of points indicated by the vertical shading and the set of points indicated by the horizontal shading is the interior of the angle formed by the ray $\overrightarrow{AB}$ and the ray $\overrightarrow{AC}$ (see Fig. 18.40).

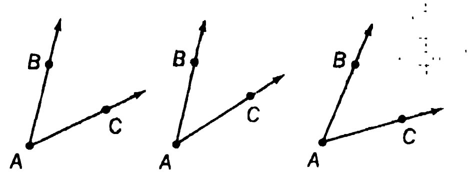


Figure 18.40

The concept of the interior of an angle has little or no direct application at the elementary level and should not be overemphasized. The brief introduction of interior should be treated as a readiness exercise for a concept that is important later as well as an opportunit to increase the understanding of regions in a plane and the concept of intersections of sets. Other more immediate facts about angles must not be neglected. Some of these are illustrated in the following activities:

1. Introduce the notation for naming angles by placing figures on the chalkboard, as illustrated in Figure 18.41.

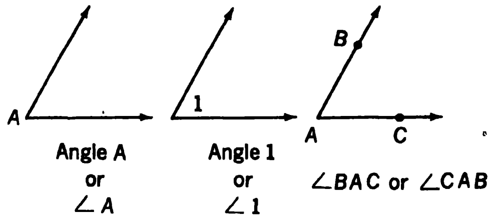


Figure 18.41

After discussing the different ways of naming angles, refer to a set of figures such as those shown in Figure 18.42. Choose angle A and ask the pupils to give two other names for this angle. Continue with other angles, using different types of names in each case.

Angle A is frequently written as $\angle A$. In Figure 18.42, $\angle A$ may also be named $\angle BAC$, $\angle CAB$, or $\angle 6$. Angle B or $\angle B$ may also be named as $\angle ABC$, $\angle CBA$, or $\angle 5$.

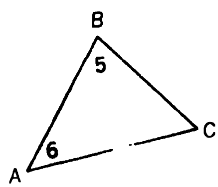


Figure 18.42

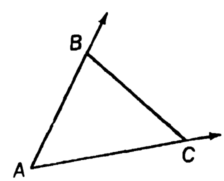


Figure 18.43

2. Refer to Figure 18.42. An alert pupil may ask how $\angle A$ can be called an angle, since there is no ray in the figure and an angle is defined as two rays with a common endpoint. Draw a figure similar to Figure 18.43. Explain that $\angle A$ is really formed by the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, as illustrated in Figure 18.43, but that it is not necessary to draw the rays, as they are implied. On occasion, when referring to an angle in a triangle, it may

be worthwhile to ask the pupils to draw the rays for angle A (or $\angle B$ or $\angle C$).

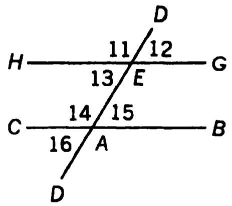


Figure 18.44

3. If the pupils need additional practice in naming angles, use a figure like the one illustrated in Figure 18.44. In this case use only two means of naming angles (as angle 11 or angle HED).

Other activities involving angles usually involve metric considerations and are discussed in connection with metric geometry.

EXERCISES

1. Which of the following is an acceptable definition for an angle? (a) an angle is formed by two rays; (b) an angle is formed by two rays with a common endpoint; (c) an angle is formed by two different rays with a common endpoint such that the two rays do not lie on a line.
2. Refer to Figure 18.45: (a) identify a polygon that is not a simple closed curve; (b) identify a nonsimple closed curve that is not a polygon; (c) identify a figure with three sides that is not a polygon; (d) identify the hexagon—is it concave or convex? (e) identify the quadrilateral—does it have a distinctive point?(f) identify a closed curve that is simple but not a polygon.
3. Refer to Figure 18.46: (a) name a point on the interior of $\angle A$; (b) name a point on $\angle A$; (c) name a point on the exterior of $\angle A$.

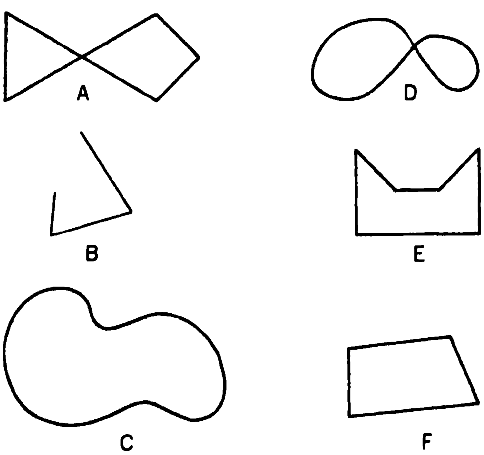


Figure 18.45

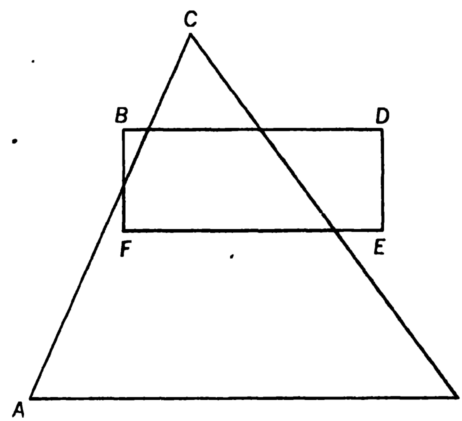


Figure 18.46

THREE-DIMENSIONAL FIGURES

A moving point generates a curve (a one-dimensional figure), as illustrated by a moving pen or pencil in the act of writing. A moving line generates a surface (a two-dimensional figure), as illustrated by a brush being used for painting a wall. A moving surface generates a solid (a three-dimensional figure), as

illustrated by a snowplow moving snow.

A line segment is bounded by two points, its endpoints. A finite portion or subset of a surface is bounded by curves, as illustrated by a football field or a state on a map. A finite portion or subset of space is bounded by surfaces, as illustrated by the interior of an ordinary rectangular box, which is bounded by six plane surfaces. The interior of a sphere is bounded by a closed curved surface, as is the interior of a football or basketball.

A triangle is formed by three one-dimensional line segments. Although a line segment has no area, the phrase "area of a triangle" is a common one and should be recognized as a convenient abbreviation for the phrase "area of the surface enclosed by a triangle."

The interior of a polygon is bounded by a closed, broken line segment and is a subset of some plane. The polygon is a one-dimensional figure, while the polygon plus its interior is a two-dimensional figure. The interior of a *polyhedron* is bounded by closed plane surfaces. The polyhedron, formed with two-dimensional plane surfaces, is a two-dimensional figure. The polyhedron with its interior is called a solid. A solid is a figure with three dimensions. Some of the familiar solids, for example, the cube, rectangular solids, pyramids, and prisms, are polyhedrons. Spheres, cones, and cylinders are not polyhedrons.

A hexagon is formed by six connected line segments called sides. Two adjacent sides have one point in common (a vertex). A vertex is the set intersection of two sets of points (two adjacent sides of the polygon).

A *hexahedron* is a polyhedron formed by six connected portions of a plane, called *faces.* A face of a polyhedron is a polygon. Adjacent faces meet in line segments called *edges.* Adjacent edges meet in a point called a *vertex* of the polyhedron. The edges of a polyhedron are the sides of polygons forming the faces of the polyhedron. The vertexes of a polyhedron are also the vertexes of polygons forming the faces of polyhedrons.

A cube is a hexahedron because it is formed by six faces. Each face of a cube is a square. A cube has 12 edges. Each of six faces has 4 edges, a total of 24 edges, but in this procedure each edge is counted twice because each edge belongs to two faces. A cube has eight vertexes. Each of six faces has four vertexes, a total of 24, but in this procedure each vertex is counted three times, since each belongs to three faces (and edges).

The following sequence of activities illustrates how some properties of polyhedrons may be discovered by pupils:

1 Discuss with the class the smallest number of line segments needed to enclose a surface and help them to discover that at least three line segments are needed. Ask them for the smallest number of sides required to form a polygon and help them to recognize that this question is the same as the previous question

2. Discuss with the class the smallest number of plane surfaces necessary to enclose a portion of space. Have some physical models of polyhedrons or some pictures of them for reference and help the class discover that the answer is four, or that the smallest possible number of faces that a polyhedron may have is four. It is desirable to have pupils give answer before they refer to the physical models to give them the opportunity to visualize geometric situations mentally.

3. Any triangular piece of paper (with all angles less than 90 degrees) may be

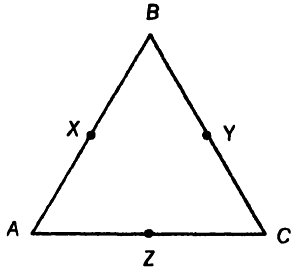


Figure 18.47

used to make a polyhedron of four sides (a *tetrahedron*). Start with triangle ABC, as illustrated in Figure 18.47. Determine the three midpoints of the three sides, X, Y, and Z. Fold firmly along the line segments $\overline{XY}$, $\overline{XZ}$, and $\overline{YZ}$. With careful and proper folding, the three vertexes of the triangle, A, B, and C, will then meet to form the fourth vertex of a tetrahedron whose other vertexes are X, Y, and Z. If this activity is done with triangles of different shapes (all acute triangles), tetrahedrons of different shapes will result. If a right triangle is used two of the faces will fold over to equal the third and form a rectangle. If an obtuse triangle is used, a tetrahedron cannot be formed. If an equilateral triangle is used, a regular tetrahedron will result (with all edges equal and all faces with the same size and shape).

4. Have the class make other polyhedrons by folding on the basis of patterns drawn on paper (see Fig. 18.48 for patterns for familiar figures).

5. Use soda straws, toothpicks, pipe cleaners, or similar materials to construct models of polyhedrons.

6. Have the class count the faces, vertexes, and edges of a tetrahedron and other polyhedrons and make a table as follows:

	Faces	*Edges*	*Vertexes*
Tetrahedron	4	6	4
Cube	6	12	8
Pyramid with square base	5	8	5

When the table has been completed for the polyhedrons that are available for the class to analyze, ask the pupils if they can discover a pattern or relationship among the number of faces, vertexes, and edges. This relationship was discovered more than 200 years ago by the Swiss mathematician Euler. This relationship is $F + V = E + 2$, where F represents the number of faces, V the number of vertexes, and E the number of edges.

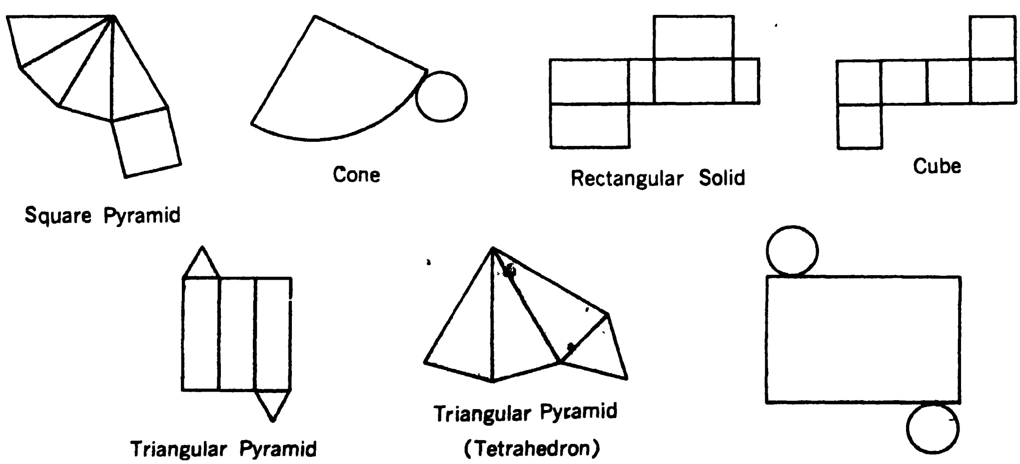


Figure 18.48

EXERCISES

1. How many dimensions may the subset of a one-dimensional figure have? a two-dimensional figure? a three-dimensional figure?
2. Which of these are pentahedrons?
   a. Squares
   b. Triangular prism
   c. Square pyramid
   d. Cube
3. Test the Euler formula given above for a pyramid with a pentagon for a base; with a hexagon for a base; with an octagon for a base.
4. Complete the following: A square is related to a polygon as a ____ is related to a polyhedron.
5. What dimension has a point? a line? a surface? a solid?
6. Explain how an angle may be considered a two-dimensional figure on one occasion and a one-dimensional figure on another.

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MEASUREMENT AND METRIC GEOMETRY

In 1955, Lancelot Hogben published his valuable book *The Wonderful World of Mathematics*.[1] Its European title was *Man Must Measure*. The work clearly shows that the history of mathematics and the history of measurement closely parallel each other. The text is colorfully illustrated and challenges the interest of elementary school children. Jean Piaget's studies of the development of numbers and measurement are useful guides in the gradation of topics in this field. An article by Coxford presents a summary of Piaget's studies.[2]

Measurement is perhaps the most important application of number that children of all ages encounter. The pupil should learn about units of measure as well as about instruments of measurement and he should have firsthand experience in applying what he learns.[3] He should also become familiar with the history and development of measur-

[1](New York: Doubleday & Company, Inc.).

[2]"Piaget: Number and Measurement," *The Arithmetic Teacher*, November 1963, 10:428–434. The article discusses the findings of Piaget with regard to stages of learning and the development of concepts of number and measurement.

[3]Helen C. Parker, "Teaching Measurement in a Meaningful Way," *The Arithmetic Teacher*, April 1960, 4:194–198.

ing instruments in order to gain an awareness of the ways in which man has devised methods of using number to describe in precise and meaningful terms the quantitive aspects of his environment.

This chapter deals with the following topics: what measurement is, learning about measures; how to teach metric geometry; perimeters, areas, and volumes; computation involving familiar measures.

WHAT MEASUREMENT IS

Meaning of measurement

If the length of a line segment is expressed as 6 inches, we have assigned a measure to that segment. The measure is 6 and the standard unit of measure is an inch. A measurement is designated as a number and not as a set of points. The number indicates how many times the unit of measure can be fitted into the quantity that is measured.

The use of number in measurement enables us to describe precisely and meaningfully the things in our environment. Measurement makes it possible for us to define, to predict, and to control. Scientists are continually looking for aspects of things to measure and are constantly devising, applying, and refining means of doing so in objective ways.[4]

Origins

In early times the means of measurement were indefinite and crude. Just as the decimal system of counting was the outgrowth of using the fingers of the hand, measures of various kinds were derived from natural events and units that were easy to manage and to understand. Thus the movements of the heavenly bodies furnished an easy way of reckoning time. The day was the time that elapsed from sunrise to sunrise; the month, the time between a certain phase of the moon and its recurrence; and the year, the time it took the sun to pass through successive changes from one position in the heavens to the same position. Short distances were measured by the number of steps taken to cover them and longer distances by the number of days' journey. Bowls and cups were used to measure the capacity of containers, and grains of wheat and barley were used to measure weights of valuables. For thousands of years barter was the means of exchange, hence definite units of value were not needed.

Development of definite units

With the passing of hundreds of years and the development of community life, there arose in a haphazard manner, in response to practical needs, various measures, which varied from locality to locality. They were adopted as convenience prompted. In some cases the names of units were similar among various places in a geographical region, but the actual quantities they represented differed from place to place because of lack of common standards. Travelers in Europe can to this day find remnants of local units of length on the walls of castles and in the market places. These units were thus made definite, but they were not standardized.

The development of commonly accepted units facilitated trade between widely separated localities. Gradually there developed well-defined units for measuring lengths, surfaces, volumes, weight, capacity, time, value, temperature, and arcs of circles. At the present time scientists are still devising methods of describing quantitatively aspects

[4]John Perry, *The Story of Standards* (New York: Funk & Wagnalls Company, Inc., 1955), p. vii.

of natural phenomena of all kinds, usually expressed as some unit of measurement.

Learning the meaning of measurement

The child must learn through experience to answer intelligently and confidently such questions as, How many? How far? How much? What does it weigh? and so on. To do so he must first learn to select an appropriate unit of measure, such as a yard, pound, or quart, as a basis and then find the answer by counting the units or by direct measurement. The major difficulty encountered in measurement is deciding on the unit to use and then the method or device to employ in finding the number of units.

The learner must become familiar with the standard units that are commonly used and with the related measuring devices. He should be taught what a *standard* is and why standard units are necessary. A standard is a unit established by law and is used in everyday life by the members of the community. The United States Bureau of Standards is the official agency concerned with the study and maintenance of standard units of measurement. Scientists have had the responsibility of establishing and maintaining standards for about 200 years. Before that, the people who established standards were the rulers, priests, and merchants.

Need for standardized units

As long as man lived in isolated places there was almost no trade or industry. It was a matter of little concern that methods and units of measure differed. However, when men began to work in groups on construction work or wished to trade among themselves, it became evident that there should be established units of measurement that would have a common meaning. In order to measure and to barter effectively or to carry on business, it became necessary to set standard units that would have the same meaning to all concerned. At first this was done for large regions within a single country, then for a country as a whole, and ultimately for groups of countries. It is believed that the Romans were the first to establish widely accepted standards of measurement. However, with the downfall of the Roman Empire these units were largely discarded.

LEARNING ABOUT MEASURES

Measures in the primary grades

In the primary grades the pupil should begin by considering the standard units of everyday usage whose objective representations can be demonstrated with concrete materials, such as foot and inch and quart and pound. He should learn about those units first by observing the uses to which they are put in the home and later by enacting these uses in social situations. He must have firsthand experience in the actual manipulation and application of measuring devices based on these units, and he should also learn the types of questions these units are supposed to answer about various kinds of amounts. For example, the ruler will help him to find the answer to the question, How many inches (or feet) long is ___ ? The scales will help him find the answer to such a question as, How many pounds (or ounces) does ___ weigh?

The pupil should next move to the consideration of less common units that he must "carry in his mind." Square measure is an example. The standard units are unit squares that the pupil

should construct at the outset and use as objective units of measure, such as the square inch or the square foot. The purpose of this work is to give meaning to the units. Once the pupil understands the unit of measure, he learns to find the area of a surface by a short-cut method of multiplication. Cubic measure is another illustration of this type of measure.

Time was originally measured with natural units, such as the day, month, or year. Later, mechanical devices, such as waterclocks, were used to find the number of intervals of time that elapsed. The discovery of the laws of the pendulum led to the development of the highly efficient modern timepiece. Children begin to learn about time through a study of the seasons, the calendar, and the clock in the classroom. They should also learn how the clock helps people to regulate the affairs of daily life, particularly in business and industry, and to plan their activities. The study of standard time belts in grades 5 and 6 provides an excellent unit of work that shows how the people of the earth solved the problems caused by the lack of a uniform time system, a situation that caused much confusion, especially in the fields of travel and communication.

Money provides a measure of value that is not stable and that varies from time to time in accordance with political, economic, and social conditions. However, without a well-supported and fairly stable monetary system, trade and industry would be almost impossible, except insofar as barter could make the exchange of goods possible. Young children often become familiar with coins before they enter school. Learning to compute with money is an activity that greatly interests children, and it should be begun in the primary grades.

In all of the work with measurements the basic question always involves a standard unit. Some units can be applied directly, others can be applied only indirectly, as has been shown. Through carefully guided learning experiences the learner should become increasingly able to apply standard units of all kinds. If some consideration is given to the history of measurement, the pupil will come to see how the present system of measures developed from diversity to uniformity.

Useful tables of measures

At one time it was believed necessary for children to memorize tables of measures. Much of the time alloted to mathematics was devoted to solving of problems based on these tables. Today, however, less frequent use is made of such specific knowledge, and therefore less stress is placed in mathematics classes on mastering the tables. It is true that certain facts, such as the number of feet in a yard and the number of minutes in an hour, are usually learned incidentally and should be memorized. However, pupils should be taught where to find tables of measures as well as facts about them in the dictionary and textbook. Frequent reference to these materials for necessary information in connection with the solution of groups of problems and other classroom activities will accustom the children to their use.

There is no evidence of a common system in the equivalent measures listed below:

2 pints = 1 quart
3 feet = 1 yard
4 quarts = 1 gallon
5 cents = 1 nickel
6 things = ½ dozen
7 days = 1 week
8 quarts = 1 peck
9 square feet = 1 square yard

10 cents = 1 dime
12 inches = 1 foot
16 ounces = 1 pound
$16\frac{1}{2}$ feet = 1 rod
32 quarts = 1 bushel
60 seconds = 1 minute
144 square inches = 1 square foot
160 square rods = 1 acre

If our system of measures had been developed on a decimal basis, the confusion involved in converting measures and in computing with them would have been eliminated. In the beginning, measurement and our system of numeration were unrelated and independent of each other. A craftsman would select a subdivision of a unit of measure that enabled him to estimate or judge the value of the smaller unit in the task at hand even though the computations involved in accurate work were difficult and complicated.

As long as measurement and its use in computation remained independent of each other, there was no conflict between the systems of measurement and numeration. When the requirements of commerce demanded that computations be made with measurements, a difficulty arose. No relationship existed between a set of measures divided binarily (base two) and the numeration system having base ten. There was no common relationship among units that would enable the user to derive one unit from the other. Even today it is necessary to rely upon memory to supply unrelated facts to find the answers to such questions as:

1. How many feet in a rod?
2. How many square rods in an acre?
3. How many cubic inches in a bushel?
4. How many cubic inches in a gallon?
5. What is the weight of a cubic foot of water?

These and many other similar questions dealing with measurement can be formulated but cannot be answered without a knowledge of isolated facts. The decimal ratio existing between places in the numeration system is missing among consecutive units of measurement. Therefore converting from one unit to another often involves difficult computation, as in finding the number of inches in a mile or the number of square feet in an acre.

The metric system

The conflict between measurement and the numeration system was most pronounced in the field of science. When science demanded increased precision in measurement, it became necessary to construct a system of measures related to the numeration system. The result was the construction of the *metric system* of measures. The ratio between consecutive linear units in this system is the same as the decimal ratio of consecutive places in the numeration system. The following listing gives the units of linear measure in both the English and the metric systems of measure.

English

12 inches = 1 foot
3 feet = 1 yard
$5\frac{1}{2}$ yards = 1 rod
320 rods = 1 mile

Metric

10 millimeters = 1 centimeter
10 centimeters = 1 decimeter
10 decimeters = 1 meter
10 meters = 1 decameter
10 decameters = 1 hectameter
10 hectameters = 1 kilometer

The English system scarcely deserves to be called a system of measures because there is no orderliness characteristic of the measures given; it is a collection of awkwardly related measures. On the other hand, the metric system was created by some of the best scientists in Europe at the beginning of the nineteenth century. The system bridged

the gap between measurement and computation. It is as easy to change from one unit to another in the metric system as to change from one number period to another. In each case the transformation is accomplished by shifting the decimal point. 750 centimeters is equal to 7.5 meters, just as 750 is equal to 7.5 hundreds.

The metric system is used almost universally in the field of science because this system unifies the units of weights and measures. In our system of measures the units for length, weight, and volume are unrelated. In the metric system there is a relationship among these elements. The centimeter is a basic unit of length; the gram, of weight; and the cubic centimeter, of volume. A cubic centimeter of water weighs a gram. Therefore units of weight and volume are related to a linear unit. From this fact it can be discovered that a receptacle of water having a volume of 1000 cubic centimeters, or a *liter*, has a weight of 1000 grams, or a *kilogram*. The number of metric cubic units in the volume of a solid is the same as the number of metric units of weight of water of that solid. In the English system of measures it is not possible to make an easy computation dealing with weight and volume of any given cubic container when the weight of a cubic inch is known.

The British government announced in May 1965 that it would switch to the metric system from the traditional English system of measurement in an apparent move to promote trade with the European Common Market. In these countries the metric system has long been in use. The announcement stated that the change would be made by industry over the next 10 years. When it is completed the United States and Canada will be the last of the major industrial countries using the so-called inch system. Many manufacturers and engineers in the United States have endorsed a change to the metric system. Their view is that the system is far more compatible with the sophisticated tolerances required in today's machinery and tools than the inch system.[5]

The superiority of the metric system is evident. One may wonder, then, why the English-speaking countries have not replaced the old system of measurement with the metric system. The two chief reasons are a reluctance to make the change and the cost factor. The estimated cost involved in replacing the machinery geared to English measurement would run into billions of dollars. In addition, the English-speaking countries are their own best customers in commercial world, and thus until recently there has not been a pressing need for change. Now that England has adopted the metric system, however, the pressure for changing to this system may increase in the United States and Canada. This should be especially true when the metric system becomes effective in England.

The chief reason for teaching the metric system is not its practical value. Rather, it is highly desirable to teach the metric system because of its structure. The system shows how computation and measurement may be integrated and unified. The pupil can compare the advantage that accrues from the use of this system when making the more complicated computations with our system of measurement. He should then be able to appreciate why our system of numeration is superior to a system in which there is not a fixed

[5]James R. Smart and J. L. Marks, "The Mathematics of Measurement," *The Arithmetic Teacher*, April 1966, 13:283–287.

ratio between any two consecutive places in a numeral.[6]

It must be remembered that most of the units of measure in the English system were adopted because they passed the tests of practical usage. Measures were devised as man found it necessary to describe the characteristics of things in specific terms, but without relationship to other measures. The metric system was devised on a different basis. To the scientist the centimeter and the meter are based on our system of numeration and correspond to the inch and the yard. However, the ordinary person finds the centimeter too small and the meter too large to express in easily understood terms the lengths of 3 or 4 feet with which he most frequently deals. There are those who maintain that measurement cannot be used in arithmetic to teach the numeration system but that the use of measures makes the work with numbers significant.

All measures are approximate

Measurement is never exact; it is always approximate. It is impossible to draw a line segment that is exactly $5\frac{3}{8}$ inches long. The segment is almost certain to be a tiny bit too short or too long, depending upon the care with which it was drawn. We can say that to the nearest inch a line segment $5\frac{3}{8}$ inches long is 5 inches long, or that to the nearest $\frac{1}{2}$ inch the segment is $5\frac{1}{2}$ inches long. The necessity for devising small units of measurement grew out of the demand for a higher degree of precision than is possible when larger units are used. Small units are divided into fractional parts as the need arises.

[6]T. J. Johnson, "The Use of a Ruler in Teaching Place Value in Numbers," *The Mathematics Teacher*, April 1952, 45:266.

Tolerance

Since no measurement is completely accurate, laws have been passed that recognize an *allowable error* of measurement in the various measuring devices. This legally recognized allowable error for a given instrument, hence a procedure, is its *tolerance*. In Handbook M29 tolerance is described as a value defining the amount of the maximum allowable error or departure from true value or performance.[7] A measuring device is considered to be accurate if it is within its tolerance; otherwise it is regarded as inaccurate. Thus the recommended tolerance (presumably adopted in most states) for a foot rule is $\frac{1}{32}$ inch. For manufacturers making new foot rules, the tolerance is $\frac{1}{64}$ inch. The tolerance for gasoline is 18 cubic inches in excess for a 10-gallon measure and 9 cubic inches in deficiency.

Three general principles apply to the idea of tolerance. First, the larger the measure, the larger the tolerance may be. Second, tolerance in excess tends to be more lenient (that is, larger) than tolerance in deficiency. And third, tolerance is less for new manufactured devices than for devices in use, in fact, half as much.[8]

HOW TO TEACH METRIC GEOMETRY

Nonmetric and metric geometry

Geometry is often divided into two categories that are not completely separated.[9] Metric geometry deals with the

[7]Published by the Bureau of Standards, Washington, D.C.

[8]B. R. Buckingham, *Elementary Arithmetic: Its Meaning and Practice* (Boston: Ginn and Company, 1947), pp. 478–479.

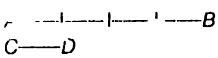
[9]C. Edith Robinson, "The Role of Geometry in Elementary School Mathematics," *The Arithmetic Teacher*, January 1966, 13:3–10.

properties of sets of points that can be measured, while nonmetric geometry deals with properties of sets of points to which measurement ideas cannot be applied. In teaching situations these two categories are intermixed. The theme of the preceding chapter was nonmetric geometry; the present chapter deals with metric geometry, which is basic work in the intermediate grades.

Chapter 18 described such figures as points, rays, line segments, and angles. We shall be concerned here with finding measures for the set of points, such as a line segment, included in certain figures.

Measuring line segments

Line segment $\overline{AB}$ is marked off in units each equal to the measure of the line segment $\overline{CD}$.

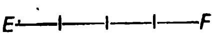


We can regard the measure of $\overline{CD}$ as the *unit of measure.* In mathematics we should not write $\overline{AB} = 4\overline{CD}$ because $\overline{AB}$ and $\overline{CD}$ are sets of points and not numbers. The symbol = shows that two numerals or sets name the same thing. To state that the measure of $\overline{AB}$ is four times the measure of $\overline{CD}$, we use the following equation:

$$m(\overline{AB}) = 4m(\overline{CD})$$

We read the equation as, "the measure of $\overline{AB}$ is equal to four times the measure of $\overline{CD}$."

The most commonly used unit for measuring length in the English system of measurement is the *inch.* The standard unit of measure of length is the *meter.* The meter determines the length of an inch. We shall consider a inch as the standard unit in our system of measure. The length of $\overline{EF}$ is approximately equal to the unit 1 inch.



This unit is divided into four congruent segments, as shown. Two segments are congruent when one is placed on the other and they match throughout.

A line has no fixed length, but we can find the length of a segment of a line that is designated by two points on the line. A ruler marked off in fractional parts of an inch is used as a measuring instrument.

The length of a line segment should not be confused with the measure of the segment. The measure of a line segment is the number that shows how many units are equivalent to the segment. The length of a line segment means the measure of the segment and the unit used in measuring. Thus, the measure of a given line segment $\overline{AB}$ is 4, but the length of $\overline{AB}$ is 4 inches, or 4 of the unit of measure.

Figure 19.1 shows a line segment $\overline{AB}$ and parts of a ruler, in which the inch is graduated into half inches, as in M, into quarter inches, as in N, and into eighth inches, as in P. The length of $\overline{AB}$ depends upon the ruler used and may be expressed as follows:

To the nearest inch	(M)	3 inches
To the nearest half inch	(M)	$2\frac{1}{2}$ inches
To the nearest quarter inch	(N)	$2\frac{3}{4}$ inches
To the nearest eighth inch	(P)	$2\frac{5}{8}$ inches

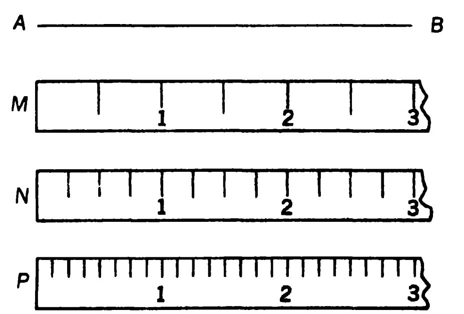


Figure 19.1

In a similar manner, the length of $\overline{AB}$ may be expressed, to the nearest sixteenth inch, as $2\frac{11}{16}$ inches, or to the

nearest thirty-second inch, as $2\frac{21}{32}$ inches. The illustration shows that no measurement is ever exact but that a measurement is always approximate.

The precision of measurement depends upon the way a measuring instrument is graduated or calibrated. The smaller the unit of measurement, the more precise is the measurement. The measurement of $\overline{AB}$ in P is more precise than the measurement in M because the unit of measure is smaller in P than in M.

In the primary grades a pupil may be expected to measure a line segment longer than an inch but less than 2 feet in length to the nearest inch or half inch. In the intermediate grades he should be able to measure segments of this kind to the nearest quarter inch. Errors of measurement would result from the incorrect use of the measuring instrument.

The value to be assigned a point between two scaled values on a ruler is determined in the same way as the value assigned a number being rounded off. If the point is nearer one scaled value than the other, the point is assigned the scale value of the point to which it is nearer. If the point is judged to be midway between two scale values, the point is given the value of the greater of the two scaled values.

Other units used in measuring length are the foot, the yard, the rod, the mile, and the meter:

1 foot = 12 inches
1 yard = 3 feet
1 rod = 5½ yards
1 mile = 5280 feet
1 meter = 39.37 inches

Measuring angles

In Figure 19.2, the measure of $\angle ABC$ is three times the measure of $\angle DEF$, hence $m\angle ABC = 3m\angle DEF$. Although

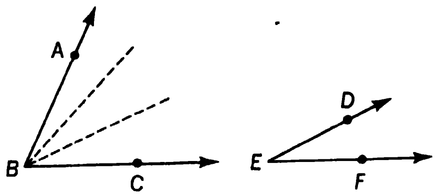


Figure 19.2

$\angle DEF$ is used as a unit of measure in the illustration, this unit is not a standard unit. Just as the inch is a standard unit for linear measure, so the *degree* is a standard unit for angular measure. We symbolize the unit of angular measure as 1° and read it as "one degree."

If we lay off 360 units of angular measure in a plane about a point as a vertex, the entire plane will be covered. The Babylonians gave us the unit for angular measure. Since the angular measure for a complete surface is 360, very probably the number of days in a year was instrumental in selecting the unit for angular measure. It should be remembered that a measure is a number. We may state that the size of an angle is 60°, but its measure is 60.

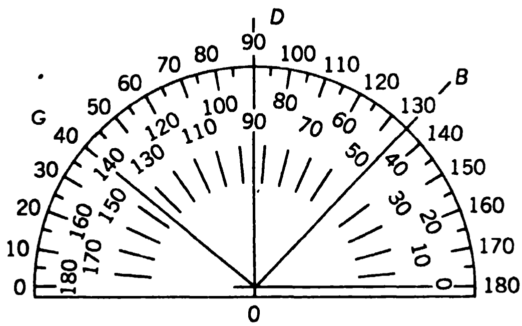


Figure 19.3

The instrument for measuring an angle is a *protractor* (see Fig. 19.3). The outer edge of a protractor contains both an outer scale and an inner scale graduated to 180 degrees. The use of two scales makes it easy to measure an angle in any position. In the diagram the zero point of the outer scale is on the

left and of the inner scale on the right. The center of the semicircle is marked on the protractor by a "crowfoot." This center is always placed on the vertex of the angle to be measured and the zero point is placed on one side of the angle. The point on the scale that is cut by the other side of the angle indicates the number of degrees in the angle. In Figure 19.3, the protractor is set to show the measure of angle BOC, which is indicated to be 45. We write this fact as follows:

$$m\angle BOC = 45$$

We read the equation as "the measure of $\angle BOC$ is 45." Tell what the measure is of the other angles shown in Figure 19.3.

It has been conventional to write the above equation as $\angle BOC = 45°$, indicating that a measure is involved. The term $\angle BOC$ refers to the set of points on the rays forming the angle. The term $m\angle BOC$ clearly indicates that a measure of the set of points between the rays is intended. The teacher should have the pupil use the expression "measure of $\angle A$" until it is clear that the pupil understands the difference between an angle and its measure.

A protractor can be used to draw an angle having any given measure. For example, to draw a perpendicular to a point on a line, place the center of the protractor at the given point and the zero point of the scale on the line. Then mark the point that corresponds to the 90° mark on the scale and connect this point with the given point on the line. The angle formed is a right angle and its measure is 90.

An *acute angle* has a measure greater than 0 but less than 90. An *obtuse angle* has a measure greater than 90 but less than 180. For Figure 19.4, have the class describe the measure of the sides of triangles (A–C) and the measure of the angles formed by the sides in triangles (D–F). Have the pupil use a ruler or a protractor to verify his answer.

The class should consider the distinguishing characteristics of the square, the rectangle, and the parallelogram.

The following questions, which deal with these figures, will help the pupil to understand the concept of a measure and of an angle.

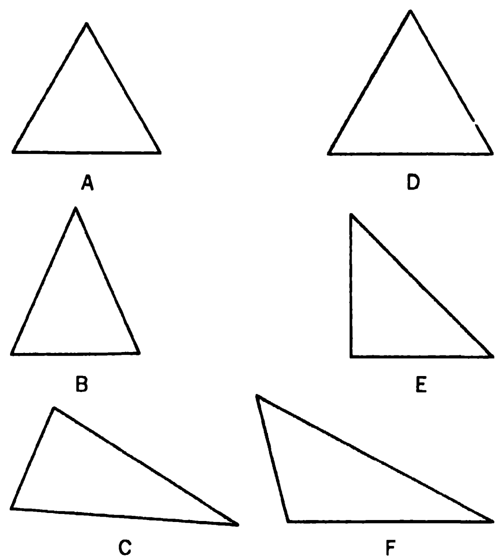


Figure 19.4

1. How many sides has a rectangle?
2. How do the measures compare of the opposite sides of a rectangle? of the sides of a square? of the opposite sides of a parallelogram?
3. What kind of angle is formed by the adjacent sides of a rectangle? of a square?
4. Is a square in the set of rectangles?
5. Is a rectangle in the set of squares?
6. If R = the set of rectangles and S = the set of squares, which of the following number sentences is true? a. $R \subset S$? b. $S \subset R$?
7. How does a parallelogram differ from a rectangle?

8. Is a rectangle in the set of parallelograms?

9. Is a parallelogram in the set of rectangles?

10. If P = the set of parallelograms, R = the set of rectangles, and S = the set of squares, select the true number sentences from the following: (a) $P \subset R$; (b) $R \subset P$; (c) $S \subset P$; (d) $S \subset R \subset P$; (e) $P \subset R \subset S$. Item (10) is intended to challenge the most able learners in the class.

Congruent geometric figures

The measures of $\overline{AB}$ and $\overline{CD}$ in Figure 19.5 are the same, therefore we say the two line segments are *congruent*. We write $\overline{AB} \cong \overline{CD}$. The symbol for congruent is $\cong$.

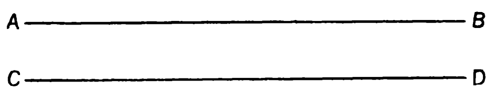


Figure 19.5

Any two plane geometric figures that have equal measures of corresponding sides and angles are congruent. In less technical terms, we may state that plane geometric figures that have the same shape and size are congruent. Which are the congruent figures in Figure 19.6?

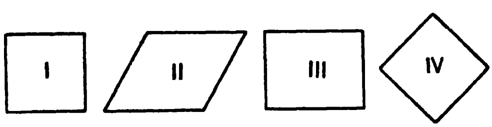


Figure 19.6

Consider the illustrations of the three kinds of triangles given on page 331.

1. What is the name of the kind of triangle that has three congruent sides?

2. What is the name of the kind of triangle that has two congruent sides? What angles in that triangle are congruent?

3. What is the name of the triangle that has no congruent sides?

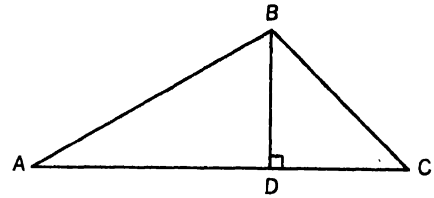


Figure 19.7

Altitudes

An *altitude* of a triangle is a perpendicular segment from a vertex of the triangle to the line containing the opposite side. In Figure 19.7, the altitude of triangle ABC is BD. The segment $\overline{BD}$ is the perpendicular from the vertex B to the side opposite, or AC. The symbol ⊾ indicates a right angle.

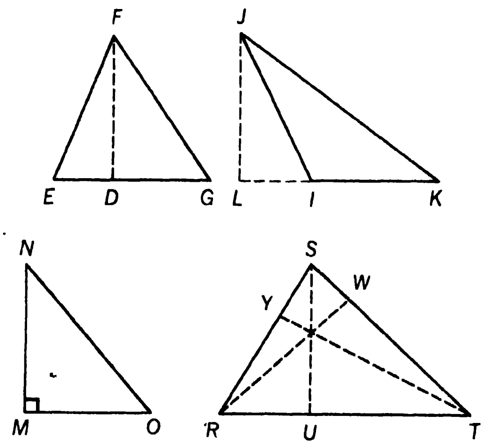


Figure 19.8

Name the altitude in each of the triangles in Figure 19.8. If there is more than one altitude, name all of them.

An altitude in a parallelogram is a perpendicular segment from a point on one of the parallel sides to the line containing the opposite side. In the given parallelograms in Figure 19.9, more than one altitude is drawn in each figure. How many altitudes can be drawn in each figure? Why?

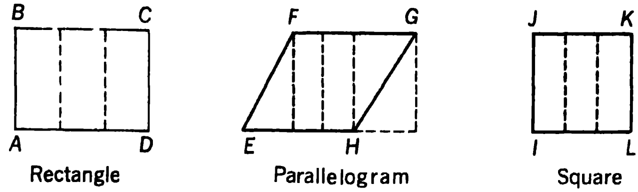


Figure 19.9

Circles and compasses

A *circle* may be drawn by tracing a set of points around a circular surface, such as a coin or the base of a bottle. A circle is the set of all points in a plane that are the same distance from a point O called the center of the circle. If A is any point on the circle, then the measure of $\overline{AO}$ is a *radius*. A circle of any given radius may be drawn by using a compass, as shown in Figure 19.10.

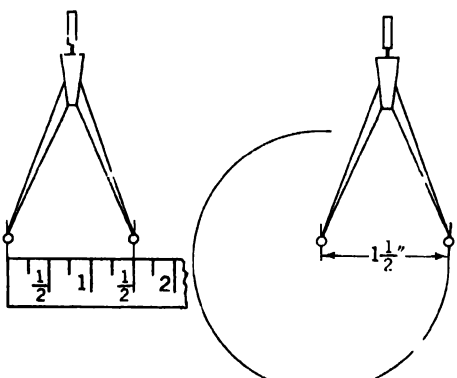


Figure 19.10

Suitable exercises such as the following may be used to help the pupil develop skill in the use of a compass.

1. Mark any point on your paper. Use that point as a center to draw a circle with a compass.
2. Use a compass to draw four circles that have different radii.
3. Draw a set of three circles that have different radii but the same center.
4. Draw a circle having a diameter of 2 inches. Draw and label a radius, a diameter, and a chord that is not a diameter. (A *chord* is a line segment connecting any two points on the circle.)

If A and B are any two points on a circle, the union of A, B, and all points on the circle between them is an *arc*. If A and B are not endpoints of a diameter, the arc having the smaller measure is called the *minor arc* and the other arc is the *major arc*.

5. Select any two points O and A on your paper. Use point O as a center and draw a circle through A. Draw the chord AB. What is the given radius? the diameter? Solve the following equation:

$$m(AOB) - \cdot m(AO)$$

6. Use a compass and a ruler to find the lengths of given line segments having different measures.
7. On a ray extending to the right, use your compass to mark off with arcs three line segments, each measuring $1\frac{1}{4}$ inches. Start with the endpoint on the left of the ray.

Constructing a perpendicular bisector of a line segment

A line segment may be bisected at right angles either by *drawing* the perpendicular or by *constructing* the perpendicular. A pupil draws the bisector when he uses a ruler to find the midpoint of the segment and a protractor to measure a right angle. He constructs the bisector when he uses only a compass and a straight edge without marking in units of measure.

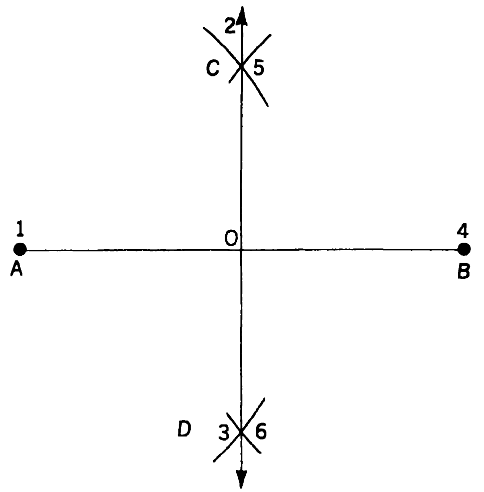


Figure 19.11

Figure 19.11 shows how to construct a perpendicular bisector of any line segment, such as $\overline{AB}$. The numerals indicate the sequence of steps for finding the points C and D. Begin by placing the compass point at 1. Have the pupil check the accuracy of his construction by using a protractor to see if each angle is a right angle and a ruler to see if the two segments are congruent.

Bisecting an angle

Figure 19.12 shows the sequence of steps to bisect a given angle, such as angle A. The pupil should bisect an angle using these steps and check the accuracy of the construction by measuring angles m and n formed by $\overrightarrow{AO}$ to see if they are congruent.

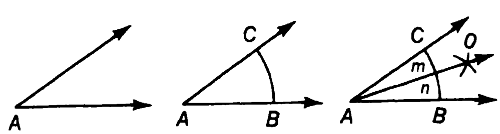


Figure 19.12

First the pupil uses a compass to locate two points, B and C. From these points as centers he makes the intersecting arcs at O. He then draws a ray from A through O.

Have the class bisect a right angle and an obtuse angle. A challenging exercise is to have a pupil bisect each angle of a triangle. If the construction is accurate, the bisectors will intersect at a common point.

Constructing a perpendicular to a line

Figure 19.13 shows how to construct a perpendicular to a line from a point P that is above the line. The small numerals indicate where the point of the compass should be placed, beginning with 1. First the pupil locates points A and B with a compass, using P as a center. Then using A and B as centers and a radius that is longer than $\frac{1}{2}\overline{AB}$, he locates M. MP is the perpendicular to AB through P. The pupils should measure the angles formed to check their construction.

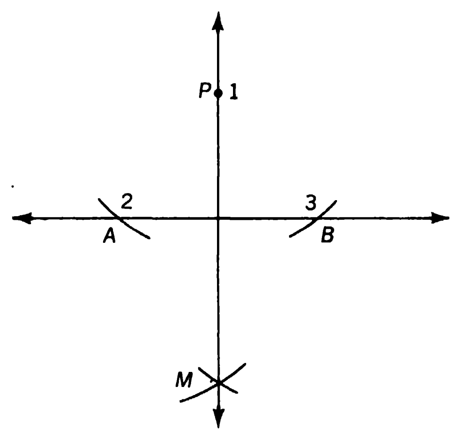


Figure 19.13

Have the class draw two lines on their papers in different positions (Fig. 19.15), and construct perpendiculars to them from points below the lines.

Figure 19.14 shows the steps to follow in constructing a perpendicular to $\overline{AB}$ from a given point on the line as point O. The numerals indicate where the point of the compass should be placed beginning with 1. The problem

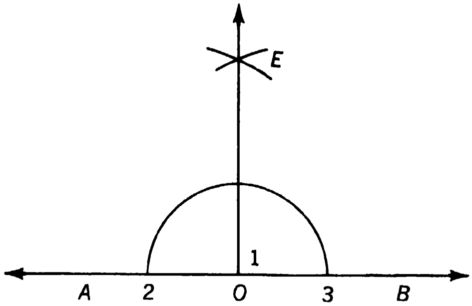


Figure 19.14

is to locate point E above or below the line which lies on the perpendicular.

Have the children carry out the steps shown. Then have them draw perpendiculars to points on lines in different positions, such as are shown in Figure 19.15.

Figure 19.15

Constructing an angle congruent to a given angle

Figure 19 16 shows the steps in constructing an angle congruent to any given angle A. The teacher should demonstrate at the chalkboard the sequence of steps in the construction. First, draw an arc BC, as shown in step 1. Draw a ray and select any point O on this ray as a center. Use a radius $\overline{AB}$ and draw an arc (2). Select the point where the arc cuts the ray, as point D. Then, using $\overline{BC}$ as a radius and D as a center, draw an arc that cuts the first arc at F (3 and 4). Draw $\overrightarrow{OF}$. The rays $\overrightarrow{OD}$ and $\overrightarrow{OF}$ are the sides of an angle that is congruent to angle A.

The class should follow the steps described in constructing two congruent angles. The angles the pupils draw will vary in size. Each pupil should use a protractor to check the accuracy of his construction.

The class can use protractors to draw an angle having any given measure at some point on a line. The more able pupil can construct congruent copies of certain familiar geometric figures, such as squares, rectangles, and equilateral triangles.

PERIMETERS, AREAS, AND VOLUMES

Finding perimeters

A polygon is a plane closed figure bounded by line segments. The *perimeter* of a polygon is the sum of the measures of the sides of the figure. A textbook problem pertaining to finding a perimeter may have the dimensions of the figure given or it may be necessary to measure the sides in order to complete the solution. The length of a side can be found by direct measurement witl ruler or by the use of a compass and a ruler. The pupil may measure the length of a side of a polygon with a compass and then lay off a segment having the same measure on a ray and continue the process for the other sides of the

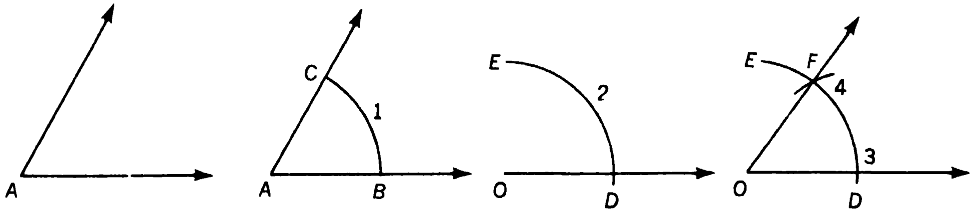


Figure 19.16

figure. He can then use a ruler to measure the segments on the ray. The sum of the measures of these segments is the perimeter of the polygon.

The following exercises suggest a suitable sequence of activities related to perimeters:

1. Find the perimeter of each of the polygons in Figure 19.17.

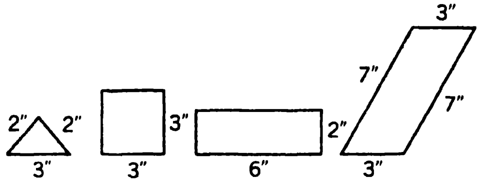


Figure 19.17

2. Draw figures having the same shape as those given in problem 1. Then measure the sides with a ruler and find the perimeter.

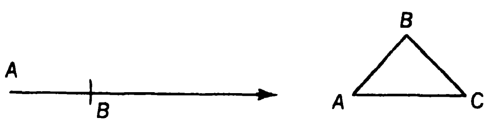


Figure 19.18

3. Consider triangle ABC in Figure 19.18. The dimensions are not given. First draw a ray. Use your compass and open it equal to the distance from A to B. On the ray, lay off $\overline{AB}$ as shown. In the same way mark off $\overline{BC}$ and $\overline{AC}$ in succession on the ray. Then with a ruler measure the length of the three line segments. This measure is the perimeter of the triangle.

4. In a similar way lay off the sides of triangle DEF (Fig. 19.19) on a ray and find the perimeter of the triangle.

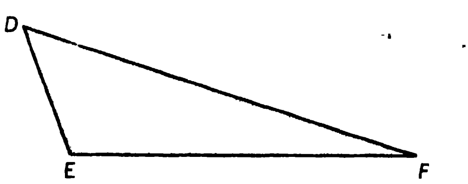


Figure 19.19

5. In the upper grades have class discussions and demonstrations to derive the formulas for the perimeters of the following (see Fig. 19.20):

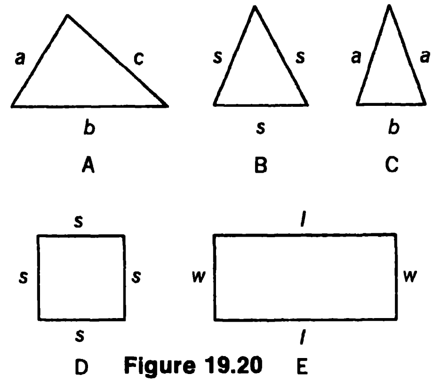


Figure 19.20

a. The perimeter of any triangle is equal to the sum of the measures of the three sides. The formula is

$$p = a + b + c$$

b. The perimeter of an equilateral triangle is equal to three times the measure of one side. The formula is

$$p = 3s$$

c. The formula for the perimeter of an isosceles triangle is

$$p = 2a + b$$

d. The perimeter of a square is equal to four times the measure of one side. The formula is

$$p = s + s + s + s, \text{ or } p = 4s$$

e. The perimeter of a rectangle is equal to the sum of the measures of the four sides. The formula is

$$p = l + w + l + w$$
$$p = 2l + 2w$$
$$p = 2(l + w)$$

The pupil should show that all three formulas for the perimeter of a rectangle are equivalent by replacing l and w with numbers and then performing the indicated operations. He should also

identify the formula $p = 2(l + w)$ as an application of the distributive property of multiplication over addition.

Unit of measure for area

In the rectangle $ABCD$ of Figure 19.21 the union of the set of points forming the figure and the set of points within is called a *region*. Often a teacher uses a rectangular surface, such as a table top or a rectangular cutout, to model a rectangle. The pupil is then asked to find the area of the rectangle. From an illustration of this kind he may form an erroneous concept of the meaning of the area of the rectangle. A rectangle has no area, but the region it encloses has an area. It is important for the pupil to understand that finding the area of a rectangle refers to the area of the region enclosed by the rectangle.

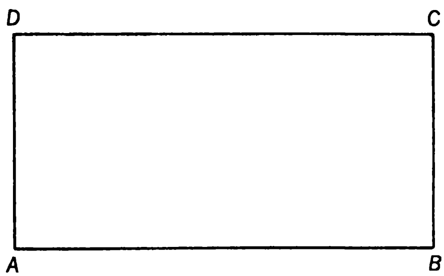


Figure 19.21

It is impossible to fill the region of a rectangle with line segments. Another region must be used to fill the enclosed space, hence a region may be used as a measure of a region. The standard unit of measure of a plane region is a surface bounded by a square having a side of 1. The standard unit for area is a *square inch*. Figure 19.22 shows a unit of area. The area of a plane figure means the measure of the region of that figure and the unit used for measuring that region. Thus, the area of the large square (Fig. 19.22) is 4 square inches, but the measure of this square is 4.

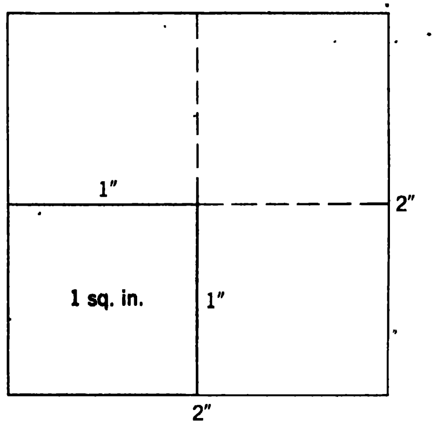


Figure 19.22

Finding the area of a rectangle

Figure 19.23 shows that each of the four plane surfaces or regions of A is congruent to the region of B. We can state that the measure of A is four times the measure of B, or $mA = 4mB$. If the area of B is 1 square inch, the area of A is 4 square inches.

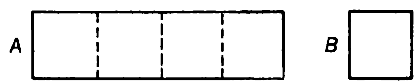


Figure 19.23

The teacher should use two types of activities to help the class discover how to find the area of a rectangular surface. First, the pupil should fill a rectangular region with squares and count the number needed. Second, he should find the number of squares in a rectangular surface that is divided in unit squares.

Each pupil should draw on paper a rectangle having measures expressed as whole numbers, such as 4 inches by 3 inches. He should then find the number of 1-inch squares needed to fill the region. He may do this either by using a 1-inch square and marking off the number needed in each row and re-

peating the process for each row or by using enough of these squares to fill the region. In either case he would count the number of squares needed. The experiment should show that there are 4 squares in a row and that the number of squares is 3×4, or 12. By repeating the process with rectangles having different measures, the pupil should discover that the number of square units in a rectangular region is the same as the product of the number of linear units in the length and in the width.

According to the second plan, the teacher should show pictures or drawings in which a rectangular surface is divided into congruent parts. Figure 19.24 shows a rectangular pan of fudge divided into squares. The pupil should count the number of squares and also find the number by multiplying the number of squares in each row by the number of rows. In a similar manner, he should find the number of units of measure in any rectangular surface.

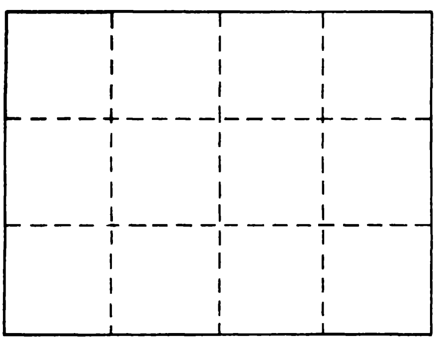


Figure 19.24

The experiment for discovering the area of a rectangular surface bounded by sides having dimensions expressed as inches can be repeated with 1-foot squares. The class should discover that the measure of a square having a side of 1 foot is 144 times the measure of a square having a side of 1 inch. In Figure 19.25, the areas in (A) and (B) are equal,

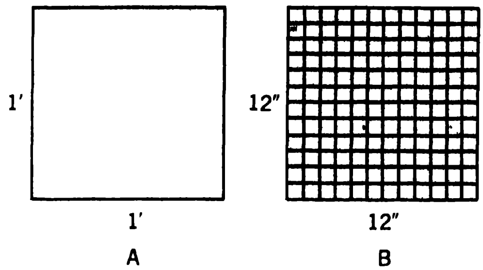


Figure 19.25

therefore 1 square foot = 144 square inches.

The teacher should draw on the chalkboard a square having a side of a yard. The region should be divided into square feet. The drawing will show that the measure of a square yard is the same as the measure of 9 square feet, or an area of a square yard is equal to an area of 9 square feet.

At this point the pupil should attempt to formulate both the rule and the formula for the area of a rectangle. The rule for the area of a rectangle may be stated as follows: The number of square units in the area of a rectangle is the product of the number of units in its length and the number of units in its width. Both dimensions must be expressed on the same linear unit.

The formula for the area of a rectangle is

$$A = lw$$

In advanced classes pupils should attempt to fill the space in his rectangle with cutouts of other figures, such as circles, equilateral triangles, or hexagons, which in most cases will prove to be impossible. This will show why the square is considered the most acceptable unit for expressing area.

For enrichment the experiment could be continued with figures other than rectangles to be filled. It will be found that equilateral triangles of certain dimensions, for example, will fill equi-

lateral triangles having certain larger dimensions but not any other figure.

The relations among the most commonly used units of area are the following:

144 square inches = 1 square foot
9 square feet = 1 square yard
1 acre = 43,560 square feet
1 acre = 160 square rods

To apply what they have learned the pupils should find the approximate areas of available surfaces, of surfaces where dimensions are shown in diagrams, and so on. Many problems having to do with finding the areas of rectangular surfaces are given in mathematics textbooks, which can be used to provide necessary practice.

The *square* is a particular kind of rectangle in which the measures of the sides are equal. In the formula $A = lw$, we may replace l by w or w by l. The formula then becomes $A = w^2$ or l^2. The letter s is generally used to represent the side of a square, hence the formula for the area of a square becomes $A = s \times s$, or s^2. Thus the measure of the area of a 4-inch square is 4×4, or 16, and the area is 16 square inches.

Finding areas of other plane figures

If time permits the teacher may develop with advanced classes the rules and formulas for the areas of other plane figures, such as the following:

Area of a triangle = $\frac{1}{2}$(base × altitude)
or $A = \frac{1}{2}ba$
Area of a parallelogram = base × altitude,
or $A = ba$
Area of a trapezoid = $\frac{1}{2}h(a + b)$
Area of a circle = πr^2

The pupil should derive the formulas given above and should also identify the mathematical properties and principles of numbers that apply when numbers replace the variables in the formulas. The formula for the area of triangle is $A = \frac{1}{2}ba$. The number properties and the mathematical principles that apply to this formula are as follows:

1. The commutative property of multiplication, as $\frac{1}{2}ba = \frac{1}{2}ab$.
2. The associative property of multiplication, as $\frac{1}{2}ba = (\frac{1}{2} \times b) \times a = \frac{1}{2} \times (b \times a)$.
3. The inverse relationship between multiplication and division. Multiplying by $\frac{1}{2}$ is equivalent to dividing by 2.
4. If a set of factors is to be divided by a number, only one factor is divided by that number. To divide the factors ab by 2, divide either a or b by 2 but not both.

The formula for the area of a trapezoid is $A = \frac{1}{2}h(a + b)$. The same four items that apply to the formula $A = \frac{1}{2}ba$ apply to the formula for the area of a trapezoid. The distributive property of multiplication over addition also applies to the formula $A = \frac{1}{2}h(a + b)$.

For purposes of enrichment for the more able learner in the elementary school, the work dealing with areas of triangles, parallelograms, and trapezoids should include the following:

1. Derivation of the formula for the area of the figure
2. Evaluation of the formula (replacing the variables by numbers and then computing)
3. Identification of the number properties and mathematical principles that apply to a formula.

Unit of measure for a solid

Figure 19.26 shows some of the more familiar types of *solids* or *solid regions*. When we measure a solid, we measure its interior region. A solid is a figure that has dimensions of length, width, and height or thickness.

Just as a square region is the unit of measure for a plane figure, a *cubic re-*

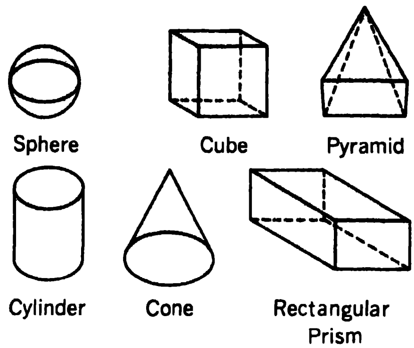


Figure 19.26

gion is the unit of measure for a solid region. The standard unit in our system of measures is the *cubic inch*. The measure of a solid region shows how many times the unit of measure will fit into that region. The *volume* of a solid is the measure of that solid and the unit used for measuring. Thus, the measure for a solid may be 12, but the volume would be 12 cubic inches, 12 cubic feet, or 12 of some other standard unit of measure.

In this text we shall limit the discussion of solids to the measurement of a rectangular *prism*. A rectangular prism is a solid of the kind shown in Figure 19.26. Each face and base of a rectangular prism is a rectangle.

Finding the volume of a prism

Figure 19.27 shows a prism (rectangular) having dimensions of 5 inches, 3 inches, and 2 inches. The figure shows one layer of the prism. Since the height is 3 inches, there will be three of these layers. The volume of each layer is 10 cubic inches, therefore the volume of the prism is 30 cubic inches.

The classroom should be equipped with approximately 125 1-inch cubes and several small rectangular boxes of different dimensions. The dimensions of a certain box are 8 inches, 5 inches, and 2 inches. A member of the class should fill the bottom of a box of this size with 1-inch cubes. The class should discover that a layer contains 5 rows of 8 blocks each, making a total of 40 blocks. Since there will be two such layers, the box will hold 80 blocks or cubes. The number in a layer is the product of the measures of the length and width. This product multiplied by the number of layers will be the number of cubes a box will hold. The number is the product of the measures of the three dimensions when each dimension is expressed in the same unit. When a pupil discovers how to find the number of cubes a box will hold he has learned how to find the volume of a rectangular solid.

If l = length, w = width, and h = height, the formula for the volume of a prism is

$$V = lwh$$

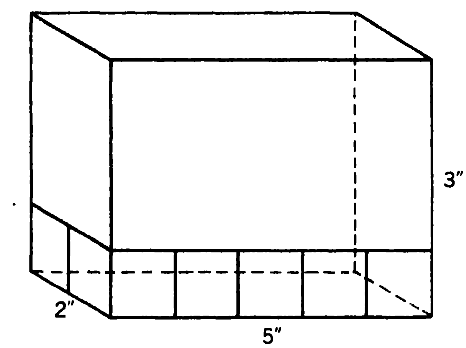


Figure 19.27

The pupil should evaluate the formula and show how the associative and commutative properties of multiplication are applied. If l = 6 inches, w = 4 inches, and h = 3 inches, the measure of the volume is $6 \times 4 \times 3$, or 72.

$6 \times 4 \times 3 = (6 \times 4) \times 3$	Binary property
$(6 \times 4) \times 3 = 6 \times (4 \times 3)$	Associative property
$6 \times (4 \times 3) = 6 \times (3 \times 4)$	Commutative property
$6 \times (3 \times 4) = (6 \times 3) \times 4$	Associative property

The illustrations show that the factors may be rearranged in any way without affecting the product. This generalization is a consequence of the associative and commutative properties of multiplication. The class had other illustrations in dealing with multiplication to show that regrouping or rearranging factors does not affect the product. The teacher should utilize every opportunity to have the class identify this mathematical principle.

Summary of measurement of figures

We have discussed the measures of figures having one, two, or three dimensions. A summary of the units of measure and the measures of these figures follows:

A. One-dimensional figures

1. *Line segments* The standard unit of measure in our system of measures for a line segment is an *inch*. The measure of a line segment is the number that tells how many times the unit can be fitted into the given segment. The length of a line segment is the measure of the segment and the unit of measure. The measure of a line segment may be 3, but the length of the segment is 3 inches.
2. *Angles*. The standard unit of measure for an angle is a *degree* (1°). The measure of an angle is the number that tells how many times an angle of 1° can be fitted into the angle. The size of an angle gives its measure and the unit of measure.

B. Two-dimensional figures

Polygons. The union of a polygon and its interior is a *plane region*. The standard unit for measuring a plane region is a square having a side of 1 inch. The measure of a region is the number that shows how many times the unit of measure can be fitted into the region. The area of a region is the measure and the unit of measure, as the area of a rectangle, is 6 square feet.

C. Three-dimensional figures

Prisms. The union of a rectangular prism and its interior is a *solid region*. The standard unit for a solid region is a cube having an edge of 1 inch. The measure of a prism is the number of times the unit can be fitted into the region. The volume of a prism is the measure and the unit of measure, as the volume of a prism, is 8 cubic inches.

COMPUTATION INVOLVING FAMILIAR MEASURES

Visualizing transformations of measures

It is easy to show the transformations of measures by means of concrete and visual materials. For example, suppose that the problem is, How many quarts are there in 7 pints? First, some child should be asked to place 7 pint containers in a row, as shown in Figure 19.28.

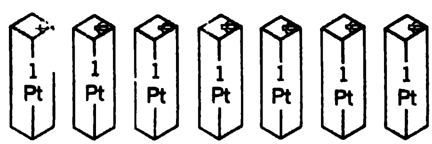


Figure 19.28

He should then be asked to tell how many pints there are in a quart. Next he should be asked to show how many groups of 2 pint bottles he can make with the 7 bottles. The arrangement should be as shown in Figure 19.29.

There are three pairs of pint bottles and an extra bottle. Therefore 7 pints are the same as 3 quarts and 1 pint.

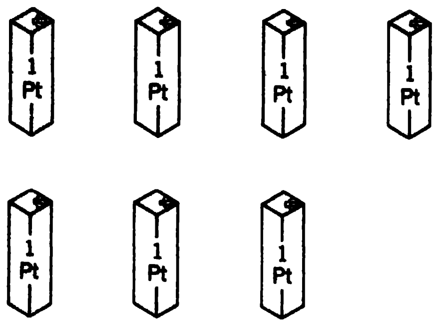


Figure 19.29

The reverse procedure of changing 4 quarts to pints can be demonstrated with a variety of materials in a similar way, as shown in Figure 19.30.

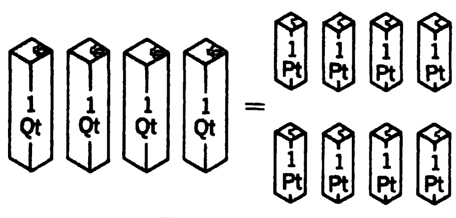


Figure 19.30

The changing of other units such as inches to feet and feet to inches, feet to yards and yards to feet, nickels to dimes and dimes to nickels, and so on, can be demonstrated in similar ways with objective materials.

Later diagrams can be used to show visually the process of transforming measures. Thus in Figure 19.31, 12 quart bottles are shown. The class knows that there are 4 quarts in a gallon. So by encircling groups of 4 bottles the class can see at a glance that 12 quarts are the same as 3 gallons. The process used is division, the ratio concept. From a number of such experiences with various measures the class discovers tha' to change small measures to larger measures we divide. In a similar way the class discovers that to change large measures to small measures we multiply.

Changing measures to fractional parts of other units

Considerable use is made of fractional equivalents in dealing with measurement. Thus a measurement of $\frac{1}{2}$ pound can be expressed as ounces by finding $\frac{1}{2}$ of 16 ounces. Similarly, $\frac{3}{4}$ hour can be expressed as minutes by finding $\frac{3}{4}$ of 60 minutes. Computation of this kind with measurements is one of the most common uses of multiplication of fractions.

A more difficult usage is the changing of a smaller measure to a fractional part of a larger unit. Thus to express 30 minutes as a fractional part of an hour, the learner must solve the example, 30 minutes = □ hour. He should be led to see that 1 minute is $\frac{1}{60}$ hour, hence 30 minutes must be $\frac{30}{60}$ hour. When renamed in lowest terms the fraction becomes $\frac{1}{2}$, hence 30 minutes = $\frac{1}{2}$ hour. He can then prove that his answer is correct by showing that $\frac{1}{2}$ of 60 minutes is 30 minutes.

Developing standards of reference

The learner can be taught how to apply measurement with which he is

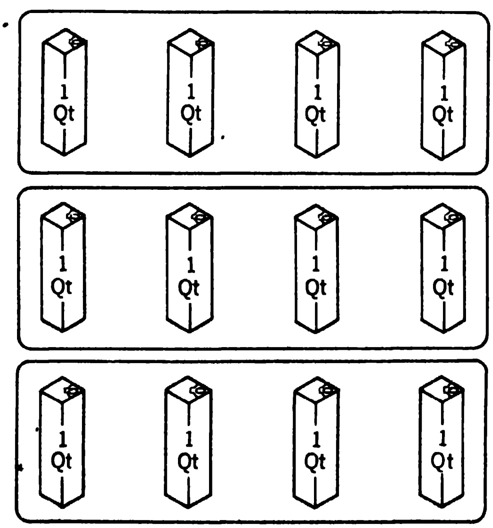


Figure 19.31

familiar as standards of reference in estimating measurements of things whose quantity he does not know. To illustrate, to develop standards of reference to be used in making estimates of length, width, and height the following laboratory procedure can be used:

1. The teacher should make a list of things of varying sizes that either can be placed on a table or are readily accessible, such as a 3-by-5-inch card, a small sheet of ordinary writing paper, a larger sheet of wrapping paper, a window or a picture frame, and a table top.
2. Next, the teacher should prepare a record form, one copy for each pupil. Each child should first estimate to the nearest half inch and then record his estimate for each item on the record form (see Table 19.1).
3. Then using a foot ruler (or yardstick) each child should measure the things listed on the record form.

After the measurements have been completed, they should be compared with the estimates. The class should discuss the range of the differences for individual children between the estimations and the true measurements, the differences among the members of the class, and possible reasons for these differences. The procedures used in estimating by children whose differences were smallest or largest should then be discussed, the purpose being to bring out the value of standards of reference to be applied in making future estimates. The class should then suggest suitable means to visualize an inch, a foot, a yard, and other measures of length.

Similar procedures can be applied to help the child to establish standards of reference for measures of weight, sizes, area, volume, time, and money.

TABLE 19.1
Record Sheet for Estimates and Measurements

	Estimates Lgh. Wdh.	*Measurements* Lgh. Wdh.
Small card		
Sheet of writing paper		
Sheet of wrapping paper		
A window (or picture)		
A table top (or desk)		

EXERCISES

1. What is a standard unit of measure?
2. Select some measuring device and look up its history.
3. Why can the measures used in our country hardly be called a system?
4. What are the merits of the metric system?
5. Demonstrate the fact that all measurement is approximate.
6. What topics in metric geometry do you think should be taught at various grade levels in the elementary school?
7. Illustrate the difference between drawing and constructing an equilateral triangle.
8. Draw any chord that is not a diameter in a circle. Construct a perpendicular bisector of the chord. If your construction is accurate, the bisector will pass through the center of the circle.
9. Draw a circle having a radius of $1\frac{1}{2}$ inches. With that radius, mark off in succession 6 arcs on the circle. Connect these points to form a hexagon.

What is the perimeter of the hexagon?

10. If Q = the set of quadrilaterals, P = the set of parallelograms, R = the set of rectangles, and S = the set of squares, write the number sentence in which each succeeding element of a set is a proper subset of the previous set or sets.

SELECTED READINGS

The most helpful materials for use in teaching about measurement are found in special articles in school encyclopedias, such as Britannica Junior, Compton's Pictured Encyclopedia, World Book, and similar series. See also the following books:

Buckingham, B. R., *Elementary Arithmetic: Its Meaning and Practice*. Boston: Ginn and Company, 1947. Chapter 13.

Clark, J., and L. Eads, *Guiding Arithmetic Learning*. New York: Harcourt, Brace & World, Inc., 1954. Chapter 8.

McSwain, E. T., and R. J. Cooke, *Understanding and Teaching Arithmetic*. New York: Holt, Rinehart and Winston, Inc., 1958. Chapter 10.

Merton, E., and L. May, *Mathematics Background for Primary Teachers*. Wilmette, Ill.: Colburn & Associates, 1966.

Perry, John, *The Story of Standards*. New York: Funk & Wagnalls Company, Inc., 1955.

Spencer, P. L., and M. Brydegaard, *Building Mathematical Competence in the Elementary School*. New York: Holt, Rinehart and Winston, Inc., 1966.

Spitzer, H. F., *Teaching Elementary School Mathematics*. Boston: Houghton Mifflin Company, 1967. Chapter 13.

Swain, R. L., and E. B. Nichols, *Understanding Arithmetic*. New York: Holt, Rinehart and Winston, Inc., 1965.

Swenson, Esther. *Teaching Arithmetic to Children*. New York: Crowell-Collier and Macmillan, Inc., 1964. Chapter 19.

PROVIDING FOR INDIVIDUAL DIFFERENCES

EQUIPMENT FOR THE MATHEMATICS CLASSROOM

The modern mathematics pupil is literally surrounded by aids to help him learn, ranging from models and measuring devices to reference books and pamphlets, visual and audiovisual aids, exhibits, programed learning materials, and other classroom resources. Some of these materials are for use in group situations while others serve as means for developing individual capacities. The variety of materials and purposes for which they are used bring to our attention the possible steps that can be taken to increase the variety of learning materials that may be used in a particular situation for a given purpose.

The considerations the teacher should bear in mind in order to provide the necessary materials to cover the wide range of individual differences in any class are suggested by the following series of questions:

1. Are the variety and quantity of resources in the classroom sufficient to permit choice?
2. Are the resources organized in such a way that they can be quickly located when needed?
3. How accessible to the learner are the resources?
4. Do the available learning devices and materials include a variety of ap-

proaches to learning in order to satisfy the needs of pupils who learn best from manipulating concrete materials and the needs of those who can handle more abstract material?

5. Will the use of this resource free the teacher to give individual help or to meet other needs of the group?

6. Will the learning resulting from use of the resource justify the time expended?

7. Is the resource sufficiently related to what children already know so they will find it easy to use in extending their experiences?

8. Do many of the materials provide a multisensory approach to knowledge?[1]

This chapter considers the following topics: kinds of materials to guide and direct learning; teaching machines and programed learning materials.

KINDS OF MATERIALS TO GUIDE AND DIRECT LEARNING

Exploratory materials

Exploratory materials include objects the learner can handle and move about. Some of these materials have a social application, such as coins, measuring cups, thermometers, and foot rulers. Other materials are specifically designed to help the learner understand some aspect of the mathematical phase of arithmetic. For example, disks are used in counting and grouping, the abacus and place-value charts to show the meaning of numbers and number operations or the role of zero as a place holder, and fractional parts of circles to help the learner discover relationships among fractions of different values.[2]

[1]See *Individualizing Instruction*, 1964 Yearbook of the Department of Supervisors and Directors of Instruction (Washington, D.C.: National Education Association), p. 122.

The use of exploratory materials that children can manipulate or arrange in various ways is an excellent way of giving them direct experience with number and its applications. These activities are especially valuable in the primary grades where a background of meanings is being developed. Counting and grouping objects such as disks, buttons, toothpicks, and small blocks are concrete ways of making small numbers meaningful and developing the concept of number groupings.[3]

The need for using exploratory materials decreases as children grow older and are able to think abstractly and to generalize. However, the teacher should never hesitate to use these materials when a demonstration with objects will clarify new work and give it meaning.

Exploratory materials for demonstrating structure of the numeration system and number operations

Throughout this book the reader has found discussions of the use of many different kinds of exploratory materials that demonstrate the meaning of numbers and number operations.

Some of the most useful of these aids are the following:

1. Pegs, tickets, or sticks to show groupings and regroupings of ones, tens, and hundreds

2. Hundred board to show meanings of numbers to 100

[2]W. J. Sanders, "The Use of Models in Mathematics Instruction," *The Arithmetic Teacher*, March 1964, 11:157–165.

[3]Marvin Karlin, "Machines," *The Arithmetic Teacher*, May 1965, 12:327–334; William H. Lucow, "Testing the Cuisenaire Method," *The Arithmetic Teacher*, November 1963, 10:435–438; Donald E. White, "The Effect of Cuisenaire Materials on Reasoning and Computation," *The Arithmetic Teacher*, November 1963, 10:439–440; James H. Zant, "The Use of New Educational Media," *The Arithmetic Teacher*, December 1965, 12:640–644.

3. Place-value charts to show the meaning of place value in numbers and of grouping and regrouping in number operations
4. An abacus to demonstrate place value and the role of zero
5. Set of cards with 100 squares, strips of 10 squares, and single squares to show numbers to 1000
6. Cutout circles and fractional parts of circles, involving especially halves, fourths, and eighths
7. Flannel board with fractional parts
8. Fractional chart for discovering equivalent fractions.

Some of the most useful exploratory materials for helping the child to discover and understand the steps in the procedures in the different operations with whole numbers, fractions, and decimals are:

1. Place-value charts
2. Cutout circles and fractional parts of circles
3. Sets of squares.

A modern abacus

The abacus has played a significant role in developing our numeration system. An instrument of this kind should be part of the equipment of the classroom in order to help the pupil understand the structure of the numeration system. There are many different kinds of abaci. One form that is an effective aid for classroom instruction may be designated a *modern abacus.*[1] Figure 20.1 shows an abacus of this type.

A modern abacus has the rods extended in a vertical position with a bridge across them to divide the rods into halves. A bead can pass from one half to the other half of each rod because the rod is so flexible that the bead can easily clear the bridge. When the

[1]Distributed by Holt, Rinehart and Winston, Inc., New York.

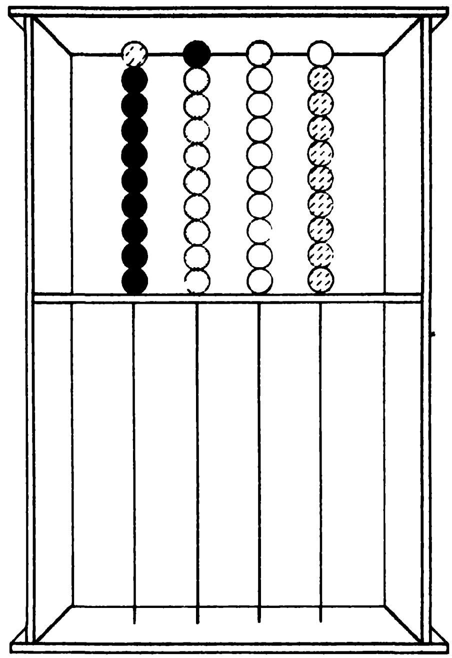


Figure 20.1

beads are in the position shown, the abacus is clear. Beads shown on the lower half of a rod represent a given number.

Each rod of a modern abacus contains 10 beads. Nine of these beads are the same color. The color of the tenth bead is the same as the color of the first 9 beads on the rod to the left. Suppose the color of the first 9 beads on the rod to the right in the ones' place is blue. The tenth bead should be a different color, such as yellow. Then the first 9 beads on the next rod in tens' place should be yellow. The tenth bead on that rod should be a different color, such as red. Then the first 9 beads on the rod in hundreds' place should be red. It is doubtful if a classroom abacus should contain more than four rods, each containing 10 beads.

A modern abacus may be used to enable the pupil to understand two characteristics of our numeration system:

1. The value of a place to the left of

any designated place is 10 times the value of that place.

2. An empty rod on an abacus holds a place in a numeral in the same way as 0 holds a place in a written numeral.

Let us consider the first characteristic. The pupil can move the 10 beads on the ones' rod and 1 bead on the tens' rod to the lower half of the rods. The two amounts represented are equal. The tenth bead on the ones' rod is the critical bead, as shown by its color. This bead has the same color as the bead on the tens' rod. In a similar manner, the same ratio between two consecutive rods can be shown by using the rods in tens' and hundreds' places or by the rods in hundreds' and thousands' places. The use of different colors should help the pupil to discover the relationship between the values of any two consecutive places on the abacus. This relationship is the same as that expressed between any two consecutive places in the system of numeration.

The second characteristic of the number system is the function of 0 as a *place holder.* The abacus in Figure 20.2 shows the numeral 3049. The 9 beads of the same color on a rod correspond to the digits 1–9 in our number system. The value represented by a rod depends upon the position of the rod. By arbitrarily assigning ones' place to the rod to the right, the value represented by each succeeding rod is 10 times the value of the rod to the right. The number of beads shown on a rod shows the number of times the value of this place is taken. Thus the first rod to the right shows 9 ones; the second rod, 4 tens; the third rod, 0 hundreds; and the fourth rod, 3 thousands. The absence of a bead on the rod in hundreds' place shows that there are no hundreds in hundreds' place. It is not necessary to have a bead to represent 0 because the rod holds hundreds' place. To write the numeral 3049 it is necessary to have 0 or some other means to show that there are no hundreds in hundreds' place. It is for this reason that 0 frequently is known as a place holder. Zero not only holds hundreds' place in the numeral 3049 but it also shows the frequency of the base, just as each of the other digits performs these functions in a written numeral.

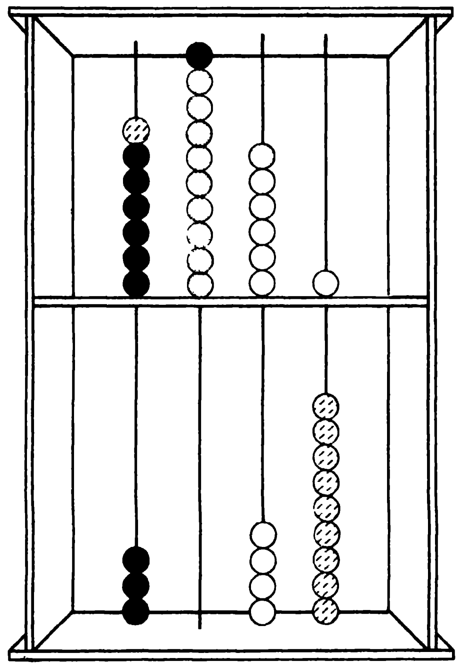


Figure 20-2

When an abacus is used to represent a number, a rod holds a given place. If the frequency of the base is 0, the rod is left vacant. It should be apparent why a modern abacus has 9 beads of the same color on each rod. It is never necessary to use more than 9 beads on a rod to represent the digits of a number on the rod of an abacus. The tenth bead of a different color is used only when demonstrating the decimal ratio between two adjacent places in our numeration system.

Kits for teachers and pupils

Ideally a kit of materials should be available for each child at each grade level above grade 1. The contents of the kit should match the materials used in teacher demonstrations. *Pupil-discovery* materials need not be as large as *teacher-demonstration* materials, since the material used by the teacher in a class demonstration should be visible from all parts of the room.

The contents of the learner's kit will vary to some extent from grade to grade because of new topics that are introduced in succeeding grades. Thus as is shown in Chapters 5 and 6, children in grades 1 and 2 should have groups of small objects to count and group as they learn the meaning of numbers, discover number groupings, and learn basic number facts These simple experiences are hardly necessary in grades 4–8. On the other hand, a pupil's kit in grade 5 should contain a set of fractional parts to be used in the study of processes with fractions, as described in Chapters 13 and 14.

The teacher faces the problem of determining a minimum list of materials for the class for instructional purposes and for *pupil discovery*.

1. A typical pupil kit of discovery materials for the primary grades should include the following:

a. Disks and small objects to be used in counting, grouping, and so forth

b. Squares – hundreds, rectangular strips of 10 squares, and single squares to learn meaning of place value

c. Ruler showing inches and half inches (smaller parts optional)

d. Everybody-show game and necessary cards.

2. A typical teacher kit for the same grade level should include the following materials:

a. Flannel board and small flannel objects to be placed on it to show numbers, groupings, facts, and so forth

b. Hundred board (optional)

c. An abacus to show place value

d. Place-value charts and markers

e. Flannel fractional cutouts.

3. A typical pupil kit for the intermediate grades should contain the following:

a. 10 strips of groups of geometric patterns arranged by twos, threes, fours, and so on, to nines, to be used in discovering products and quotients

b. Ruler showing fractional parts of inch, including sixteenths

c. Fraction kit

d. Squares grouped as ones, tenths, and hundredths to show decimals

e. Everybody-show game and necessary cards.

4. A typical teacher kit of demonstration materials for the intermediate grades should contain:

a. Abacus

b. Place-value charts – whole numbers and decimals

c. Fractional equivalents – cardboard or flannel

d. Flannel board

e. Chart of fractional equivalents

f. [illegible] larged squares similar to pupil square[illegible] o show decimals.

Detailed directions for making the following essential materials are given in the Appendix: (1) charts of fractional equivalents; (2) flannel board; (3) fraction kits; (4) place-value charts; (5) squares to show whole numbers and decimals.

Use of materials

The value derived from equipping a classroom as a learning laboratory depends on the use made of the available materials. The pupil may merely manipulate materials as directed by the

teacher without gaining any insight into what is being presented. The teacher, however, may teach the children how to use concrete material to discover relationships among the quantities involved. Obviously the experiences the pupil has had in dealing with the various kinds of materials will determine the particular type he needs at his level of thinking. In the early stages of learning too much emphasis should not be placed on abstract, symbolic materials. Ordinarily experiences with exploratory and visual materials should precede the work with symbolic materials to make the underlying concepts more meaningful and to reduce reading difficulties. The proper functioning of each kind of material is essential to a meaningful program.

In the primary grades considerable use should be made of exploratory and visual materials. As the pupil grows in his ability to deal intelligently with numbers, he should show a corresponding growth in ability to deal with symbolic materials. In a learning laboratory the teacher directs the learning activities, asks questions to guide the thinking of the children, and leads the discussion. Discoveries are made by the individual learner. The basic understandings are developed by joining or separating sets or by comparing sets. The pupil then translates the operations with sets into mathematical symbols.

It is the process of learning rather than the end product that is vital. A problem that the pupil is ready to attack is the ideal basis of learning. Numbers help him to analyze the situation and to formulate the relationships that are involved. Through the use of exploratory materials and visual aids he can test one or more possible ways of arriving at a solution and ultimately discover a method of finding an answer that is meaningful to him. He is then ready to learn the systematic algorism that is explained in the textbook. Subsequently he can apply the new step in a variety of situations to broaden its meaning.

Many teachers evaluate learning in mathematics in terms of the skills that have been mastered and give almost no consideration to the means used by the learner to master them. The result is that they tend to overlook the importance of understanding in learning. If the process of learning is such a vital element in the instructional program, the teacher should use a variety of materials and experiences that will make it quite certain that the learner will acquire an understanding of what is being taught.

Instruments of measurement

The use of instruments of measurement in concrete situations is an excellent learning activity. Through these experiences the children become familiar with the various units of measurement and devices for applying them The construction of usable measuring instruments, such as a ruler, clock, or thermometer, gives the child meaningful and accurate concepts of standard units of precision and fractional parts of those units.

In the primary grades the teacher should arrange learning activities in such a way that the child will have frequent opportunities to observe the use of measuring devices by others and also to use the simpler instruments himself. Later the child can learn to read scales on instruments of various kinds and thus become familiar with fractional parts of units and increased precision of measurement.

Some of the measuring devices that should be available in a learning laboratory are the following:

1. *Quantity:* abacus, adding machine, number charts, tallying devices, street numbers, fact finders, counting block
2. *Lengths:* ruler, yardstick, tape measure, meter stick, standards for measuring height, micrometer, speedometer, odometer
3. *Time:* calendar, clock, watch, stopwatch, sundial, shadow stick, candle clock, hourglass, timetable, standard time chart
4. *Value:* coins, bills, checks, wampum, tax tokens, stamps, price tags
5. *Weight:* postal scales, balances, spring scales, pressure gauges, height-weight charts, pictures of scales for weighing large amounts, labels showing weights of things
6. *Area:* square-inch cards, square-foot cards, maps
7. *Volume:* pint, quart, gallon measures; cup, teaspoon, tablespoon; cooking measures; peck and bushel measures; rainfall gauge; cubic-inch blocks
8. *Temperature:* thermometer, clinical thermometer, cooking thermometer.

Illustrations and pictures

Pictures can be made the basis of discussion of the uses of mathematics in daily life when direct contact is not feasible or is difficult to arrange. The study of pictures is especially valuable in the primary grades, for it enables the teacher to show the children in a short time many different situations in which mathematics is applied.

Modern mathematics textbooks and workbooks contain a wealth of illustrative materials of this kind to enrich and extend meanings. Most museums have excellent collections. At all grade levels teachers can also accumulate and file for future use many excellent pictures, photographs, and other visual aids to supplement the textbook and to introduce new developments in mathematics. The drawing of pictures based on the use of mathematics in some situation is a stimulating experience in quantitative thinking. The study of maps, home plans, and pictures of business forms also enriches learning. It is evident that the use of pictures and visual aids is one of the most effective ways we have of stimulating pupil interest in a topic and making it meaningful.

The teacher should consider carefully the purposes of pictures and other visual aids in instructional materials. Pictures in textbooks may be classified as three types, (1) compositional, (2) associative, and (3) functional.[5]

A picture classified as compositional is primarily a "filler." It gives a flashy appearance to the textbook, but makes little if any contribution to the meaning of the materials being presented. The discussion is not geared to the picture, and the pupil gets little help from it.

A picture may be classified as associative if it deals with some activity included in a group of problems or in a discussion of a topic, but no particular use is made of the picture in the text. For instance, if a page of problems is about the uses of mathematics on a farm, there may appear on the page a picture of some rural scene or activity.

A picture or illustration classified as functional is one that makes a definite contribution to the discussion. It may serve as the basis of the work and add to the meaning of the situation. A functional picture ordinarily contains information on which problems are based or facts that are needed to solve one or more of the problems. For example, the picture may contain a price list to which the pupil would be expected to refer to find the prices of articles mentioned in

[5] F. E. Grossnickle, "Illustrations in Arithmetic Textbooks," *Elementary School Journal*, October 1946, 47:84–92.

the problem. An examination of mathematics textbooks indicates that the use of functional pictures is increasing quite markedly.

The discussion of simple, clear pictures of everyday uses of mathematics is an important element of a well-rounded readiness program in the primary grades. During this type of discussion reading difficulties are eliminated and meanings and oral vocabulary can be stressed. Pictures enable the children to perceive relationships that otherwise would be difficult to grasp.

Graphs, maps, diagrams, business forms, and clippings from newspapers and magazines are excellent types of functional illustrations. They can be made the basis of groups of questions that can be answered only by referring to the illustrations for necessary information.

Motion pictures and television

Numerous experiments on the use of films and television have shown that the total learnings, both direct and indirect, are greater than those attained by any other medium.[6] The results are best when films are used in connection with other methods of instruction. Motion pictures contribute much to the significance, richness, and accuracy of the concepts being learned, especially those related to the social applications of mathematics. The experiences become more meaningful, thinking is made more effective, and verbalism is reduced. Films make it easy to arouse and maintain interest. There apparently is greater retention of information, particularly in the case of children of average and lower intelligence.

Motion pictures and filmstrips are an excellent substitute fór field trips and excursions. They can be used to clarify important points connected with visits and situations that arise in the course of classroom discussions. Many school systems now have excellent collections of visual aids and films that are used frequently by teachers to add vitality to mathematics instruction. A few have television programs in mathematics, for example, Portland, Detroit, and Pittsburgh.

The value of teaching by television can be shown by the results of an experiment conducted in Pittsburgh during the school year 1955–1956. A total of 20 television classes and 19 control or comparison classes in grade 5 in school districts near Pittsburgh participated in the experiment. The subjects taught daily by television included reading, mathematics, and French. We shall deal with the results obtained in mathematics. There were 655 pupils in the television group and 696 pupils in the control group. The two groups were equated for mental ability and ability in arithmetic. The results and conclusions reached were as follows:

1. The mean grades at the end of the experiment in arithmetic were about one month greater for the control group than for the TV group.

2. Significant gains made by the comparison pupils over the TV pupils in arithmetic were made by those in upper middle and middle IQ groups. Comparison pupils and TV pupils in lower IQ group did equally well in arithmetic.

3. Teachers and pupils agreed that the chief value of TV teaching in comparison with regular classroom teaching was its ability to bring abundant enrichment to pupils.

4. Both teachers and principals agreed that the chief values of regular classroom teaching over TV teaching pertained to

[6]Helen K. Struve, I. Brune, and R. C. Glazier, "Arithmetic via Television," *The Arithmetic Teacher*, October 1956, 3:162–168.

various pupil-teacher relationships and adjustments to the individual differences of pupils.[7]

Several sets of audiofilms have been published to be used in teacher education programs:

SMSG Films. Modern Learning Aids, New York, New York 10022 (30 films with lectures by L. Moredock), 1964

The Need for Modern Mathematics. Science Research Associates, Chicago, Illinois 60611 (1965)

Mathematics for Tomorrow. National Council of Teachers of Mathematics, distributed by Audio-Visual Sound Studio, National Education Association, 1201 Sixteenth Street, N.W., Washington, D.C. 20036 (1966)

Using modern mathematics

Teachers should contact producers and distributors for lists of films that are available for school use. The following list of selected titles suggests the wide variety of films that can be secured:

Motion pictures and filmstrips

1. *Teaching Numbers and Number Operations*

Base Ten. Encyclopedia Britannica Films, Wilmette, Illinois

Counting. Encyclopedia Britannica Films, Wilmette, Illinois

Discovering Numerals. Film Associates, Santa Monica, California

Doing and Undoing in Mathematics. Film Associates, Santa Monica, California

Inequalities. Encyclopedia Britannica Films, Wilmette, Illinois

Inverse. Film Associates, Santa Monica, California

Place Value, Ones, Tens, Hundreds. Coronet, Chicago, Illinois

Set Relations. Encyclopedia Britannica Films, Wilmette, Illinois

2. *Applications of Mathematics*

Arithmetic in the Food Store. Coronet, Chicago, Illinois

Closed Curves. Encyclopedia Britannica Films, Wilmette, Illinois

Gravity, Weight, Weightlessness. Film Associates, Santa Monica, California

How Man Learned To Count. Association Films, New York, New York

Making Change for a Dollar. Coronet, Chicago, Illinois

The Story of Our Money System. Coronet, Chicago, Illinois

The Story of Weights and Measures. Coronet, Chicago, Illinois

What Time Is It? Coronet, Chicago, Illinois

3. *General*

Geometry: Curves and Circles. Film Associates, Santa Monica, California

Using and Understanding Numeration. Society for Visual Education, Chicago, Illinois

Seeing the Use of Number Series. Eye Gate House, Jamaica, New York

Using Modern Mathematics. Society for Visual Education, Chicago, Illinois

Using Numbers. Encyclopedia Britannica Films, Wilmette, Illinois

Mathematics for the Primary Grades. Association Films, New York, New York

Most large-city school systems and all state departments of education have a film library. A list of films available for distribution can be found in *Education Film Guide* and *Educational Media Index*, H. H. Wilson, 950 University Avenue, Bronx New York.

Symbolic materials

Textbooks and workbooks contain the most important symbolic materials

[7] *Pittsburgh Schools Bulletin* (Pittsburgh: Division of Curriculum Development and Research, 1957), 4–5:177, 180.

used in teaching of mathematics. A good textbook with its accompanying teacher's manual is an invaluable guide in planning the mathematics program.

A modern mathematics textbook is set up as a learner's book. It is well illustrated with functional pictures and illustrations. It contains a carefully arranged, systematic, step-by-step development of the number processes, which are clearly and meaningfully presented. The modern textbook development is geared to the use of exploratory and visual materials, such as have been described. The textbook contains the necessary visualization and simple, adequate explanations of each new step to be learned.

The textbook contains an abundance of practice materials to establish and maintain the basic skills in computation and problem solving. The practice materials include exercises to develop skill in reading and interpreting tables, charts, graphs, diagrams, maps, and other kinds of quantitative materials found in the curriculum.

A modern textbook contains many illustrations and problems dealing with everyday applications of mathematics that are within the range of the interests and needs of most children. It is a source of materials for enriching the work in mathematics for the more able learners (see Chap. 24). To assist the teacher in appraising the achievements of the children, the modern textbook provides the necessary testing procedures for evaluating the progress made and for diagnosing and correcting learning difficulties (see Chaps. 21 and 22).

Evaluation and selection of textbooks

The basic consideration in evaluating a series of mathematics textbooks should be: To what extent does the series fit into, or make possible, a modern program of instruction? An effective mathematics program requires the use of exploratory, visual, and symbolic materials, such as have been described in the preceding sections of this chapter.

The choice of a textbook should never be made until a systematic overview of a number of books has been completed to determine their strengths and limitations. The list of items following suggests some of the important kinds of information that should be considered in making an evaluation. The headings in the list indicate the areas to which special attention should be given.

Items to be considered in examining and selecting textbooks are:

1. Consideration of structure of mathematics (arithmetic, geometry, algebra)
2. Rigor of presentation (clarity, proof, reasons for principles)
3. Vocabulary (names of significant ideas)
4. Definitions and undefined terms (adequacy, clarity)
5. Correctness of statements and contact
6. Theorems and proofs (geometry)
7. Generalizations provided for (as aids to learning)
8. Ordering of content (sequence, gradation, organization)
9. Tests, exercises, reviews (kinds, frequency, distribution)
10. Illustrative examples (adequacy of explanations, color)
11. Teachability (reading difficulty, spacing of content)
12. Optional topics (enrichment).[8]

[8]Philip Peak, "Aids for Evaluation of Mathematics Textbooks," *The Arithmetic Teacher*, May 1965, 12:388–394.

In addition to the above items the committee suggested the importance of considering the following characteristics of materials being evaluated:

1. General format (cover, illustrations, spacing)
2. Index and references (and glossary)
3. Usability (durability, size)
4. Services of the publisher (guides, consultants, catalogs)
5. Teachers manuals and teacher helps.

The following references contain discussions of practical procedures to be used in the analysis and evaluation of mathematics textbooks:

Dooley, Mother M., "The Relationship between Arithmetic Research and the Content of Arithmetic Textbooks, 1900–1957," *The Arithmetic Teacher*, April 1960, 7:178–183.

Downing, M., "A Comparison: Textbooks, Domestic and Foreign," *The Arithmetic Teacher*, November 1963, 10:428–434.

Heddens, J. L., and L. Smith, "Readability of Mathematics Books," *The Arithmetic Teacher*, November 1964, 11:466–468.

Lerch, H. H., and C. T. Mangrum, "Instructional Aids Suggested by Textbook Series," *The Arithmetic Teacher*, November 1965, 12:543–546.

Sherman, H. S., and R. E. Belding, "Are Soviet Arithmetic Books Better Than Ours?" *The Arithmetic Teacher*, December 1965, 12:633–637.

Smith, K., and J. Heddens, "Readability of Experimental Mathematics Materials," *The Arithmetic Teacher*, October 1964, 11:391–394.

Wesley, J. L., and V. Robinson, "Quantitative Concepts in Selected Seco 1 Grade Social Studies Textbooks," *Elementary School Journal*, December 1964, 64:159–162.

The interested student should also look under the topic "arithmetic" in *Encyclopedia of Educational Research* (New York: Crowell-Collier and Macmillan, Inc., 1960), pp. 63–77.

In making a preliminary study of several series of textbooks, a teachers' committee should select for study those items in the major list that they regard as most important. A list of pages and illustrations that deal with these items should then be compiled for each book. This information should be made the basis of the selection of the three or four series to be more fully considered before making a final choice. If it is desired to give point ratings to the textbooks, items should be selected from the main list that can be rated in this way. The weighting of each item should be agreed on by the members of the committee.

The basis of a rating may be either a subjective judgment as to the quality of the content related to an item in the list or an evaluation based on an analysis of numerical data found by counting the actual number of times selected items appear in the textbook. In the former case such questions are asked as, How effective is the use of exploratory material? How effective is the use of visual materials? How well is practice distributed? An item may be rated from excellent to poor.

When quantitative data are desired, the basic question is, How many illustrations, examples, tests, or problems does this book contain? On the basis of such facts a judgment as to the merit of the contents of several textbooks can be made, based on a direct comparison of the quantitative data secured for several series. Weaknesses in certain textbooks have often been brought to light by quantitative analyses of their contents.

Supplementary practice materials

Modern textbooks and workbooks contain such a wealth of practice materials that it seldom is necessary for the teacher to prepare additional exercises. However, when a pupil needs practice on simpler exercises than those provided in the basic textbook, the teacher should either make use of materials found in the textbook for a preceding grade or else prepare the necessary supplementary materials as they are needed. The authors believe that sets of textbooks and workbooks for a span of at least three grades should be available for every teacher. In that way instructional materials can be adapted to the child's needs and to his level of development. Thus if a child whose reading ability is low needs special help in problem solving, the teacher may use groups of simpler problems in a textbook for a lower grade level.[9]

Teachers should be very critical of the supplementary materials they prepare. Hastily prepared materials used in classrooms often reflect little insight into the nature of learning difficulties. Prepared materials in textbooks, constructed according to rigorously applied specifications by specialists, are preferable to those devised by the overworked teacher on the spur of the moment.

Oral work in arithmetic

Much of our thinking with numbers is done without direct use of paper and pencil. Thus when we read that the population of a certain city is 487,000, we usually transform this figure, to facilitate thinking, to 500,000 or to a half a million. We do this kind of thinking without using paper and pencil. In daily life most of the computations involve calculations that are simple enough to be done mentally. This fact makes it necessary to include considerable work with oral or unwritten computations in arithmetic practice exercises.

Oral arithmetic emphasizes the use of number relationships and other important aspects of the number system. Extensive use is made of approximations and estimations of values. Instead of trying to multiply by 49 or 68, we use the rounded numbers 50 and 70 because they are easier to handle.

Oral arithmetic is probably most important in the primary grades because it eliminates both reading and writing difficulties and focuses the attention of the class on the number relationships involved, especially in problem solving. The work in oral arithmetic also provides an excellent means of reviewing and practicing number facts and processes. The efficiency of this exercise can be increased by having the children number 20 lines on a page and then write the answers to a list of facts, examples, or problems without using paper or pencil in making necessary computations. This plan requires pupils to pay close attention to the work at hand and to select the important points involved.

Using paper and pencil

When we compute carefully with paper and pencil, as in taking a test or in practicing a number operation, we usually do so because an exact answer is required and so that we can subsequently go over our work to check it. We also use paper and pencil to write down facts and information that we derive from an analysis of the quantitative aspects of a situation. Subsequently we manipulate the data in performing ne-

[9] W. K. Durr, "The Use of Arithmetic Workbooks in Relation to Mental Abilities and Selected Achievement Levels," *Journal of Educational Research*, April 1958, 51:561–571.

cessary calculations to arrive at a solution. We usually write down estimates and approximations arrived at mentally as a record of our thinking. This is especially useful in the teaching of number work in the primary grades. A written record of numeration and facts focuses attention on the facts and reinforces learning. Listing facts on paper tends to systematize them and thus to make learning more vivid and more permanent.

In grades above the primary level, writing out procedures used in learning a step in a number operation tends to clarify thinking. Drawing a picture or making a diagram often adds to the meaning of a situation and brings out basic number relationships, such as those in finding the perimeter of a rectangle. Chalkboard or paper can be used to demonstrate and record various possible ways of solving a problem so that the procedures derived by the children can be compared and evaluated.

Supplementary books and reference materials

In most of the sets of school encyclopedias, such as *The World Book*, *Compton's Pictured Encyclopedia*, and *Britannica Junior*, there are authentic, well-written discussions of many topics studied in arithmetic. *The World Almanac* is another valuable source of information on many topics. The preparation of oral or written reports by selected children based on these materials is an excellent activity. All of the children should be encouraged to explore these and similar reference books because of the interest that will be aroused.

Many children's books have been published that deal with applications of arithmetic, for instance, the story of money, the arithmetic of the weather, aviation, and the like. The contents are largely informational in nature. Many places of business publish pamphlets containing valuable information about their activities, much of it quantitative in nature. The teacher should draw on the resources of the local library for books and bulletins about topics being studied that will enrich the work for the children and familiarize them with all the available sources of information.

Catalogs, governmental bulletins, special reports, and similar materials also provide excellent sources of information about problems that are being investigated in class. The teacher should use a convenient filing cabinet to arrange these materials by topic for ready reference. These should be supplemented from time to time as new materials become available.[10]

Bulletin boards and displays

Many teachers add vitality to the learning experiences by posting on bulletin boards clippings from newspapers, magazines, and similar sources. Clippings may also be brought to class by the children, such as advertisements, weather reports, cartoons, pictures, maps, and interesting reports of civic and industrial affairs containing statistical information. School magazines and periodicals often contain interesting articles in which number plays an important role. The discussion of these materials makes children sensitive to the many uses of number in everyday affairs.

[10]See E. J. Berger and D. A. Johnson. *A Guide to the Use and Procurement of Teaching Aids for Mathematics* (Washington, D.C.. National Council of Teachers of Mathematics, 1959), p. 400. Clarence E. Hardgrove, *The Elementary and Junior High School Mathematics Library* (Washington, D.C.: National Council of Teachers of Mathematics, 1960), pp. 1–20.

WRONG ANSWERS

QUESTION 1

We have learned that the numbers multiplied together to form a product are called the factors of that product.

There are some interesting and important results when the same number is used as a factor several times. A knowledge of this process is essential to an understanding of exponents, logarithms, the slide rule, the operation of certain computers, and of many other powerful mathematical tools.

Here's an example of the same number used as a factor more than once:

In the multiplication

$$3 \times 3 = 9$$

the number 3 appears as a factor twice. Of course we can use the same factor more than twice.

What is the product when 2 is used as a factor 3 times? Press the button corresponding to the right answer.

Answer	Button
6	C
8	B
9	A

YOUR ANSWER: When 2 is used as a factor three times, the product is 6.

You merely used 2 and 3 as factors. $2 \times 3 = 6$. This is incorrect.

We want the product you would get if you used the number 2 as a factor three times. In other words, we want the result of the multiplication

$$2 \times 2 \times 2 = ?$$

WRONG ANSWER C

YOUR ANSWER: When 2 is used as a factor three times, the product is 9.

Your answer is incorrect. You have used the number 3 as a factor twice.

$$3 \times 3 = 9$$

However, you were asked to use the number 2 as a factor three times.

$$2 \times 2 \times 2 = ?$$

WRONG ANSWER A

YOUR ANSWER was "8"

You are correct.

$$2 \times 2 \times 2 = 8$$

The mathematical symbol meaning "form the product reached by using 2 as a factor three times" is

$$2^3$$

It means

use the number 2^3—this many times as a factor

Similarly, $2 \times 2 \times 2 \times 2$ could be written as 2^4, 3×3 as 3^2, etc.

What would 3^4 be?

Answer	Button
$3^4 = 3 \times 4 = 12$	C
$3^4 = 4 \times 4 \times 4 = 64$	B
$3^4 = 3 \times 3 \times 3 \times 3 = 81$	A

RIGHT ANSWER

Figure 20.3

TEACHING MACHINES AND PROGRAMED LEARNING MATERIALS

Characteristics of teaching machines

Experimental psychologists have found that when a pupil understands what he is learning and knows immediately whether he is right or wrong he learns most easily and retains what he has learned.

The basis of programed instruction in mathematics is the attempt to apply the findings of the science of learning to the task of teaching. Intensive research is being done in various centers with the support of private and public funds to develop the necessary methods and means of effective programs of mathematics instruction. Special types of textbooks, workbooks, and teaching machines have been constructed. It has been found in case after case that these materials produce marked improvements in the speed and effectiveness of learning.[11]

Scientists have observed that learning with these materials proceeds best when the following five conditions are met:

1. What is to be learned is clearly presented one small step at a time and with simple explanations that can be easily understood (see the sample frame of instructions in Fig. 20.3). The student advances from simple to more complex and more difficult material only after he is fully prepared to do so. Since

[11] David R. Cram, *Explaining "Teaching Machines" and Programming* (San Francisco: Fearon Publishing Company, 1961).

the preceding material has always been understood and mastered, the learner is confident of his progress at every point. The sequence of materials is so designed that it includes adequate cues and helps to learning and the necessary distributed practice to assure retention.

2. An active response by the learner is required. Writing is the most widely used method of response. The pupil learns by doing something. He may work a typical example or solve a problem and in various ways use his previous knowledge. With programed materials the learner must "construct" an active response, such as filling in a blank, answering a question, selecting an answer from a group of possible answers, and so forth. Psychologists have found that an active response insures total recall and not mere recognition of "wrong" alternatives. The student does not in fact learn anything that he does not understand and actually respond to actively.

3. There is an immediate presentation of the correct answers to each response before the next step is presented when the pupil's response is appropriate and correct; prompt confirmation of this fact is a powerful reward. Experimental psychologists have observed that learning that is immediately rewarded is most likely to be thoroughly retained. Approval of work done and knowledge of success are effective aids to further advance. They reinforce learning. If an error is made, the learner is fully informed before he is allowed to proceed on an unsound basis. In a word, the effect of the error in learning is removed before it can become cumulative.

4. With well-prepared materials there is a low error rate. The carefully graduated steps in the development work and the provision for clear instructional aids to a large extent eliminate the likelihood of errors. Learning that is free from error is not only simpler and easier but also has the side effects of improved pupil morale and a high level of motivation. It is believed by programers that it is possible for learners to learn to avoid errors without actually having to make them. Good programing seeks to strengthen correct behavior in real situations in which alternatives are numerous and are often unstated.

5. Self-pacing by the pupil of his rate of progress is provided in programed learning. In ordinary classroom instruction, whether by lecture or visual means such as television, films, or slides or by discussion of the content of a textbook page, the teaching is invariably directed at the class as a whole, ordinarily the average student. Under such conditions relatively few of the class can make an active response. The slow may not understand, they strain to keep up with the others, they become discouraged and fall behind; on the other hand, the rapid learner is held back and his potential power is not challenged or developed. Programed instruction is learner-centered, and each student is encouraged to work at his own best rate. This individualized procedure permits the learner to pause for reflection and analysis without penalizing him by "bondage to a relentless pace." He can work at the pace he finds comfortable and at the same time he is given incentives to advance rapidly by content reinforcement.

Preparation of programed materials

The construction of such programed materials for mathematics requires careful step-by-step sequencing of instructional content, the preparation of a clear, simple explanation for each step in the development, and the interweaving of

new ideas and those previously presented to assure growing insight and surer retention. These materials must be tested repeatedly to make sure that the error rate is low and that the pupils are in fact learning. At frequent intervals achievement tests covering what has been learned should be administered to evaluate the program as it develops. In its early stages it cannot be expected that programed materials can be so constructed that errors will not be made or that learning efficiency cannot be improved. In any case, the experience of repeated success and growth in the mastery of what is being learned creates and maintains a high level of motivation.

The use of programed instruction in mathematics has a promising future. It provides a sound basis for adapting the work in the classroom to the wide range of individual differences among the students. The successful use of these materials in the upper grades and in colleges at the present time should encourage teachers of grades 1–6 to utilize sets of similar materials in mathematics that are now being published in ever-increasing numbers. Active experimentation with self-instructional materials and devices prior to adoption by schools is to be encouraged.

Sources of programed materials

Persons concerned with the selection of instructional materials in mathematics should secure catalogs from publishers and distributors of teaching machines, textbooks, and pupil workbooks. The following list contains a variety of sources from which to obtain information:

1. Leading publishers of school textbooks are exploring the field.
2. A list of commercially available programs and devices may be obtained from the Department of Audio-Visual Instruction, National Education Association, Washington, D.C.
3. California Test Bureau, Monterey, California
4. Encyclopedia Britannica Films, Wilmette, Illinois
5. Field Enterprises Corporation, Merchandise Plaza, Chicago, Illinois
6. Teaching Machines Corporation (TE-MAC), a division of Grolier, Inc., 575 Lexington Avenue, New York, New York
7. Teaching Machines, Inc., Albuquerque, New Mexico
8. *A Guide to Programmed Instructional Materials Available to Educators* Washington, D.C.: Government Printing Office, 1962.

As an aid to research the Department of Audio Visual Instruction, National Education Association, in 1960 published a 700-page sourcebook containing a compendium of titles on the subject, "Teaching Machines and Programmed Learning," edited by Arthur H. Lumsdaine and Robert Glaser. The volume contains the original article by B. F. Skinner of Harvard University in which the use of automation procedures and programed learning was first discussed. This source should be consulted by those interested in the development of programed learning materials and devices.

The following articles are recent reports dealing with the use of programed materials in mathematics:

Bonghart, F. W., J. C. McLaulin, J. B. Wesson, and L. Pickart, "An Experimental Study of Programmed versus Traditional Elementary School Mathematics," *The Arithmetic Teacher*, April 1963, 10:199–204

Dutton, W. H., "Individualizing Instruction in Elementary School Mathematics for Teachers," *The Arithmetic*

Teacher, March 1966, 13:227–231

Fincher, G. E., and H. T. Fullmer, "Programmed Instruction in Elementary Arithmetic," *The Arithmetic Teacher*, January 1965, 12:19–23

Meadowcraft, B. A., "The Effects of Conventionally Taught Eighth Grade Mathematics Following Seventh Grade Programmed Mathematics," *The Arithmetic Teacher*, December 1965, 12:614–616

Smith, L. W., "The Use and Abuse of Programmed Instruction," *The Mathematics Teacher*, December 1965, 58: 705–708

The following publications contain general discussions on the construction and use of programed materials:

Hughes, J. L., *Programmed Instruction in Schools and Industry*. Chicago: Science Research Associates, Inc., 1962

Programed Instruction, Sixty-sixth Yearbook of the National Society for the Study of Education, Part 2. Chicago: University of Chicago Press, 1967

Skinner, B. F., "Teaching Machines," *Science*, October 1958, 78:969–977

Teaching Machines and Programmed Instruction, Edward Fry, ed. New York: McGraw-Hill, Inc., 1963

Teaching Machines and Programmed Instruction, A. A. Lumsdaine and Robert Glaser, eds. Washington, D.C. National Education Association, 1960

Sources of instructional devices and materials

The following is a list of well-known companies that distribute instructional devices and materials. Catalogs are available on request.

Creative Playthings, Cranbury, New Jersey 08512

Cuisenaire Company of America, Inc., 246 East 46th Street, New York, New York

Ginn and Company, Boston, Massachusetts

J. L. Hammett Company, Kendall Square, Cambridge, Massachusetts

Holt, Rinehart and Winston, Inc., 383 Madison Avenue, New York, New York 10017

Houghton Mifflin Company, 2 Park Street, Boston, Massachusetts

Ideal School Supply Company, Oaklawn, Illinois

Jam Handy Organization, Detroit, Michigan

Judy Company, 310 North Second Street, Minneapolis, Minnesota

W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago, Illinois

EXERCISES

1. Members of the class may volunteer to prepare or gather some of the various kinds of instructional materials described in this chapter as part of a mathematics exhibit. Prepare a teacher's kit for the grade level at which you are to teach.
2. "Markers and a number line are the only supplementary aids necessary to teach number concepts in a modern program," said a teacher. Evaluate this statement.
3. Some abaci contain as many as 20 beads on a rod. From the standpoint of an effective instructional aid, compare the value of an abacus of this kind with a modern abacus (see p. 367).
4. Try to obtain and observe a mathematics film and then to evaluate it. How effective was the film, judging from your reactions and those of the observers?

5. Outline a plan to be used by a faculty for evaluating and selecting a series of mathematics textbooks.
6. Why might a mathematics corner be of value?
7. How should the teacher organize the work in geometry to make the most effective use of the materials for teaching geometry.
8. Try to secure and examine a teaching machine or some form of programed material. Evaluate its usefulness.

SELECTED READINGS

DeVault, M. V., *Improving Mathematics Programs.* Columbus, O.: Charles E. Merrill Books, 1961. Chapters 9 and 10.

Instruction in Arithmetic, Twenty-fifth Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1960. Chapter 10.

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Spencer, P., and M. Brydegaard, *Building Mathematical Competence in the Elementary School.* New York: Holt, Rinehart, and Winston, Inc., 1966.

The Teaching of Arithmetic, Fiftieth Yearbook of the National Society for the Study of Education, Part 2. Chicago: University of Chicago Press, 1951. Chapter 7.

APPRAISING OUTCOMES OF ELEMENTARY MATHEMATICS

In order to determine the effectiveness of the mathematics program, teachers can apply a variety of evaluative procedures, ranging from measurement of ability in elementary mathematics to informal observation of behavior in the classroom and elsewhere to personal interviews. In recent years special emphasis has been placed on developing methods of appraising outcomes in areas that do not easily lend themselves to objective measurement, for example, understanding of the structure of the numeration system and number operations, the ability to use number effectively in quantitative situations, and attitudes toward mathematics. These new procedures have brought to the attention of teachers important values that have sometimes in the past been overlooked.

This chapter will discuss the following topics: the appraisal process; selection and construction of appraisal instruments; methods of evaluation; interpreting the results of appraisals, evaluating the instructional program; improving the mathematics program.

THE APPRAISAL PROCESS

The nature of appraisal

Appraisal is a continuous process that is concerned with the evaluation, study, and improvement of all aspects of the instructional program in mathematics, including pupil achievement. Ideally this process should be carried on cooperatively by all who are concerned with the care and development of children.[1] On the basis of information secured by suitable evaluative procedures, judgments can be made concerning the effectiveness of the mathematics program and the extent to which it meets the needs of the children and of the community as a whole. An analysis of pupil growth and community needs reveals the strengths and weaknesses of the mathematics program and can point to the means for assuring effective educational practices.

Basic steps in the evaluative process

The essential steps in the process of evaluation are discussed in the following paragraphs.

1. All major goals and values of the mathematics program must be determined and accepted. These reflect the ideals and wishes of the community. Objectives were discussed in Chapter 2.
2. The objectives, both immediate and ultimate, should be based on a systematic analysis of individual and group needs. They should be clarified and formulated in terms of desirable behavior on the part of the individuals and groups concerned.
3. Steps must then be taken by appropriate procedures to collect evidence of achievements and growth with respect to the established goals and values as revealed by changes in the behavior of the learners.
4. There should be an examination of the school environment and of instructional practices, including the curriculum and the kind of instruction that are used to achieve these goals. The contacts and experiences of children both in and out of school should be analyzed.
5. The synthesis and interpretation of all of these findings concerning pupil growth and educational practices is the final step in evaluation. This evaluation may lead to redefinitions of goals and values, as may appear desirable, and to the planning of improved ways and means to attain the accepted objectives.
6. The schools should act throughout to secure the interest and cooperation of parents and all community agencies concerned with the growth and development of children in evaluating the total mathematics program and in planning its improvement. Choices should be based on the informed judgment of the groups involved as to the situation that exists and the likelihood that proposed procedures will bring about desired changes. The continuing study of local problems and experimentation with ways of solving them are areas in which teachers should be expected to play an important role.

SELECTION AND CONSTRUCTION OF APPRAISAL INSTRUMENTS

There are five basic steps in developing methods of evaluating the important outcomes of the mathematics program. They may be listed as follows:

[1] *The Impact and Improvement of School Testing Programs*, Sixty-second Yearbook of the National Society for the Study of Education (Chicago: University of Chicago Press, 1963), Part 2, Chaps. 1 and 2.

1. Formulate the objectives clearly

Chapter 2 analyzed the specific outcomes of instruction in elementary mathematics. The outcomes to be evaluated should include not only skill in number operations and in problem solving but also knowledge of the meanings of numbers and the basic vocabulary, understanding of structure, ability to think critically and to apply learnings in all curriculum areas, maturity to use effective study and work habits, and resourcefulness in using numbers in dealing with quantitative situations.

To be most helpful, a general analysis of objectives should be further broken down in order to indicate the specific objectives for each stage of growth or grade level. Such an analysis is of value both to the teacher and to anyone who is attempting to devise a suitable means of appraisal. It should be recognized that the learning of mathematics is a long process of development and that specific objectives and goals should be adjusted to the growth process. Pupils do not all progress from stage to stage at the same rate. The listing of objectives according to levels of growth rather than grade levels is necessary for adequate provision to be made in individual differences in rates of learning.

2. Clarify the objectives

The objectives must be further defined by describing them in terms of pupil behavior that represents changes in the direction of the desired objectives. For example a test of understanding of the system of numeration should be based on an analysis of what this item means, as discussed in Chapter 5. In constructing or selecting procedures to be used in appraisal, these questions should be considered: Does the kind of behavior involved in this procedure or instrument relate to an important objective of mathematics? and, What evaluative means can be used to appraise other important outcomes?

3. Collect test situations and items

A test should consist of situations that are representative of the variety of situations in which the pupil ordinarily uses the skills, information, and other items to be tested. The test situations should be practicable from the standpoint of time, effort, and facilities required. They should sample the defined behavior under a variety of conditions so that dependable conclusions may be drawn as to the typical performance of those tested.

4. Record the behavior

Some sort of record of the pupil's behavior is necessary so that his performance can be evaluated. Procedures involving the use of paper and pencil furnish one kind of record, such as ordinary examinations, objective tests, or daily written work. Reports of observations of pupil behavior, records of responses in test situations or in free activities, things produced by the learner, anecdotal records, photo samples, tape recordings, check lists in analyzing behavior and similar means are also used in recording behavior. The form of record to be used depends on the nature of the behavior to be evaluated. A cumulative record of previous behavior and other information about the individual will greatly facilitate a diagnosis. If records are to be practical and useful, they should not require much time and effort for interpretation.

5. Evaluate the behavior and interpret the record

To evaluate a pupil's behavior, his performance should be appraised in

terms of the important objectives of instruction. The question should be raised: What is the individual's status with respect to a particular objective? The chief problem here is the establishment of standards for evaluating observed performance in different kinds of test situations and for various kinds of reports. In some cases appraisal is relatively simple, as in testing the ability of a grade 4 child to add two three-place numbers. This can be done by finding the percentage of a group of representative examples that were answered correctly. On the other hand, the evaluation of a pupil's understanding of the decimal system of numeration or of his ability to use the four operations in various quantitative situations is difficult because objective means of describing pupil achievement relative to these objectives are lacking. These traits must be appraised by less formal procedures.

The problem of setting up norms of achievement presents many difficulties. The present general practice is to consider the average score for children of a given chronological or mental age group or of a given grade as the normal achievement of children of the group. Because of the wide range of differences in the abilities and interests of the individuals in the group, this method of arriving at a norm is of doubtful validity. In setting up a goal for an individual child, the primary consideration should be the nature of the objective and the evidence there is that the learner is making optimum progress in the direction of the goal. The teacher's purpose should be to guide the pupil "from where he is to where he ought to be," as judged by the achievements of similar children and his own potentialities, that is, his level of expectancy. In many schools the norm used to measure subsequent progress in corrective work is the individual's past performance. In general it is recognized that if the behavior is in harmony with the accepted objectives, it is given a high rating.

Evaluation is facilitated by using objective types of data so that, as far as possible, subjective judgment and personal bias are eliminated. When available, a form of test should be used that can be easily administered and scored. In recent years considerable progress has been made in the development of evaluative procedures for appraising outcomes that were formerly regarded as unmeasurable.

School records contain a wealth of information about the intelligence, school history, health, interests, and home background of children that is of great value in interpreting the data gathered about pupil achievement and behavior in mathematics by the above appraisal procedures.

METHODS OF EVALUATION

Procedures for evaluating the outcomes of the mathematics program

A great many different kinds of techniques are being used to appraise the behavior and characteristics of the learner that are related to mathematics. Some of these procedures are of recent origin, while others have been used for many years. Table 21.1 presents a list of learning objectives and the techniques used to evaluate these objectives. The most valuable methods are the following:

1. Standard tests and objective test procedures
   a. Standard tests
      (1) Achievement tests
      (2) Readiness tests

TABLE 21.1

Outcomes of Learning and Techniques to Evaluate Them

Outcomes	*Evaluative Techniques*
The learner is: 1. Developing a clear understanding of the structure of numbers and of the decimal system of numeration a. Understands the significance of place value in numbers b. Understands grouping and regrouping of numbers in operations c. Understands the number line d. Understands number properties	Objective tests of understanding Observation of daily work Interview with learner Anecdotal records about contributions Demonstration by learner
2. Becoming skillful in fundamental operations and the ability to apply them a. Has control of knowledge of basic number facts b. Understands the meaning of the four number operations and their interrelationships c. Has skill in performing computations d. Can solve real and vicarious problems	Standard tests Informal tests, from textbooks or teacher-prepared Observation of behavior Analysis of daily written work Interviews to test understanding Anecdotal records
3. Developing competence in utilizing systems and instruments of measurement and quantitative procedures in dealing with problems of daily living a. Can read and use the ruler b. Has skill in using measurements to describe and define quantitative aspects of objects, events, and ideas c. Constructs and interprets methods for communicating by graphic and tabular means	Problem-situation tests Objective tests Behavior records and ratings Rating of product of student's work Interview with learner Reports of responses in other curriculum areas
4. Developing meaningful concepts in algebra and geometry a. Uses manipulative and visual aids effectively b. Uses mathematical sentences intelligently in solving problems c. Knows how to solve equations	Observation of daily work Interviews with learner Questionnaires Analysis of reports of methods of study
5. Developing desirable interest in and attitudes toward mathematics[a] a. Makes voluntary contributions of significance to class discussions b. Reads widely about mathematics and its uses c. Is resourceful and inventive in dealing with quantitative aspects of problems and situations	Interest inventory Ratings of interest in activities and toward curriculum content Observation of behavior Self-rating devices Interview with learner Questionnaires Anecdotal records
6. Developing desirable behavior patterns and good citizen traits as a result of group activities a. Reveals qualities of leadership b. Participates effectively in group work, committee assignments c. Is able to attack real problems systematically and effectively	Observation of behavior Problem-situation tests "Guess who" tests Rating scales Interviews with learner "What would you do" tests Tape recordings

[a]See Richard Madden, "New Directions in the Measurement of Mathematical Ability," *The Arithmetic Teacher*, May 1966, 13:375–379.

(3) Diagnostic tests, dealing with specific phases

b. Unstandardized short-answer objective tests

(1) Simple recall or free response

(2) Alternate response

(3) Multiple choice

(4) Completion

(5) Matching

2. Evaluation by less formal procedures

a. Analysis of behavior in some problematic situation

b. Use of behavior records

(1) Controlled conditions involving check lists, rating scales, time studies, recordings

(2) Uncontrolled conditions, involving anecdotal records, diaries, reports by self and others, observation of behavior in the classroom and elsewhere, records of social agencies

c. Inventories and questionnaires about attitudes, interests, activities, methods of study

d. Interviews, conferences, personal reports

e. Analysis of the qualities and merits of some product, such as a graph or an oral report

f. Sociometric procedures to study social relations

Characteristics of standardized tests

The principal characteristics of a standardized test are as follows:

1. The contents of the test are selected and arranged systematically according to accepted specifications.

2. The conditions under which the test is to be administered, the directions to be followed in giving it, and the time allowance for the test are all standardized to insure uniformity.

3. The specified method of scoring the test is definite and objective, so that as far as possible the personal judgment of the tester is eliminated.

4. Standards or norms based on the performances of large numbers of typical pupils are provided, making it possible to evaluate and interpret the scores of an individual pupil.

It should be kept in mind that norms are based on average results for average pupils taught by average teachers with average materials, and hence they are at best measures of mediocrity. These standards are usually surpassed where there is high-grade instruction.

The application of informal evaluative procedures requires the observer to select suitable techniques from among those available. When informal evaluation is used, there are no standardized procedures or norms, and the interpretation of the data secured must be based upon estimates of values and subjective personal judgments.

The most widely used standardized tests in elementary school mathematics, grades 3–6, that are available include the following:

California Arithmetic Test, California Test Bureau

Contemporary Mathematics Test, California Test Bureau

Functional Evaluation in Mathematics, American Guidance Service

Iowa Tests of Educational Development, Science Research Associates, Inc., 259 East Erie Street, Chicago, Illinois 60611

Metropolitan Achievement Test, Harcourt, Brace & World, Inc., 757 Third Avenue, New York, New York 10017

Sequential Tests of Educational Progress (STEP), Educational Testing Service, Princeton, New Jersey 08540

SRA Achievement Series, Science Research Associates, Inc., 259 East Erie Street, Chicago, Illinois 60611

Stanford Achievement Tests, Harcourt, Brace & World, Inc., 757 Third Avenue, New York, New York 10017

The Iowa, SRA, and Stanford tests include test items in reading graphs and tables in the study skills section of the test battery.

A number of publishers of mathematics textbooks for grades 3–6 have developed sets of unstandardized tests that parallel their textbooks. Such tests are useful for teaching and diagnostic purposes. Most of the following include some items dealing with contemporary mathematics, an important item in testing:

Addison-Wesley Publishing Company, Inc., Reading, Massachusetts 01867

Holt, Rinehart and Winston, Inc., 383 Madison Avenue, New York, New York 10017

Science Research Associates, Inc., 259 East Erie Street, Chicago, Illinois 60611

Scott, Foresman and Company, Glenview, Illinois 60025

Silver Burdett Company, Morristown, New Jersey 07960

Purpose of standardized tests

The purpose of standardized achievement tests is to provide a basis for determining the effectiveness of the mathematics program as a whole, as measured by the progress made by individuals and groups toward the achievement of accepted goals. The data supplied by a dependable survey test also provide the teacher with information on which to base the instructional program.

At the present time there are available a number of standardized mathematics tests that are useful for survey purposes. By careful selection it is possible to secure fairly satisfactory measures of such outcomes as computational skill; ability to solve verbal problems; knowledge of mathematics vocabulary and technical terms; knowledge of social applications of mathematics; ability to read maps, graphs, and tables; functional quantitative thinking; and understanding of the system of numeration and number operations.

Criteria considered in selecting appraisal methods

The following criteria should be considered in selecting measurement and testing methods:

1. The value of the characteristic or aspect of growth tested Does the test claim to measure an aspect or characteristic of pupil growth in which you are interested? Are the educational outcomes tested of undoubted value and significance?

2. Validity Does the test actually measure what it purports to measure?

3. Reliability Does the test measure accurately? Will the children tend to get the same scores if measured again?

4. Ease of administration and scoring Are the directions for administering the test clear and easy to follow? Is the test fairly easy to score? Are the tabulation forms clear?

5. Provision and usability of standardized norms Does the test yield scores that are well defined and adequately standardized? Are the scores readily understandable?

The discussion of diagnostic tests is deferred to Chapter 22.

Testing outcomes related to modern mathematics

The only standardized test that deals to any extent with the contents of con-

temporary mathematics other than computational procedures is the Contemporary Mathematics test for grades 4 and 5 published by the California Test Bureau. The test deals with structure and modern mathematical devices and is regarded by the publisher as a supplement to the California Achievement Test, which deals with traditional arithmetic. Some of the unstandardized tests distributed by textbook publishers contain a sampling of items related to modern mathematics. This means that the schools must usually prepare informal tests.

The following are examples of objective test items that can be used to appraise outcomes related to modern mathematics. A number of basic mathematical concepts are covered and the examples include multiple choice, completion, specific-answer, and optimal-answer items.

A. Sets[2]

1. Which of the following expressions is a set?
   a. $22 + 7 = 29$
   b. $\{8 - 4 = 4\}$
   c. 7, 9, 11, 13
   d. $\{2, 4, 6, 8, \cdots\}$
2. The set $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}\}$ is called
   a. a finite set
   b. an infinite set
   c. an empty set
   d. an equal set
3. $\{2, 3, 4\}$ is called a subset of
   a. $\{1, 2, 3, 4\}$
   b. $\{2, 4, 6, 8\}$
   c. $\{2, 4, 8, 12\}$
4. What is the set { } called?
5. A line is an infinite set of ____.
6. A plane contains an infinite set of points. Yes____ No____

B. Understanding the numeration system[3]

1. In the numeral 3796 in what place is the 7 written? (a) ones; (b) tens; (c) hundreds; (d) thousands.
2. Which of the following names the same number as 47? (a) 477; (b) 4 tens + 7 tens; (c) 47 tens; (d) 40 + 7.
3. The zero in 5308 means (a) no tens; (b) 30; (c) no ones; (d) 3 hundreds.
4. The Roman numeral XI names the number as (a) 4; (b) 9; (c) 6; (d) 11.
5. The numeral 234 is written in (a) base four; (b) base two; (c) base ten; (d) base three.
6. Which of the numerals below has the least value? (a) 10_4; (b) 10_3; (c) 10_2; (d) 10_8.
7. Which of the numbers below will make this sentence true: $3268 = 3000 + \square + 60 + 8$? (a) 2; (b) 200; (c) 368; (d) 68.
8. The numeral .38 is the same as (a) 38 ones; (b) $\frac{3}{8}$; (c) 38 tenths; (d) 38 hundredths.

C. Understanding of properties[4]

A good type of test to use to evaluate

[2]See Robert Von Brock, "Measuring Arithmetic Objectives," *The Arithmetic Teacher*, November 1965, 12:537–542; Robert H. Koenker, "Measuring the Meanings of Arithmetic," *The Arithmetic Teacher*, February 1960, 7:93–96, Russell A. Kenny, "Mathematical Understandings of Elementary School Teachers," *The Arithmetic Teacher*, October 1965, 12:431–442.

[3]See Frances Flournoy, Dorothy Brandt, and Johnnie McGregor, "Pupil Understanding of the Numeration System," *The Arithmetic Teacher*, February 1963, 10:89–92, W. H. Dutton, *Evaluating Pupil Understanding of Arithmetic* (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1964); Roland F. Gray, "An Approach to Evaluating Arithmetic Understanding," *The Arithmetic Teacher*, March 1966, 13:187–191. The last-named article discusses the use of the interview technique.

[4]See Florence Flournoy, "Applying Basic Mathematics Ideas in Arithmetic," *The Arithmetic Teacher*, February 1964, 11:104–108. The article reports test data for 18 multiple choice items related to mathematical properties.

A *Illustration*		B *Principle*	
1. $4 + 0 = 4$	__	I. Commutative property of addition	(4)
2. $(4 + 5) + 7 = 4 + (5 + 7)$	__	II. Commutative property of multiplication	(7)
3. $7 \times 1 = 7$	__	III. Associative property of addition	(2)
4. $4 + 5 = 5 + 4$	__	IV. Associative property of multiplication	(5)
5. $3 \times (4 \times 5) = (3 \times 4) \times 5$	__	V. Distributive property	(6)
6. $3 \times 46 = 3 \times 40 + (3 \times 2)$	__	VI. Identity element for addition	(1)
7. $2 \times 4 = 4 \times 2$	__	VII. Identity element for multiplication	(3)

knowledge of a group of related items is the matching test. The exercise above illustrates this type of test.

On each blank write the number of the statement in (A) that matches the statement in (B). (The key is given in parentheses in the column at the right.)

Below is a series of suitable multiple choice items that can be used to test understanding of number properties:

1. If $78 + 69 + 147 = 294$, then $147 + 69 + 78 = \square$: (a) 194; (b) 492; (c) 284; (d) 294.

2. 46×25 has the same product as 40×25 plus $\square$: (a) 4×25; (b) 6×25; (c) 40×5; (d) none of these.

3. When we know that $8 + 5 = 13$, then by the commutative property of addition we know that (a) $(5 + 3) + 5 = 13$; (b) $5 + 8 = 13$; (c) $13 + 1 - 8 + 5$; (d) $13 + 0 = 13$.

4. Which of the following is not true? (a) $27 + 48 = 48 + 27$; (b) $48 - 27 = 21$; (c) $27 \times 48 = 48 \times 27$; (d) $48 \div 27 = 27 \div 48$.

D. Mathematical sentences

1. In which of the following does $\square$ hold a place for a missing sum? (a) $8 + 4 = \square$; (b) $8 - 4 = \square$; (c) $8 \div 4 = \square$; (d) $8 \times 4 = \square$.

2. In which of the following does n hold a place for a missing addend? (a) $9 - n = 6$; (b) $n \div 5 = 7$; (c) $n \times 5 = 25$; (d) $12 \div n = 3$.

3. In which of the following does n hold a place for a missing product? (a) $7 \times n = 7$; (b) $9 \times 8 = n$; (c) $7 + 9 = n$; (d) $n - 7 = 15$.

4. In which of the following does n hold a place for a missing factor? (a) $4 \times 7 = n$; (b) $6 \times n = 30$; (c) $n \div 3 = 12$; (d) $4 + 7 = n$.

5. The mathematical sentence $3 + 4 = 4 + 3$ is called an ____. [Equation]

6. What is the value of $\square$ in the equation $4 + \square = 7 + 3$? (a) 3; (b) 6; (c) 14; (d) 5.

E. Knowledge of mathematical symbols

1. Which of the following mathematical sentences is true? (a) $7 < 9$; (b) $6 + 5 = 7 + 2$; (c) $8 > 8 + 0$; (d) $25 < 24 \times 1$.

2. Which of the following mathematical sentences is false? (a) $4 + 7 = 7 + (3 +$; (b) $8 < 6$; (c) $7 > 5$; (d) $26 + 0 = 26$.

3. Which of the following mathematical sentences is true? (a) $2 \times 38 = 56 \times 4$; (b) $38 \times 2 > 2 \times 38$; (c) $2 \times 48 < 48 \times 2$; (d) $3 \times 48 > 2 \times 48$.

4. Which of the following mathematical sentences is false? (a) 1 ft. < 1 yd.; (b) 2 qt. = 4 pt.; (c) 2 hr. = 120 minutes; (d) $14 \neq 2 \times 7$.

F. Problem solving

1. Consider this problem: *How much do 4 oranges weigh if 1 weighs 5 ounces?* Which mathematical sentence be-

low would you use to find the answer to the problem? (a) $4 \times 5 = \square$; (b) $4 + 5 = \square$; (c) $\square \times 5 = 4$; (d) $5 \div 4 = \square$.

2. Consider this problem: *Jack is 52 inches tall, Tom is 55 inches tall, and Bob is 59 inches tall. What is their average height?* (a) $52 + 55 + 59 = \square$; (b) $(52 + 55) - 59 = \square$; (c) none of these; (d) $(52 + 55 + 59) \div 3 = \square$.

3. *May bought 3 apples at 5 cents apiece and 1 candy for 40 cents. How much did she spend in all?* Which mathematical sentence would you use to find the answer to the problem? (a) $(3 \times 5) + 40 = \square$; (b) $3 + 5 + 40 = \square$; (c) $40 - (3 + 5) = \square$; (d) $40 - (3 \times 5) = \square$.

G. Geometry

1. How many lines can be drawn through any given point? (a) 1; (b) 2; (c) 4; (d) so many we can't count them all.

2. A point has what dimensions? (a) length; (b) width; (c) no dimension; (d) depth.

3. The area of a rectangle 4 inches long and 2 inches wide is (a) 6 inches; (b) 2 square inches; (c) 8 square inches; (d) 12 inches.

I. ○; II. □; III. ▭; IV. ▱.

4. Which of these is not a parallelogram?

5. We call a geometric figure that has only one endpoint a: (a) line; (b) ray; (c) line segment; (d) side.

6. A triangle has two right angles. [Yes__ No__]

7. A quadrilateral has ____ sides. [Completion]

Arithmetic attitude scale[5]

10.5	Arithmetic thrills me and I like it better than any other subject.
9.0	I would like to spend more time in school working arithmetic problems
8.1	Arithmetic is very interesting
7.0	Sometimes I enjoy the challenge presented by an arithmetic problem
5.9	Arithmetic is as important as any other subject
4.6	I don't think arithmetic is fun, but I always want to do well in it
3.7	I don't feel sure of myself in arithmetic
3.0	I can't see much value in arithmetic
2.5	I have always been afraid of arithmetic
1.5	I have never liked arithmetic

Pupils can be asked to check the item in the scale that most nearly describes their attitude. It may be of interest to know that in one investigation it was found that children in grades 5 and 6 rated arithmetic as one of the best-liked fields of study.[6] Boys rated arithmetic first and girls rated it second, slightly below reading. Chase asked the pupils to rank arithmetic among all of the subject areas given in a list. This procedure can be used by any teacher. When arithmetic is rated low in interest by children, the teacher should study the possible causes and take steps to improve the situation.

Evaluation by less formal procedures

For reasons of space it is impossible to describe here in detail the less formal procedures that can be used in evaluation of outcomes related to arithmetic instruction. The illustrative analysis on page 387 defines six important outcomes of arithmetic along with several evaluative techniques, including tests for ascertaining pupil performance. Other outcomes can be appraised by similar procedures.

[5]Adapted from an attitude scale developed by W. H. Dutton and reprinted in "Attitudes of Junior High School Pupils toward Arithmetic," *School Review*, January 1956, 64:18–22; see also H. Bassham and others, "Attitude and Achievement in Arithmetic," *The Arithmetic Teacher*, February 1964, 8:66–72.

[6]W. I. Chase, "Subject Preferences of Fifth Grade Children," *Elementary School Journal*, December 1949, 50:208–211.

INTERPRETING THE RESULTS OF APPRAISALS

The interpretation of any evaluation data should be done with caution even when standardized procedures are used and when agreed-upon norms and dependable standards are available. When these are lacking, as in the case of the informal procedures, special care should be exercised in the interpretation of the results. Consideration must be given to the general characteristics and capacity of the individuals and their possible achievements. Teachers should always interpret the results of standard tests in terms of the background and experience of the children. Unfortunately, it is usually true that a group of children from an underprivileged area cannot be expected to achieve as high a level as a group living in a more favored area. In general, however, knowledge of test results greatly aids the teacher in applying the less formal types of evaluation in observing hildren in the classroom.

The analysis given below suggests methods for analyzing and interpreting the results of survey tests in arithmetic at various levels in the school system, beginning with administrative officers and concluding with the classroom teacher:

1. Superintendent and central staff
   a. Analysis of city-wide results grade by grade in comparison with expected performance
   b. Comparison with results of previous years
   c. Consideration of the consolidated distributions of class results
   d. Overview of results for vari us schools
   e. Consideration of possible causes of these variations in results among the different schools
   f. Planning next steps for improving the educational program
2. Staffs of individual schools in cooperation with consultants
   a. Comparison of results for the school as a whole with city-wide scores and with standard scores at various grade levels
   b. General trends or progress from grade to grade as compared with results for previous years
   c. The deviation of each grade and class from expected levels of attainment in relation to the mental ability of the children, their social background, and health
   d. Consistency of levels of attainment in the various areas tested
   e. The range of test results for each area within individual classes
   f. The overlapping of test scores at consecutive grade levels
   g. Identifying strengths and weaknesses of the school's program on the basis of the test results
   h. Considering possible next steps
3. Individual teachers in cooperation with principal or consultant
   a. Overview of the results for the class as a whole, sharing the information with the pupils
   b. alysis of the progress made by indivi ual pupils based on comparison with previous tests, preferably summarized in graphic profile form
   c. Critical comparison of educational levels achieved in relation to the mental ability of individuals
   d. Consideration of factors that might throw light on variations in achievement of individuals and deviations from levels of e pectancy
   e. Consideration of discrepancies between test results for individual pupils and teacher's estimates
   f. Analysis of the test items that have possible diagnostic value

g. Planning the ways in which to use the data most effectively in the public relations program.

The steps to be taken to evaluate information secured by less formal appraisal procedures are similar to those used in dealing with test results. Evaluation will necessarily be less precise and definite than is possible with the use of standard tests, because comparable norms are lacking.

EVALUATING THE INSTRUCTIONAL PROGRAM

There are three aspects of the instructional program that should be evaluated as part of a comprehensive appraisal program: (1) the curriculum, (2) the methods of classroom instruction, and (3) the materials and equipment available.

The curriculum

The curriculum should be evaluated in terms of criteria, which should grow out of group discussion and study of modern trends in elementary mathematics.

The curriculum, either as found in the classroom or as discussed in instructional guides, may not be well organized; the objectives may be too limited and may not include the wide variety of outcomes discussed in Chapter 2; the gradation and sequence of subject matter may be faulty; the content may be highly academic, formal, and unrelated to the needs and interests of the children; the contents may be out of line with recent developments in elementary mathematics; it may not be adapted to development and rates of growth; standards set up may not be flexible and the contents of the curriculum may not be adapted to differences in the abilities of the children. Special consideration should also be given to the process by which the curriculum is being developed. The program may be limited to a textbook, by the prescriptions of the course of study, and by regulations set up by the school authorities. Apparently little may be done to assist teachers to study the needs of the children and the community as a basis of selecting curriculum content.

Evaluating methods of classroom instruction

The evaluation of methods of teaching used in classrooms should be a cooperative undertaking in which teachers take an active part. Such criteria as the following should be set up cooperatively by the group and then applied in appraising instruction:

1. Procedures should be used that will make number and number operations meaningful to the children.
2. The procedures should stress the understanding of number and efficiency in quantitative thinking.
3. The work in elementary mathematics should be associated with the activities of the school day and not only with a particular class period.
4. Instruction should provide adequate time in school for the systematic practice needed to develop competence and skill in the use of number and quantitative procedures.
5. The diagnosis and treatment of learning difficulties in arithmetic are essential.
6. Provisions should be made to adapt instructional procedures to individual differences in ability and in rates of learning.
7. The school should provide for mental health and social adjustment through successful work on challenging activities.
8. There should be a continuous program of evaluating learning which in-

forms child, teacher, and parent about the individual's growth in arithmetic.

It may be demonstrated by careful appraisal that many of these criteria are being successfully applied. On the other hand it may be discovered that instruction is highly formal and that the work is largely limited to an intensive drill program dealing with the mastery of computational skills. Little attention may be given to other important outcomes. The practice that is provided may be on operations that have not been made meaningful to the children. Then the pupils do not understand what they are being required to learn. The teaching procedures may not be in line with the findings of research and may be unskillfully applied. Little attempt may be made to group the children according to their needs.

The discussion of the methods of diagnosing learning difficulties in arithmetic, of the factors contributing to them, and methods of treating them is the theme of the next chapter.

Evaluating materials of instruction

The availability and adequacy of materials can be evaluated by applying such criteria as the following:

1. Manipulative and exploratory materials, objects, and visual aids should be used to make number and number operations meaningful to children.

2. Supplementary reading materials should be used to explore and extend the vocabulary and background of mathematics.

3. Special materials especially intended to arouse and maintain interest in mathematics should be made accessible to teachers and children.

4. Efficient scientifically constructed practice materials and testing procedures should be used to develop and maintain basic knowledge and skills.

It may be found that the instructional materials and equipment are fully adequate to meet the needs of a learning laboratory as described in Chapter 3. On the other hand, it may be decided that there are serious limitations in the quality and variety of what is available. The materials of instruction may be too difficult or they may be poorly organized, constructed, and arranged. The contents may not be presented attractively or clearly; the subject matter may lack interest; the supplies may not include essential concrete teaching materials and diagnostic tests. Limited supplementary material and few visual aids may be available and little use may be made of places of business, museums, libraries, and similar civic centers to enrich and broaden learning. These situations arise partly because these agencies may be indifferent, even unwilling to permit the schools to use them, partly because the school has not integrated them into learning experiences.

Systematic procedures for evaluating the mathematics program

In recent years the widespread interest in the evaluation of mathematics programs has led to the development of systematic methods of evaluation and their application in all parts of this country. The most widely used approaches are subjective ratings of the arithmetic program, checklists, consensus studies, and group discussion. The following references should be consulted for helpful suggestions and procedures by teachers, consultants, and others who wish to study and improve their arithmetic programs:

Brueckner, L. J., *Improving the Arithmetic Program.* New York: Appleton-Century-Crofts, 1957.

Elementary Evaluative Criteria. Boston: Boston University, 1953.

Evaluating the Elementary School. Atlanta, Ga.: Southern Association of Colleges and Secondary Schools, 1951.

Ragan, W. B., *Modern Elementary Curriculum*, 2nd ed. New York: Holt, Rinehart and Winston, Inc., 1966.

IMPROVING THE MATHEMATICS PROGRAM

Selecting items to be improved

The group concerned having evaluated the existing program, should then list the problems, difficulties, shortcomings, and needs revealed by the appraisal. New departures that might be considered for introduction into the local situation should also be listed. Through discussion the group should list their needs, problems, and new departures which seem most urgently to require alteration. These items can then become the objectives of an improvement program.

Planning the improvement program

Once the problems are selected and defined, the next task is the cooperative planning and organization of the actual activities of diagnosis and solution.[7] The improvement program should be flexible and adapted to the needs of individuals and of groups having common problems and needs. The program may involve any of a number of subsidiary techniques,[8] including:

1. Conferences with individuals and small groups for the planning of any and all kinds of projects
2. A series of local study groups, general or limited teachers' meetings
3. A local workshop with facilities and personnel available at stated times
4. Extension courses, summer school work, leave of absence for study or travel
5. Cooperatively developed bulletins, usually with references and study guides
6. Experimental work, either individual or group, for the development of new materials, new evaluational devices; for try-out of materials
7. Committee and study groups to examine student interests, attitudes, problems, and needs
8. Committee work on curriculum improvement or course of study writing
9. Visiting teachers in local and outside schools according to plans devised by teachers and staff
10. Visits and conferences by supervisory personnel, usually on call and for cooperatively determined purposes
11. Cooperatively determined programs of directed observation and directed teaching
12. Committees and study groups to examine new texts, to select texts and materials
13. Exchange of teachers between schools and between systems.[9]

An illustrative improvement program

The program of curriculum research in developmental arithmetic in New York City, described by Eads, illustrates

[7] Dan Tredway, "The Secondary Teacher and Elementary School Mathematics," *The Mathematics Teacher*, April 1965, 58:313–316.

[8] W. H. Burton and L. J. Brueckner, *Supervision: A Social Process* (New York: Appleton-Century-Crofts, 1955), p. 133.

[9] C. E. Hardgrove and B. Jackson, "*CUPM* Report on the Training of Teachers of Elementary School Mathematics," *The Arithmetic Teacher*, February 1964, 11:89–93.

cooperative procedures for improving arithmetic instruction.[10] The program began because of general dissatisfaction of gifted children with their progress in arithmetic at all grade levels. The main problems were lack of understanding of subject matter and vague number concepts. In the first year experimental work began in the primary grades to develop methods and materials to teach meaningful number concepts and processes. The results of this experimental work were then introduced in grade 1 on a city-wide basis. Each succeeding year the next higher grade was included in the program. Trained classroom teachers assisted regular teachers to apply the procedures. Workshops were organized that developed and demonstrated experiences, materials, procedures, content, drill devices, procedures for evaluating pupil learning, and similar materials. Bulletins and guides for teachers were also prepared and distributed. This experimental approach is the essence of a continuing program to improve arithmetic instruction. Methods suggested could be used in any school.

[10]Laura K. Eads, "Learning Principles That Characterize Developmental Mathematics," *The Arithmetic Teacher*, October 1957, 4:179–181.

EXERCISES

1. Why should the evaluation of the mathematics program be regarded as a cooperative undertaking?
2. Examine local school records to determine the data given that would assist in interpreting test results in mathematics.
3. What are the values and limitations of standardized tests? What standardized tests are used in local schools?
4. Why is it necessary to use less formal methods of evaluation in many instances?
5. Try to get a measure of attitudes toward mathematics of children in some class using Dutton's scale or a ranking procedure.
6. Examine a mathematics textbook to determine the types of tests that are included. Evaluate the tests.
7. Secure the results of a mathematics test for some class. Analyze the errors and indicate the kinds of remedial measures you would apply.
8. Why may test results prove to be at a level considerably below normal? What can be done to improve conditions related to curriculum, instruction, and materials?
9. What is being done by the local schools to improve the program in elementary mathematics?
10. Have a student give an evaluation of a widely used standardized test in arithmetic as reported in one of the Mental Measurement Yearbooks. The third and fourth yearbooks are published by Rutgers University Press, while the fifth and sixth are published by the Gryphon Press, New Brunswick, N.J.

SELECTED READINGS

Brueckner, L. J., *Improving the Arithmetic Program.* New York: Appleton-Century-Crofts, 1957.

Brueckner, L. J., and G. L. Bond, *The Diagnosis and Treatment of Learning Difficulties.* New York: Appleton-Century-Crofts, 1955. Chapters 2 and 8.

Dutton, W. H., *Evaluating Pupils' Understanding of Arithmetic.* Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1964.

Evaluation in Mathematics, Twenty-sixth Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1961.

The Impact and Improvement of School Testing Programs, Sixty-second Yearbook of the National Society for the Study of Education, Part 2. Chicago: University of Chicago Press, 1963. Chapters 1, 2, 3.

Mathematics in General Education, Report of the Committee on the Function of Mathematics in General Education for the Commission on the Secondary School Curriculum. New York: Appleton-Century-Crofts, 1940.

Spitzer, H., *Teaching Elementary School Mathematics.* Boston: Houghton Mifflin Company, 1967. Chapter 14.

THE DIAGNOSIS AND TREATMENT OF LEARNING DIFFICULTIES IN MATHEMATICS

The school should arrange a variety of functional learning experiences that, if effective, will lead to the well-rounded growth and development of all wholesome aspects of the learner's personality. The chief problem is to provide fully and efficiently for individual differences among the children. The continuous study of the pupil by the teacher by means of carefully selected evaluation techniques as well as self-appraisal by the learner himself are important elements of a well-conceived guidance program. Whenever there is any realistic evidence that growth and development are not proceeding satisfactorily, it becomes necessary to identify the nature and causes of the deficiency by appropriate diagnostic procedures so that the necessary corrective and remedial meas-

ures can be taken as soon as possible. Thus appraisal, guidance, and diagnosis are intimately intertwined parts of a continuing process of guiding learning.

The guidance function of the school requires the creation, with that part of the environment under its control, of conditions that are most likely to be conducive to wholesome growth; and in that part of the environment not under the school's control, the securing of the cooperation of the pupils and of all members of the social group in creating an environment that stimulates and sustains the growth of all. The school should help the pupil to set up standards of attainment and behavior by which he can at all times evaluate his conduct.

When instruction is effectively organized and the work is meaningful to the children, only some of them will encounter difficulties. The teacher should continuously scrutinize the work of individual pupils to discover points of weakness so as to prevent the accumulation of deficiencies that may interfere with progress. In spite of the best efforts of the teacher, some children do not understand what they are learning, and they invent strange procedures that appear to them to be correct but are actually faulty and inefficient. In some cases such factors as moving from one community to another, excessive absence due to illness, physical handicaps, and emotional disturbances may interfere with learning. The teacher should make every effort to analyze learning difficulties and should take steps that are likely to bring about an improvement.

This chapter deals with the following topics: the use of tests in teaching; individual and trait differences; levels of diagnosis; techniques of diagnosis; the treatment of learning difficulties.

THE USE OF TESTS IN TEACHING

Standard tests

Most teachers have found standard achievement tests useful for a variety of purposes. Modern tests focus attention on important educational objectives and clarify them for teachers and pupils. The results of well-constructed tests place each pupil on a scale of ability in a particular field or skill, varying from an inferior performance to an outstanding one. This information aids the teacher in determining each pupil's relative status in the whole group. Tests also enable the teacher to measure the pupil's growth over a period of time. This information should be shared with the individual. It should be realized that many factors such as an emotional upset or some temporary distraction make it difficult to measure the changes in individual pupils with certainty. The use of the results of standard tests of intelligence in interpreting achievement test scores makes them more meaningful.

Informal testing

Informal testing provides an excellent means for checking on the amount of subject matter a pupil has learned. Test results stimulate learning by enabling pupils to think of their achievements in objective terms. They also serve as an excellent motivating device by revealing evidences of growth. Tests can be constructed so that they reveal specific learning difficulties. They serve as a helpful means for locating areas of learning that should be reviewed. The teacher can use evaluative data to divide a class into groups for instructional purposes. Tests enable the teacher to measure the effectiveness of steps that

are taken to adjust the instructional program to the strengths and weaknesses of individual pupils. Test data also provide an effective basis for a well-rounded, balanced presentation of pupil progress for parents.

Use of tests in the guidance of learning

Testing should be used as the need arises in the teaching-learning situation. Before teaching is begun, the teacher should gather information about the mental ability of the children, their readiness for new work, their special abilities and interests, and their strengths and weaknesses in all curriculum areas, particularly their reading ability. This can be done by use of pretests, by analysis of data on the pupil's premanent record card, and by observation of behavior during class discussions. In the course of learning, progress should be appraised from time to time by the informal methods of appraisal discussed in Chapter 21.

The diagnosis and treatment of learning difficulties that appear from time to time should be a continuing process that is undertaken whenever the need arises. The teacher can use the evaluative data to guide and motivate the learner by discussing his performance with him. In many school systems, cases in which serious difficulties and deficiencies develop that the regular teacher cannot effectively deal with in the classroom are referred to guidance clinics for special study and diagnosis. On the basis of their findings these clinics make suggestions as to the steps to be taken that are most likely to lead to improvement.

A number of mathematics textbooks contain well-constructed test exercises for the guidance of learning, including inventory tests, readiness tests, diagnostic tests, and progress tests. These materials are invaluable supplements to a standard testing program.

INDIVIDUAL AND TRAIT DIFFERENCES

Range of individual differences

Perhaps the most important fact that has been revealed by educational measurement, as far as instruction in mathematics is concerned, is the wide range in achievement and intelligence levels in any typical class in our schools. Equally significant are the differences in results from school to school. There apparently is an increase in variability in both achievement and intelligence from grade to grade and age by age. Cook[1] showed that the range in arithmetical computation and reasoning is between six and seven years on the grade 6 level. This range is somewhat less than the range for other curriculum areas, such as reading.

The data in Table 22.1 illustrate the differences in achievement levels in typical classes in arithmetic. They show the variations in the scores on tests in arithmetic and reading in the 1957 edition of the California Arithmetic Test of the 3 children in grade 5.1 in a small eastern city. The medians for the group in arithmetic were practically equivalent to the test standards. In reading, the median, grade 5.7, was approximately six months above the norm. However, there was a very wide range in ability in both arithmetic and reading. The range in ability in arithmetic reasoning was from grade 3.0 to grade 6.9, or approximately 4 years, for arith-

[1] W. W. Cook, *Educational Measurement* (Washington, D.C.: American Council on Education, 1951), pp. 10–12.

TABLE 22.1

Variability in Arithmetic and Reading Ability in Grade 5.1 in a Small School System (73 Cases)

Grade Level	*Arithmetic Reasoning*	*Arithmetic Fundamentals*	*Total Arithmetic*	*Reading*
8.0 and up	–	–	–	1
7.5–7.9	–	–	–	2
7.0–7.4	–	–	–	2
6.5–6.9	2	–	–	9
6.0–6.4	11	1	–	11
5.5–5.9	13	11	15	7
5.0–5.4	16	29	29	18
4.5–4.9	21	19	21	13
4.0–4.4	8	10	6	6
3.5–3.9	0	3	2	1
3.0–3.4	2	0	0	3
Medians	5.3	5.1	5.1	5.7

metic fundamentals from grade 3.5 to grade 6.4, or 3 years, and for total arithmetic from grade 3.5 to grade 5.9, or 2.4 years. In reading, the range was still greater, from the 3.0 grade level to above the grade 8.0 level.

Trait differences

When the results of general achievement tests in reasoning and fundamentals are broken down into scores on tests of more specific items or traits, the range in the ability of the children in each trait is also very wide. The data in Table 22.2 illustrate this point. The data are for 100 children in grade 6.1 with IQ's from 90 to 110, selected at random from the whole country. Raw scores and grade scores are given for seven traits and for general ability in

TABLE 22.2

Variability in Scores on Sections of the California Arithmetic Achievement Test of 100 Children in Grade 6.1 with IQ's Ranging from 90 to 110

	Raw Scores			*Grade Scores*		
	LOWEST	HIGHEST	MEAN	LOWEST	HIGHEST	MEAN
I. Reasoning						
A. Meanings	2	13	8.6	2.7	8.0+	5.7
B. Signs and Symbols	5	14	11.8	3.4	7.4	6.2
C. Problems	1	12	7.0	2.6	8.0+	5.4
Totals on A, B, C	16	35	27.4	3.9	7.7	5.9
II. Fundamentals						
D. Addition	4	15	10.0	3.6	8.4	6.0
E. Subtraction	1	15	9.6	2.4	8.0+	5.9
F. Multiplication	3	15	7.5	4.5	8.0+	5.8
G. Division	3	15	8.7	4.4	7.9	6.1
Totals on D, E, F, G	21	52	35.7	4.8	7.4	6.0

arithmetic reasoning and fundamentals. For example, the range on test A is from grade 2.7 to grade 8.0+, on problems from grade 2.6 to grade 8.0+, and in subtraction from grade 2.4 to grade 8.0+. The ranges on tests of single traits are wider than for general abilities included under totals.

Providing for individual differences

There is no one plan of classroom organization that solves the problem of individual differences in teaching mathematics. What the teacher does and the procedures that are used determine the success of any plan of organization for the teaching of elementary mathematics. Six of the most widely used plans are the following[2]:

1. Ability grouping: A plan whereby the children of a given grade are divided into two or more sections and then placed in different classrooms on the basis of mental age and IQ.[3]

2. Completely individualized instruction: A plan whereby each pupil proceeds at his own rate from one type to another. This plan is best known as the Winnetka (Illinois) Plan, where the program is being effectively applied to the learning of the number operations. This part of the program is paralleled by the study as a class of applications of mathematics.

3. A combination of the whole class plan and the small achievement grouping: This plan preserves the uniform forward movement of the class as a whole. The successive topics are introduced to the whole class at the same time, but intraclass groups are used to differentiate the methods, materials, and aspects of the topics that are presented. Supplementary materials are used for practice and to provide enrichment activities. The whole class stays with a topic until all of the children can undertake a new topic.

4. Splitting a class into two achievement groups for instructional purposes: The basis of the division is general performance on standard tests. Ordinarily the better half of the class is in one section, the lower half in the other. Teachers believe that this plan makes it possible for instruction to be better adapted to individual differences.

5. Division of the class into three groups early in the year: Under this plan the three groups progress at different rates. The groups are divided early in the year on the basis of general mathematics achievement and instructional needs. Each such group starts with the topics and level of difficulty at which the teacher believes it can experience reasonable success. The teacher must plan separate programs for each group in accordance with readiness and the capacity to move forward.

6 Helping individual pupils in a class as they move through a uniformly paced program: The chief problem the teacher faces is to provide for individual differences by making special efforts to give individual help to slow learners and to provide some enrichment for the fast learners.[4]

[2]Frances Flournoy and Henry J. Otto, "Types of Class Organization for Meeting Individual Differences," *Meeting Individual Differences in Arithmetic*, Publication No 11 of the Bureau of Laboratory Schools (Austin, Tex.. The University of Texas, 1959), pp. 12–19.

[3]Grant C. Pinney, "Grouping of Arithmetic Ability—An Experiment in the Teaching of Arithmetic," *The Arithmetic Teacher*, March 1961, 8:120–123. George McMeen, "Differentiating Arithmetic Instruction for Various Levels of Achievement," *The Arithmetic Teacher*, April 1959, 6:113–120.

[4]H. H. Lerch and F. J. Kelly, "A Mathematics Program for Slow Learners at the Junior High School Level," *The Arithmetic Teacher*, March 1966, 13:232–236, Ramon P. Ross, "Diagnosis and Correction of Arithmetic Underachievement," *The Arithmetic Teacher*, January 1961, 10:22–27.

School resources for pupils with special needs

The bulletin *School Resources for Pupils with Special Needs* of the Los Angeles school system analyzes the resources that are provided for this purpose by one of our largest cities. The following are the headings under which the various services are discussed. They show the wide variety of provisions a growing American community has found it necessary to make.

1. Adjustment within schools for pupils with academic or behavior problems
2. Provisions for intellectually gifted pupils
3. Services for pupils needing special help in reading
4. Services for mentally retarded pupils
5. Services for pupils with speech and hearing disorders
6. Services for aphasic pupils (communication disorders)
7. Services for pupils who are socially maladjusted
8. Services for pupils who are emotionally disturbed
9. Services for pupils who are educationally handicapped
10. Services for pupils with home or community centered problems
11. Services for pupils with postural defects
12. Services for visually impaired pupils
13. Services for pupils with auditory impairment
14. Services for physically handicapped pupils
15. Special programs.

Under the last heading are listed a number of educational programs that are being offered in designated schools under state or federal auspices on an experimental basis to meet the constantly expanding needs of individual pupils, such as extended day programs, preschool classes, and Saturday classes.

Sources of difficulty in learning mathematics

Collier, as an outgrowth of an extensive investigation dealing with causes of difficulties in arithmetic, lists the following possible blocks to mathematical understanding and reasoning ability:

1. Emphasis on memorization and drill at the expense of understanding and thinking
2. Not enough concrete experiences in situations to help the learner to develop meaningful concepts
3. Poor understanding on the part of teachers, resulting in poor instruction
4. Fear and dislike of arithmetic. How much the teachers' attitude toward a subject area affects the learners' attitude "we really don't know"
5. Children not physically, mentally, or emotionally ready
6. Lack of understanding of important mathematical terms
7. Pupils see little reason for learning arithmetic
8. Bad reputation of arithmetic.[5]

Collier suggests that more experiences taken from everyday life should be utilized in the classroom. Arithmetic will be a more enjoyable experience and understanding will be facilitated, he believes, if the teacher will take advantage of more and better instructional aids, and he stresses the importance of helping every child experience early and continuing success in mathematics.

LEVELS OF DIAGNOSIS

We shall discuss three levels of diagnosis that may be identified as (1) general diagnosis, (2) analytical or differential diagnosis, and (3) case study procedures used in the study of individual children.

[5]Calhoun H. Collier, "Blocks to Arithmetical Understanding," *The Arithmetic Teacher*, November 1959, 6:262–268.

General diagnosis

By general diagnosis is meant the systematic use of comprehensive survey tests and other types of general evaluation procedures, such as those discussed in Chapter 21. The data thus secured give the teacher and the staff of the school information about the general level of pupil performance on aspects of mathematics that are needed in a well-managed school system. These data together with information about the school history of the learners, their characteristics and behavior, their social background, and similar data taken from available school and social records are of value in surveying conditions that may affect the growth and condition the development of children.

Analytical diagnosis

By analytical diagnosis is meant the use of systematic procedures for identifying specific weaknesses in mathematics and related curriculum areas for the group as a whole or for some particular individual.[6]

Case study procedures

By case study procedures is meant the application of diagnostic techniques that will enable the teacher to study in detail the performance or achievement of an individual pupil who has an evident learning difficulty. These studies are used to determine, as specifically as possible, the nature and seriousness of the difficulty and the underlying causes. As will be shown, a number of case study procedures that have been developed in educational and psychological clinics can readily be adapted and applied by classroom teachers.

[6] C. I. Chase, "Formal Analysis as a Diagnostic Technique in Arithmetic," *Elementary School Journal*, February 1961, 61:282–286.

Place of each level of diagnosis

The three levels of diagnosis can be illustrated by a brief statement of methods that were used to identify and diagnose the nature of a specific mathematics disability in the case of a grade 5 boy.

Level 1: general diagnosis On a general achievement test it was found that Bob's scores were considerably below normal in mathematics computation and reasoning. His scores in reading were high and his IQ was 110, as measured by an individual test. The general diagnosis was that Bob had difficulty in mathematics, but more information was needed to determine the nature of his handicap.

Level 2: analytical diagnosis The results of an analytical test that included separate tests of each of the number operations, understanding of number, and problem solving showed that Bob's scores in addition, subtraction, and multiplication of whole numbers, understandings, and problem solving were satisfactory, but that his score in division of whole numbers was quite low. Thus a specific area of difficulty was identified. The test scores themselves did not indicate definitely what was wrong with his work in division, nor did they reveal Bob's personal reactions to his difficulties. More specific and detailed information was also needed to determine what modifications of instructional procedures would have to be made to deal with Bob's learning difficulty.

Level 3: case study procedures A thorough individual diagnostic study was made of Bob's work in division to determine where corrective work should

begin. First, his knowledge of the basic facts in subtraction and division was tested both orally and in written form. His responses were rather slow. There also were numerous instances of counting, guessing, and omissions of answers, especially for the more difficult division facts. To determine the types of division examples that were causing him difficulty, he was asked to work a set of examples that represented the entire process of division. He was asked to work aloud certain examples in the test that contained errors not easily determined by inspection so that incorrect thought processes would be identified. He was also asked questions to test his understanding of the operation of division, his methods of estimating quotients, his attitude toward mathematics, and similar matters as the situation required. Available school records were also examined to check the results of this study, including his school history, the results of vision and hearing tests, his interests, and his social background. On the basis of this information his specific difficulties were identified and an improvement program was planned.

Using the findings of diagnostic study

On the basis of the findings of a diagnostic study, the teacher must decide what modifications in instructional procedures are necessary. If a case presents complexities more difficult than those indicated in Bob's case, the teacher should refer the case to a specialist, if available. This would be especially true if difficulties in reading seem to be involved.

Types of cases

As has been demonstrated, there is wide variation in mathematical ability in the typical class. When the results for individual children are examined, the following types of cases emerge.

Normal or above progress The achievements of these children are in line with what children of their ability and level of development ordinarily achieve, in some cases considerably above. For these children the regular program is satisfactory, although it is desirable to strengthen and enrich the program to secure even better results, especially for the more able learners.

Simple retardation These children are performing at a level somewhat below what may be expected of them, but there is no apparent disability requiring special treatment. They often lack necessary experience and background, but under careful guidance their work can be considerably improved. Transiency and long periods of illness are often factors causing difficulty.

Specific disability cases These children have specific weaknesses that interfere with successful growth. For example, lack of progress in subtraction of whole numbers may be due to lack of knowledge of many of the basic facts or lack of understanding of the process of regrouping. This deficiency contributes to lack of success in work in division by two-place numbers because of errors in subtraction that lead to incorrect answers. This specific difficulty can be detected by suitable diagnostic procedures and corrected by reteaching the process as may be necessary. Both diagnosis and treatment can usually be undertaken by the classroom teacher.

Complex disability cases These children for a variety of reasons have made little progress in mathematics. Frequently they have acquired a dislike

for the work because of inability to learn.[7] Sometimes because of lack of interest they make no effort to master the facts and skills involved. They may fear the subject and become emotionally upset when working on it. Often they do not understand the work, and for a variety of reasons they have serious deficiencies in underlying skills, such as a reading disability that interferes with ability to solve verbal problems. Often they are normal or above in mental ability. These cases present such serious problems that the services of specialists are necessary to make a diagnosis and to assist the teacher to plan a corrective program. Unless a dependable diagnosis is made, treatment cannot be effective. In cases of extreme mathematics disability, the problem is sometimes complicated by severe reading deficiencies. In such cases treatment on a clinical basis may be necessary.

TECHNIQUES OF DIAGNOSIS

We shall now discuss practical techniques that can be used by the teacher to identify and diagnose learning difficulties at each of these three levels.

General diagnosis

The results of standardized tests are the most efficient means of securing a measure of a pupil's general level of achievement. To be of greatest value the tests should be administered early in the school year.

Where standardized achievement tests are not available, teachers should administer early in the year informal inventory tests of the work done in previous grades, such as are available in some mathematics textbooks, or tests that they themselves prepare. The inventory test may be so constructed that a quick analysis of the total scores and of those for each section of the test will not only afford a fairly satisfactory measure of the individual pupil's general level of ability but also indicate to both teacher and pupil the processes in which the pupil is strong and those in which carefully planned review, even reteaching, may be necessary.

Graded series of progress tests in processes and problem solving that can be administered at regular intervals during the year are also available in some textbooks, usually at the ends of chapters, or are published in separate pamphlets. Such tests serve as an excellent motivating device.

Analytical diagnosis

Perhaps the first step in making a diagnosis of difficulty in some area of work with whole numbers in which a deficiency appears to exist is to determine how well the children know the basic number facts. Two easily applied procedures may be used.

Answer–strip method The teacher should first prepare a list of 25 of the basic number facts on some operation, for example, addition. The facts should be arranged at random. Each child should then receive a blank slip of paper, number 25 lines on the paper, and then write the answers as the teacher dictates the facts. The rate of dictation should be adjusted to the maturity level of the children. Three or four seconds per fact is satisfactory for grade 4. The children should be told to leave a space blank if they do not recall an answer in the time allowed. This procedure reduces the likelihood of counting and roundabout procedures in arriving at answers. The teacher can dictate the

[7] See Collier, "Blocks to Arithmetical Understanding."

answers while the children score the papers. A copy of the test paper with answers should then be distributed. Each child thus discovers the facts on which he should do special work. The teacher should examine the papers of children with a number of errors to determine the extent to which guessing has been applied.

Controlled dictation method The teacher should prepare a set of number facts to be tested. The groupings should then be duplicated on slips of paper. Each child should receive a copy. The teacher should then read the facts on the test paper one at a time at the rate of a fact every 4 seconds. The pupils should be required to write the answers at the rate at which the facts are read. Omissions will indicate that the answers are not known or cannot be recalled within the time allowed. Incorrect answers often indicate guessing. The teacher can dictate the answers as the pupils check their papers. The test paper thus becomes the pupil's record of the facts not answered correctly or for which answers were omitted.

Factors in evaluating ability in mathematics

In estimating ability in mathematics, the teacher should consider at least six characteristics of performance: (1) rate of response, (2) accuracy, (3) altitude or level of development, (4) quality of work, (5) area of experience or range of ability, and (6) methods of thinking and performance.

Rate of response The pupil's rate of work is a valuable index of skill and control of a particular function, such as knowledge of addition facts. A slow rate of response is often symptomatic of learning difficulty. The rate at which a pupil can give answers to number facts can be measured by testing how many answers the pupil can write in a given time, such as a minute.

Accuracy The greater the proportion or the number of correct answers, the higher is the level of performance. Thiele has suggested the following standards of accuracy for average children in grades 4–6 for the facts in the four processes when only the answers are written on a test paper containing the facts.[8] If a child cannot write all of the answers correctly on a test paper containing 20 addition facts in 1 minute, his performance is not satisfactory.

Grade	*Addition*	*Subtraction*	*Multiplication*	*Division*
4	20	20	20	15
5	25	25	25	20
6	30	30	25	25

Altitude and level of development The altitude of ability is measured by the degree of difficulty of the tasks a pupil can perform successfully. The more difficult the tasks a pupil can perform, the higher is his altitude of ability and his level of development. Most standard achievement tests measure altitude. Levels are expressed in age and grade standards.

Quality The quality of general merit of ability can be appraised by observing the "smoothness" of a child's performance in working a set of examples or by evaluating a concrete product of his work, such as a graph, scrapbook, report, written work, and the like. Stand-

[8]Exact Science Department, Board of Education, Detroit, Michigan, File No. 5499, p. 4.

ards for evaluating materials of this kind should take into consideration such items as:

1. The authenticity of the facts or representations
2. The arrangement and organization of materials
3. The care and neatness of materials
4. Evidences of originality and resourcefulness
5. Richness and variety of content.

Learners should participate in setting up standards for evaluating quality of work.

Area of experience or range of ability

In appraising the area or range of ability, it is necessary to determine the breadth of an individual's learning at each general level of difficulty and the extent to which he has mastered all of the essential abilities in some basic process such as division by two-place numbers. A general test in mathematics does not supply adequate information on the basis of which to evaluate the many specific mathematics abilities; their status must be determined by measurements of each specific ability. At the same time, consideration should be given to such outcomes of learning and instruction as interests, attitudes, appreciations, and insights related to mathematics.

Methods of work In evaluating any performance, an important factor is the merit of the methods of work and the efficiency of the thought processes the learner employs. A pupil's achievement is often lower than it should be because of the inefficiency of his work habits and methods of study. The existence of faulty methods of work are to be suspected when a pupil's performance is slow and clearly inferior to what can be expected of a pupil of his mental ability. Faulty methods of work can often be discovered by observing his behavior while he is performing some task, such as taking a test. For example, a pupil may be able to write the correct answers on a test of the addition facts, but observation may show that he uses a variety of methods of counting to find the sums, such as counting with his fingers, tongue, feet, pencil, and so on. An interview with the learner may reveal that he does not have an effective, systematic plan of studying the number facts he does not know.

Standardized analytical tests

When the purpose of diagnosis is to determine with greater exactness the specific phases or element of some process in which a weakness or deficiency exists, diagnostic tests of an analytical type should be used. For example, to locate weak spots in division of whole numbers by the standard algorism method, the following elements derived from an analysis of the steps in working the example shown in (4) should be tested:

1. Knowledge of the even and uneven division facts, as $2\overline{)19}$.
2. Ability to divide by one-place numbers as an indication of knowledge of the steps in the division process itself.
3. Ability to estimate quotient figures correctly, first those types in which the estimated quotient is the true quotient, as in $21\overline{)193}$; then those in which the estimated quotient must be corrected, as in $27\overline{)195}$.
4. Ability to multiply, as in finding 7×27 in the example.
5. Ability to subtract to find the remainder, if any.

$$\begin{array}{r} 7 \\ 27\overline{)195} \\ 189 \\ \hline 6 \end{array}$$

A comprehensive series of analytical diagnostic tests in operations with

whole numbers, fractions, decimals, and per cents is available.[9] The series includes the following tests:

1. Tests of the basic facts (5 tests)
2. Tests in the four operations with whole numbers (5 tests)
3. Tests in common fractions (7 tests)
4. Tests in decimal fractions (4 tests)
5. Test in per cent (6 sections)
6. Test in operations with measures (5 sections)

Each diagnostic test has cross-references to self-helps, which appear on the back page of each test, to be used for any necessary corrective work. This combination of diagnostic tests and self-helps provides the essential elements of an effective improvement program in mathematical operations. The fact that both the means of diagnosing a specific deficiency and the kind of corrective measure to apply to that deficiency are given make these materials valuable for the classroom teacher.

Developmental diagnostic tests

Analytical diagnostic tests that are closely integrated in the developmental program should appear at the end of each new unit of subject matter. These appear in some modern mathematics textbooks. Such tests should be administered at regular intervals in order that weak spots in the new work may be diagnosed and promptly corrected. The steps in developing such diagnostic tests are as follows:

1. There should be a breakdown of a major unit, such as addition of fractions, into a series of subunits. For example, a complete series of four developmental diagnostic tests in that process should be based on the following subunits:

Subunit I. Addition of fractions having like denominators—no regrouping in the sum, as in $\frac{1}{3} + \frac{1}{3}$ of $3\frac{1}{4} + 2\frac{1}{4}$

Subunit II. Same as in (I), with regrouping in the sum, as in $\frac{3}{4} + \frac{3}{4}$ or $4\frac{1}{3} + 2\frac{2}{3}$

Subunit III. Addition of fractions having unlike but related denominators, as in $\frac{1}{4} + \frac{1}{2}$ or $3\frac{1}{2} + 4\frac{7}{8}$

Subunit IV. Addition of fractions having unlike and unrelated denominators, as in $\frac{1}{4} + \frac{2}{3}$ or $2\frac{5}{6} + 3\frac{3}{4}$

2. Next should be listed a series of the specific types of examples in order of increasing complexity for each subunit. The step-by-step development of types for subunit (I) are as follows:

a.	$\frac{1}{3} + \frac{1}{3}$	Sum not renamed or regrouped
b	$\frac{1}{4} + \frac{1}{4}$	Sum renamed or regrouped
c.	$4\frac{1}{8} + 2\frac{2}{8}$	Mixed numbers involving (a)
d.	$3\frac{1}{4} + 3\frac{1}{4}$	Mixed numbers involving (b)
e.	$4\frac{1}{2} + 2$	Addition of a whole number and a mixed number

3. In a developmental diagnostic test there should be no less than three examples of each type to insure a reliable diagnosis. A suitable diagnostic test for subunit (I) is given below.

Diagnostic Test in Addition of Fractional Numbers Having Like Denominators

	A	B	C
1.	$\frac{1}{4}$ $\frac{2}{4}$	$\frac{3}{8}$ $\frac{4}{8}$	$\frac{2}{5}$ $\frac{2}{5}$
2.	$\frac{1}{8}$ $\frac{3}{8}$	$\frac{1}{4}$ $\frac{1}{4}$	$\frac{3}{6}$ $\frac{1}{6}$

[9] A series of specific diagnostic tests developed by L. J. Brueckner is published by the California Test Bureau, Los Angeles, California (1955). Sample sets of the tests can be secured from the publisher at a nominal cost.

3.	$\begin{array}{r} 4\frac{2}{4} \\ \underline{2\frac{1}{4}} \end{array}$	$\begin{array}{r} 5\frac{3}{8} \\ \underline{4\frac{2}{8}} \end{array}$	$\begin{array}{r} 6\frac{1}{3} \\ \underline{7\frac{1}{3}} \end{array}$	4. $\begin{array}{r} 2\frac{3}{8} \\ \underline{4\frac{1}{8}} \end{array}$	$\begin{array}{r} 3\frac{1}{4} \\ \underline{4\frac{1}{4}} \end{array}$	$\begin{array}{r} 5\frac{1}{6} \\ \underline{4\frac{1}{6}} \end{array}$
5.	$\begin{array}{r} 3\frac{1}{2} \\ \underline{2} \end{array}$	$\begin{array}{r} 1 \\ \underline{1\frac{1}{4}} \end{array}$	$\begin{array}{r} 2 \\ \underline{\frac{2}{3}} \end{array}$			

4. Developmental diagnostic tests should be keyed to suitable learning aids in a textbook or workbook and to special helps on difficult spots. The teacher may wish to prepare more detailed supplementary materials.

Similar developmental diagnostic tests for all operations with whole numbers, fractions, and decimals are invaluable aids in teaching, particularly because they assist the pupil as well as the teacher to determine the progress that is being made and the points where further study may be necessary. When these tests are not included in the textbook in use, the teacher can quite easily develop them by following the plan described above.

An analysis of the results

A developmental diagnostic test will enable the teacher to locate the sources of pupil difficulty in a number operation. A more penetrating approach to diagnosis is needed to discover the nature and causes of the learning difficulty. Answers to such questions as the following ought to be sought by the teacher:

1. How adequate is the pupil's grasp of meaning and understanding of the basic procedures?
2. What are his attitudes toward mathematics? How aggressive is his attack on the learning of mathematics?
3. At what level of thinking is the pupil attempting to operate? Is the level of his maturity of operation appropriate to his mental ability, power of concentration, and depth of perception?
4. To what extent does the pupil regard the computational procedures as purely mechanical skills to be mastered? Is he able to apply them effectively in quantitative thinking?
5. Is he dependent on rigid step-by-step procedures with paper and pencil in making computations? Or is he able to perform some operations mentally and also to use short-cut procedures when the numbers have properties that suggest them?
6. What are the limits that the pupil can be expected to reach in computational skills? What standards of rate and accuracy should be set up for him as goals to be achieved?

The diagnostic testing techniques described in the preceding sections will help teacher and pupil to locate specific areas of weakness. More penetrating procedures must be used by the teacher to determine the exact nature of a temporary difficulty or one of long standing, and if possible, the causes of unsatisfactory performance so that the proper corrective treatment may be applied.

Illustrations of faulty methods of work

Because they do not know the basic number facts or because they do not understand the operations involved in working examples, children often devise roundabout, inefficient methods that are uneconomical and wasteful. These methods may work with small numbers but they are too difficult to use with larger numbers. Several illustrations of faulty methods of work follow:

$$\begin{array}{r} 81 \\ \underline{-37} \end{array}$$

Case A A grade 4 boy reported as failing in mathematics found the answer to the subtraction example at the right by counting back from 81 to

37 by ones. He had devised a way of keeping a mental record as he counted. He used this method in all cases when regrouping was required. When asked to subtract two larger three-place numbers, he balked and refused to attempt to find the answer.

Case B A grade 4 girl persistently worked subtraction examples in which regrouping was necessary by subtracting the lesser number of ones named in the sum from the greater number of ones named in the known addend. This seemed to her to be a procedure logically correct.

$$\begin{array}{r} 72 \\ -26 \\ \hline 54 \end{array}$$

Case C A grade 6 boy added before multiplying. Thus he found $8 \times 4 = 32$; he named 2 in ones place and added the 3 tens to the 6 tens; then he found $8 \times 9 = 72$. This is a case of transfer from regrouping in addition.

$$\begin{array}{r} 64 \\ \times 8 \\ \hline 722 \end{array}$$

Case D This pupil does not understand the role of 0 as a place holder, as is shown in the incorrect work in the division example.

$$\begin{array}{r} 6\ 7\,\text{r}5 \\ 6\overline{)3647} \\ 36 \\ \hline 47 \\ 42 \\ \hline 5 \end{array}$$

Case E A grade 4 boy, IQ 118, 1.4 years below grade in mathematics, had devised many ingenious but awkward methods of finding answers to problems, as illustrated by the following:

He was asked to read the problem below and to give the answer:

If 2 loaves of bread cost 42 cents, how much does 1 loaf cost? Quick as a flash he said, "21 cents." When asked, "How did you find the answer?" he replied, "Well, 20 and 20 are 40, and 1 and 1 are 2, so 21."

The example $2\overline{)42}$ was written on the chalkboard, but he could not work it.

Other similar cases could be described, but these five suggest the nature of faulty work in mathematics that can be diagnosed by suitable methods. The first four are specific disability cases, while the fifth is a complex disability case who had difficulties with all phases of arithmetic operations.[10]

Significance of variable and persistent faulty responses

When pupil responses are random, erratic, and inconsistent, the teacher should realize that a lack of basic understanding and often indifference are indicated. The need of reteaching is apparent. When there is evidence of consistency in faulty reactions, there is evidence that learning actually has occurred but that steps must be taken to establish basic understandings and to correct thinking patterns.

The underlying sources of difficulty undoubtedly are lack of comprehension of the ideas that are the basis of understanding and failure to master number facts and computational skills and techniques.

Illustrations of case study procedures Case study procedures are clinical in nature and are most suitably applied in the study of the work of individual pupils or groups of pupils who have difficulty in the same area. However, procedures such as those listed above may also be applied informally by the teacher whenever it becomes necessary in the course of regular instruction to identify and remedy a difficulty that may prevent satisfactory

[10]Similar lists of the most common faults in operations with fractions and decimals may be found in L. J. Brueckner and G. L. Bond, *Diagnosis and Treatment of Learning Difficulties* (New York: Appleton-Century-Crofts, 1955), Chap. 8.

mastery of some particular step in the development of an instructional unit. Diagnostic procedures should always be applied when work in drill exercises is inaccurate or slow or when little progress is being made.

Illustrations of the methods of applying to mathematics diagnosis the various techniques that can be used to determine shortcomings of various kinds are listed below. First are given the types of informal procedures that any teacher can use. At the end of the list are mentioned standardized tests especially constructed for clinical purposes whose administration requires special training.

1. Analysis of written work to discover faulty responses, such as:

a. Numerals written incorrectly, as reversal in the primary grades

b. Types of examples worked incorrectly

c. Nature of computational errors made in tests and in regular daily written work; zero difficulties

2. Analysis of oral statements

a. Faulty thought processes are revealed by having the pupil "state aloud" his steps in working difficult examples or problems

b. Reading difficulties are revealed when the pupil reads the problem aloud

c. Having the pupil tell how he would solve a problem reveals faulty thinking

3. Personal interview to secure information by asking the pupil

a. His thought processes in working an example

b. To test his understanding of a number operation

c. About methods of solving a problem

d. About interests, attitudes, and methods of work

4. Questionnaires and inquiry blanks

a. Securing interest ratings of topics in mathematics

b. Reports from classmates, parents, and teachers

c. Study habits and methods of work

5. Observation in the course of daily work

a. Evidence of the use of counting and other inefficient methods of work

b. Rate of work

c. Study habits; use of reference books

d. Factors affecting performance, such as health, vision

e. Methods of using some measuring device

6. Analysis of available records

a. Anecdotal records

b. School cumulative records

7. Administration of diagnostic tests given in textbooks or workbooks or prepared by the teacher.

Steps to be followed in case studies

The steps to be followed in making case studies are as follows:

1. Administer an informal survey test containing a graded series of examples in the operations being studied to locate areas of deficiency and to determine the pupil's level of development.

2. Administer a properly constructed analytical diagnostic test in each operation in which the screening tests indicate a real deficiency.

3. When the work in these tests reveals weak spots, apply the following case study procedures to discover the underlying difficulties:

a. Examine the written work in the test to determine faults, errors, incorrect procedures, poor form, etc.

b. Have the pupil work the incorrect examples again on another paper to see if the fault persists. Observe also his methods of work, his attitudes, and symptomatic behavior.

c. In case of doubt as to the thought processes used, have the pupil do the work aloud and observe his thought processes. Record illustrations of his procedures.

d. In case of doubt, ask the pupil questions to get at subtle, hidden difficulties that he may not be able to express orally, also to test his understanding of a step.

e. If you identify an apparent weakness in an underlying operation, for example, in the subtraction involved in division, administer a diagnostic test in subtraction to see how serious the difficulty is.[11]

4. Repeat the above steps for any or all operations in which the pupil has difficulty.

THE TREATMENT OF LEARNING DIFFICULTIES

Dealing with learning difficulties

There usually is no single cause or condition that creates a learning difficulty. The teacher's understanding of the case will be greatly broadened, however, if information is secured about possible physical and sensory defects, mental ability, personal and social adjustment, interests, and motivation. Special consideration should also be given to environmental and instructional factors that may contribute to the difficulty. Transiency of population is an important social factor affecting the organization of instruction.

The causes of learning difficulties are legion. The teacher who would correct a learning difficulty in mathematics should take the necessary steps to determine what is wrong with the learning and what adjustments in instruction should be made rather than waste time looking for explanations in the child's personal and educational history.

Because of lack of space it is not possible to give here a detailed description of corrective measures the teacher can apply. A group of general principles may be presented, however, that may be regarded as basic in the management of an improvement program.

1. Treatment should be based on a diagnosis and should be individualized.

2. Secure the cooperation of the learner so that he will be likely to attack his problems aggressively and willingly. Explain to him the nature of his difficulty and its significance. Describe also the steps to be taken to bring about an improvement.

3. Attack specific deficiencies of the learner directly. Begin reteaching at the point where there is likely to be success in the corrective work from the start so that the learner will take satisfaction in the progress he makes. Give special attention to the treatment of reading disabilities.

4. Take steps to correct any physical, emotional, and environmental factors that are likely to interfere with progress.

5. Proceed on a tentative basis in the correction of weaknesses and do not hesitate to modify the steps taken when progress is slow and uncertain. Make extensive use of exploratory materials and visual aids to make the work meaningful to the learner, especially those of low mental ability.

6. Select instructional procedures and materials that are of demonstrated value in making the operations meaningful for the learner.

7. Integrate the corrective and developmental program so that the learner

[11] Brueckner and Bond, pp. 223–225.

will feel that he is not isolated and that he still is a member of the group.

8. Take steps to assure the growth of all aspects of the learner's personality. Do not focus on the correction of deficiencies to such an extent that positive values such as interests, attitudes, and appreciations are neglected.

EXERCISES

1. What is a standard achievement test? What standard tests in arithmetic are administered in local schools? How do local classes compare with standards?
2. What types of tests are included in the mathematics textbooks that are used locally? in other areas?
3. Try to secure the test results of some class and analyze the results to determine the range of individual differences in achievement. How do the results compare with those given in Table 22.1?
4. Compare the performance of several children on achievement tests in several curriculum areas to find how their scores vary in the different traits tested. Are the profiles similar? Discuss some of the more interesting profiles.
5. Why is the range in test scores on the grade 5 level likely to be greater for reading than for mathematics?
6. What is meant by general diagnosis? analytical diagnosis? case study procedures?
7. If possible, illustrate each of the four types of cases that are described on page 406. What conditions do you think may lead to what are defined as complex disability cases? Do you know of such a case? If so, describe the child's behavior and his difficulties.
8. Apply the two methods of testing knowledge of basic number facts that are described on page 407. Score the papers and report the findings to the class. Which method do you prefer?
9. Show why the six characteristics of a performance given on page 408 should be considered in evaluating the work of a pupil. How can each characteristic be evaluated?
10. What is a developmental diagnostic test? How should it be constructed?
11. Examine available textbooks to see if they contain diagnostic tests of the developmental type discussed in this chapter.
12. What diagnostic methods may be used by the teacher to determine the kinds of difficulties an individual pupil experiences in number operations?
13. How can the teacher find time for the diagnosis and treatment of learning difficulties in mathematics?
14. Explain the difference between evaluation and diagnosis. What is the role of standard tests in each?

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MATHEMATICS FOR SLOW LEARNERS

Children who rank in the lowest quarter of the school population in mental ability and achievement are steadily receiving more attention from educators, psychologists, and social agencies. Many of these pupils are low achievers in mathematics. In recent years large sums of money have been spent by government agencies and private foundations to study methods of helping low achievers not only to do better in school but to make a satisfactory social adjustment. Many pupils are problem learners because of their limited rate and level of development. Such pupils often present adjustment problems because of the innumerable failures and frustrating experiences they have encountered while growing up, particularly unsatisfying educational situations in the school. They also frequently become behavior problems because the majority of them are raised in slum areas where certain kinds of deviate behavior are often condoned.

The composition of this group of slow learners is heterogeneous both educationally and intellectually. They can be divided into three broad and rather distinct groups, namely, the mentally de-

ficient, the mentally handicapped, and the slow learners. The first two groups constitute approximately 2–5 per cent of the total school population. In most communities special provisions are made for these children. Many of them are potentially educable to a limited degree and are capable of working in unskilled and semiskilled jobs. The remainder of the group, about 14–20 per cent, are the slow learners for whom the schools must develop more appropriate educational programs and experiences, particularly in mathematics. Statistics indicate that a large number of school dropouts are low achievers in mathematics and reading.

The United States today needs the potential man power of all students, including the low achiever in mathematics. Low achievers, however, will not qualify for very many occupations unless they learn more mathematics than they are learning at the present time. It is believed that through a carefully planned program the mathematical level of many slow learners can be developed to the extent necessary for a salable skill. The low achiever should obviously receive the mathematics instruction necessary for a rich, cultural citizenship.

This chapter discusses the following topics: characteristics of slow learners; organizing the educational program for slow learners; curriculum adjustments for slow learners; instructional adjustments for slow learners; adjusting instructional materials for slow learners; the treatment of learning deficiencies.

CHARACTERISTICS OF SLOW LEARNERS

The general characteristics of slow learners in mathematics may be listed as follows:

1. Their IQ's range from approximately 75 to 90.
2. Their rate of growth in mathematics is considerably below that of the other children.
3. They have a low functional reading ability.
4. They cannot remember basic mathematical concepts and principles.
5. They can make only simple generalizations about mathematical relationships.
6. They find it difficult to solve verbal problems.
7. Often they lack interest in mathematics and develop blocks, tension, and faulty attitudes that make them ineffective learners.
8. Often they come from culturally and socially disadvantaged homes.

In the average community where the school serves children from all social and economic levels, a class of 30 unselected children can be expected to contain from 3 to 5 slow learners. In subcultural areas of large cities, however, the picture is quite different. As many as half of the children in these schools, where the mean IQ is 80–85, can be designated as slow learners.

The slow learner begins at a slow rate the first year of school and continues to fall farther and farther behind as he grows older. Deviate and antisocial behavior are often characteristic of the low achiever. Such a student often drops out of school long before graduation, thus severely limiting his mathematical background. A recent study indicates that 46 per cent of school dropouts have IQ's of less than 90, while only 21 per cent of high school graduates have IQ's below this level.[1]

[1] H. L. Voss, A. Wendling, and D. S. Elliott, "Some Types of High School Dropouts," *Journal of Educational Research*, April 1966, 59:363–368.

ORGANIZING THE EDUCATIONAL PROGRAM FOR SLOW LEARNERS

Slow learners are often retarded because their physical and social environment was deficient in their formative years. This problem has been recognized in many localities in recent years, and prekindergarten classes, for example, Operation Head Start, have been established for disadvantaged children. In such programs a play environment is established where the children can be active, in contact with children of all levels of ability, and can use their imaginations to the fullest extent. Preschool children learn much about the activities of the community and about living things, nature, and mechanical action. Such group and individual experiences help them to develop intellectually as well as socially. At the present time scientific data are lacking to show the value of such programs.

Traditionally children enter the kindergarten when they are 4 and 5. Normal children remain in the primary school for four years before being advanced to the next higher level. Slow learners usually enter the primary school at the same time as normal children, but since they are intellectually less prepared for school, they may require up to five years to complete the program. Children of superior ability often complete the program in two or three years. The normal child completes grade 6 in six years. When a child enters the ungraded primary school, he is not enrolled in the kindergarten or in a grade but is simply registered as a new beginning pupil. Grades and grade concepts are abolished. A student is assigned to a group of children of his apparent level of development. Regrouping may occur whenever the children in a particular group show such a disparity of development that they no longer derive equal benefit from the instruction provided that group. Regrouping may be intraclass and group classes. Grouping in mathematics depends on the level and rate of development of the pupil's mathematical understanding and skills. Sometimes special help is needed to remedy weaknesses revealed by diagnosis. The length of time a child should remain in the primary program depends upon his growth. New skills, knowledge, and concepts should be introduced at the rate of the child's development. Eventually slow learners in the various groups tend to be grouped together.

The work in the intermediate school should be organized on the same basis.[2] This is now being done in a number of areas through the introduction of the ungraded elementary school. Grouping must be flexible, and regrouping of an individual may occur whenever it seems advisable. The pupil should be placed with a group on the basis of his understanding of the content and level of instruction for that particular group.

CURRICULUM ADJUSTMENTS FOR SLOW LEARNERS

Mathematical concepts and skills are constantly used in everyday life by both children and adults. Without a knowledge of basic mathematics the individual would not be able to maintain himself in today's complex industrial society. Many of these necessary concepts and skills are well within the learning ability of slow learners and should be included in the mathematics program that is provided for them.

[2]Bruce E. Meserve, "The Teaching of Remedial Mathematics," *The Mathematics Teacher*, May 1966, 59:437–443.

The curriculum should be arranged to provide for continuity of child development with a minimum of strain and tension, and it should be organized so that there is a reasonable likelihood that successful learning will take place. The available evidence as to the social utility and learning difficulty of number operations should be carefully considered in the selection and gradation of subject matter.

The mathematics program should include a well-integrated treatment of the mathematical and social phases of the subject and should deal with topics and processes of undoubted social value to the average individual. Such content is within the experience of slow learners. The more difficult operations with fractions, decimals, and per cents such as are required in technical work should be deferred to levels beyond the elementary school. Here the need for mastering such operations in vocational and prevocational courses will motivate most interested slow learners to make a special effort.

Mathematics should be taught in close association with other school work in which the use of quantitative procedures serves to clarify the situation and to make it meaningful for the learner.

Systematic provision should be made to adapt the curriculum to differences in the needs, abilities, and interests of the learners as well as to differences in the rates at which they learn. The component skills in mathematics can be arranged in the order of their learning difficulty and complexity. This method may then serve as a guide for the teacher in adjusting the work to the ability of the learners. In the ungraded elementary school, the program is organized at all levels so that each student can progress at his own rate. The teacher must make frequent use of diagnostic procedures to discover a pupil's needs and to determine whether or not he is making optimum progress. In a number of schools there are parallel track plans in mathematics consisting of programs ranging from a minimum program suited for the slowest learner to enriched programs for the more able. Provision is made for adjusting the time required to the rate at which the children can master the content. In secondary schools such differentiated programs in mathematics are very common.

Problem solving should be taught concomitantly with number operations. Poor achievers in problem solving are significantly lower than high achievers in general mental ability, reading ability, and in skill in number operations. The teacher should recognize these weaknesses and should make adjustments in the work where problem solving deals with measurements and other everyday applications of number operations. The reading skills that are peculiar to problem solving should be stressed.

It is known that slow learners can master many of the simpler elements of algebra and geometry. At the present time very little information is available concerning the learning difficulty for slow learners of most topics in algebra and geometry in the elementary school. Much experimental research is necessary to determine suitable content for low achievers in these two areas of mathematics. In general, topics in algebra and geometry should be taught when the need arises in classroom activities and when the teacher believes that the children will profit from the experience. The slow learner should be helped to see how the use of mathematical sentences clarifies word problems.

It has been shown that the ability of many children to deal with mathematical concepts in other fields of study is at a low level. The schools must therefore organize the mathematics program so that it deals more effectively with the mathematics aspects of all curriculum areas, particularly science and social studies.

INSTRUCTIONAL ADJUSTMENTS FOR SLOW LEARNERS

The methods by which slow learners master the concepts and skills of mathematics are not unique or strikingly different from those used by children with greater learning ability. Slow learners, however, cannot learn skills as rapidly as children of higher levels of ability, and the instructional procedures used with this group must make much more extensive use of concrete, socially significant experiences and materials than those used with normal children.

The simplification of the curriculum is undoubtedly the most fundamental step that can be taken. Slow learners do not have the mental capacity to master the more difficult topics in division of whole numbers and operations with fractions, for example, addition and subtraction of fractions having unlike and unrelated denominators. If these topics are to be taught in the intermediate grades to the slow learners, they should be presented for informational purposes only. Mastery cannot be expected. If subsequently the individual's vocational choice requires that he learn how to perform these computations, he will at least be aware that these operations are possible.

The following are some of the most useful methods of adjusting instruction to slow learners:

1. Make greater use of concrete social situations to give meaning to operations and to enrich the experiential background of the children.
2. Provide extensive opportunity for the children to work with concrete materials in the development of number facts and in the demonstration of the meaning of a process or topic.
3. Use a wide variety of visual aids such as pictures, diagrams, and similar materials to enable the learner to visualize the situation involved and to grasp the meaning of the steps to be taken in a new operation.
4. Encourage slow children to invent procedures that may be meaningful to them.
5. Be sure that there is a well-graded development of new work so that only one new difficulty is introduced at a time. Give the slow learner the opportunity to generalize.
6. Spread the presentation of a new process or topic over a longer interval of time than normal children require.
7. Allow more time for practice exercises to develop skill and vary them through the use of games and a variety of applications of skills in social situations so as to avoid monotony. More frequent reviews are required.
8. Delay the introduction of new topics until it is clear that the pupil has acquired the underlying skills and concepts essential to their mastery. A readiness program is important.
9. If possible, assign to slow pupils only those activities and problems in textbooks that are not likely to frustrate them.
10. Do not expect all pupils in this group to achieve the same standard. Differentiated standards of achievement may be necessary.
11. Check frequently by observation the work habits of the children and by questioning uncover evidences of diffi-

culty, faulty methods of work, and lack of comprehension.

12. Give diagnostic tests systematically to locate weak areas at an early stage of learning. Reteach in a simpler way as may be necessary.

13. Prevent the practicing of errors and faulty procedures by insisting on the understanding and mastery of each step before new work is presented.

14. Give considerable guidance in directed reading activities to develop the reading skills connected with the use of the textbook and supplementary materials.

15. Make use of experience units in which there is a wide variety of individual activities. See to it that the slow learners select and carry out assignments that are within their interests, and at which they are likely to be successful.

In short, the rate of progress should be adjusted so that each learner will work comfortably, successfully, and with a minimum of tension.[3]

ADJUSTING INSTRUCTIONAL MATERIALS FOR SLOW LEARNERS

In the traditional graded elementary school children were usually grouped on an age basis. They were expected to proceed at a uniform rate and to learn the same content. If they were unsuccessful, they were required to repeat the work of the grade, several times if necessary, until they had mastered the content.

Many serious problems arose because all children of a given grade were required to use the same textbook regardless of its adequacy, the level for which it was intended, and its relationship to pupil needs and interests. The results of modern research on learning indicate that such an approach is absurd and futile, since it ignores the fact that children differ widely in mental ability, in the progress they make in the various content areas of the curriculum, and even in mastery of the elements of a single curriculum area such as mathematics.

As a result of recent research, new materials of instruction have been developed that to some extent should help the teacher to adapt instruction to the needs of individual pupils. Instead of a single textbook for a given grade or level of achievement, books of two or three levels of difficulty have been prepared. One authority believes that there should be a three-level series, a book of average difficulty for the group of children of average ability, one more challenging for the top group, and a much simpler book for the slow learners. In the book designed for slow learners, according to this view, the development should be more detailed and considerably simplified, and the book should be geared to a lower reading and vocabulary level. There should be many illustrative examples and worked-out problems for pupil reference.

Workbooks are sometimes provided for slow learners that contain detailed explanations, supplementary practice materials, and special helps in the reading skills required in problem solving. The teacher can select from the contents the special kinds of work that will meet the needs of individual pupils.

Many textbooks suggest the use of a variety of supplementary learning aids for slow learners, for example, place-value charts to make clear the sequence of steps in working examples in a number operation with whole numbers and

[3] H. H. Lerch and F. S. Kelly, "A Mathematics Program for Slow Learners at the Junior High Level," *The Arithmetic Teacher*, March 1966, 13:232–236.

cutouts of fractions to aid in understanding the meaning of fractions and how they are manipulated in computations. The more able learners can master the work in mathematics with very little use of concrete manipulative materials.

Supplementary readers, storybooks, pictures, and illustrations of the uses of mathematics in daily life are valuable means of vitalizing the learning experience for low achievers. Excursions to places of business add to the experiential background of the children and make their study of the applications of mathematics more meaningful.

In the kindergarten and grade 1 social experiences should be planned in which the uses of number arise in a natural way. In the kindergarten the children of low ability should have ample opportunity to play with sets of objects and toys, to group them in various ways, to combine and separate sets, and to compare sets. They should use sets of blocks to construct small replicas of buildings in the community. In connection with their activities they should learn about the use of simple measuring devices. They should dramatize simple activities that take place in the community in which numbers play a role. Games should be used to provide various kinds of valuable practice with numbers and number facts. Slow learners should take part in activities of this kind throughout the primary grades. In the course of these activities the teacher will become aware of the differences in the levels of growth of the children and will be able to group them according to their needs.

THE TREATMENT OF LEARNING DEFICIENCIES

In planning a remedial program for dealing with a deficiency in some phase of mathematics, consider the following:

1. Is the condition revealed in fact a disability requiring the attention of one or more specialists? The nature and severity of the disability should be established by suitable diagnostic procedures.
2. Who should deal with the disability – the regular teacher, a special teacher, a social worker, a clinical diagnostician, a psychologist, or a physician?
3. What kind of instructional program is most likely to correct the deficiency?
4. How can this program be managed most satisfactorily? Consideration should be given to location, scheduling, grouping, materials, using available services.
5. What changes in conditions present in the learner are necessary? Is correction of visual or auditory defects, malnutrition, low level of interest in arithmetic, fear, and so on, called for?
6. What changes in the environment – in the home, classroom, community – are necessary?

Plans for organizing remedial instruction

The following are brief descriptions of remedial plans that have proved effective:[4]

1. The regular teacher is responsible for both the diagnosis and the corrective measures required, which can be applied on a group basis, even in the case of individuals whose problem is not complex.
2. A specialist is available to assist teachers in planning ways of diagnosing and treating learning difficulties.
3. Learners seriously retarded in some phase of mathematics are referred to a remedial teacher in the same building for one or two hours a week for spe-

[4]See Lauren Woodby, ed., *The Low Achiever in Mathematics* (Washington, D.C.: U.S. Office of Health, Education and Welfare, 1965).

cial help. They also attend regular classes.

4. Children from several schools are sent to a remedial teacher in some conveniently located center for a part of each day for special help, especially if there appears to be a reading disability that interferes with progress in mathematics.

5. Children may be referred directly to a guidance clinic or center for special help when severe disability exists.

6. Children with extremely serious and complicated learning difficulties associated with achievement, physiological, social, and emotional problems may be sent to a special school for clinical treatment.[5]

7. Vacation classes and special programs may be set up during the summer to give additional help to children who are slow learners. These classes can organize instruction in such a way that provision is made for meeting the needy individual children.

Difficulty of correcting unsatisfactory conditions

It has been repeatedly demonstrated that training and practice ordinarily produce marked changes in specific traits. For example, teaching procedures that stress rate of work will under normal conditions produce a marked improvement in the rate at which a pupil can write or recite the answers to sets of number facts. If a pupil does not respond to instruction, the teacher must locate the difficulty and apply corrective measures.

There are marked differences in the ease with which desirable changes can be made. Some deficiencies, such as mental defects due to heredity, disease, or birth injury, cannot be corrected by any known technique. Hygienic measures can in most cases greatly alleviate physiological weaknesses such as faulty vision and hearing, malnutrition, and glandular disturbances. Many of the minor learning difficulties that a pupil experiences in mathematics, for example, counting, disappear with the passing of time and with growth in control of the basic skills. Some faults, such as failure to learn basic skills at each grade level, are cumulative, and unless they are given adequate attention tend to become more serious as the student progresses. The redirection of interests, attitudes, and emotional and social adjustment is often extremely difficult because of the inability of the school to control unfavorable conditions in the community. The correction of these faults is usually an individual problem and should be approached from this point of view. The important thing is that all agencies of the community that are concerned with the development of children should cooperatively attack the problem growing out of unfavorable factors.[6] It is known that faulty attitudes toward learning are the result of disintegrating influences in the home and elsewhere.

Unfortunately, little research is available in the area of mathematics that

[5]See E. M. Bower, *Early Identification of Emotionally Handicapped Children* (Springfield, Ill.: Charles C Thomas, Publisher, 1960), Chap. 6; G. T. Donahue and Sol Nichtern, *Teaching the Troubled Child* (New York: Crowell-Collier and Macmillan, Inc., 1965); H. S. Fremont, "Some Thoughts on Teaching Mathematics to Disadvantaged Groups," *The Arithmetic Teacher*, May 1964, 11:319–322.

[6]See Daniel Schreiber, ed., *The School Dropout* (Washington, D.C.: National Education Association, 1964); David Montague, "Arithmetic Concepts of Kindergarten Children from Contrasting Socioeconomic Areas," *Elementary School Journal*, April 1964, 64:393–397; Solomon O. Lichter, *The Dropouts* (New York: Crowell-Collier and Macmillan, Inc., 1962.

deals with the relationship between social and emotional factors and progress in arithmetic.[7] Because of the close connection between reading and mathematics, it is quite probable that the kinds of conditions that cause reading deficiencies are operative in the case of difficulties with mathematics.

Improving pupil attitudes

Bassham and others conducted a well-planned study to investigate the relationship between pupil attitudes toward mathematics and their achievement in the subject.[8] The study included five grade 6 classes of 159 pupils whose mean IQ was 101.37, whose reading comprehension grade was 6.99, and whose arithmetic achievement grade was 7.14. The tests used were the Kuhlmann-Anderson Intelligence Test, the Iowa Tests of Basic Skills, and the Dutton Scale for Measuring Attitudes toward Arithmetic. The basic conclusions reached may be summarized as follows:

1. The relationship between attitude and classification as over- or under-achieving according to intelligence and reading comprehension was shown to be significant.

2. The wide variability in weighted achievement at both extremes of the distribution of attitude scale scores would indicate that prediction of achievement on the basis of attitude score for individuals would be rather hazardous.

Bassham made the following suggestions for reexamination of the mathematics program:

1. Increasing the clarity of subunit learning objectives is often helpful in favorably modifying attitude toward mastery. By the time the group has reached a good understanding of what is to be learned, the primary learning often has been accomplished. This is not a new concept, yet lack of clarity in the subunit learning objectives may be easily found, from the postdoctoral to kindergarten levels. We want to "get on with the job" before we have the team harnessed to the plow.

2. Increasing the closeness of the relationship between agreed-upon subunit learning objectives and testing for attainment of those objectives tends to favorably influence attitude toward mastery. The pupil who understands both what he is to learn and how he can know when he has learned it, tends to feel more secure. This does not mean that testing should consist only of pupil regurgitation of factual material or that tests must be easy, but the pupil should know in advance how he will be tested and test items must be fair as judged by comparison with the subunit learning objectives. Also, detailed discussion of means to be used in testing for mastery of specifics often proves to be very helpful in clarifying the subunit learning objectives themselves (and sometimes it helps the teacher construct better tests, too).

3. Increasing the availability to the student of objective, nonthreatening evidence of progress is helpful in favorably modifying attitudes toward mastery of material. The use of the teaching machine, pretests, and group checking for learning purposes only are examples of this technique.

4. The sense of oneness in purpose between members of the group and between the teacher and the group is one of the most powerful factors in modifying attitude toward mastery. Oddly enough, while attitudes are very personal "inner feelings," the members of a closely knit reference group moving toward well-defined goals tend to absorb the group values very readily.

[7]See "Education for Socially Disadvantaged Children," *Review of Educational Research*, December 1965, Vol. 35, No. 5; "Education of Exceptional Children," *Review of Educational Research*, February 1966, Vol 36, No. 1.

[8]Harrell Bassham, Michael Murphy, and Katherine Murphy, "Attitude and Achievement in Arithmetic," *The Arithmetic Teacher*, February 1964, 11:66–72.

EXERCISES

1. List the characteristics of the typical slow learner.
2. Why are children with IQ's of less than 75 not included in the group of slow learners?
3. Comment on the statement that slow learners should be taught in segregated groups.
4. A 10-year-old slow learner with an IQ of 80 has a mathematics age of 8 years. Is his achievement below normal for his age?
5. What is a three-track curriculum?
6. What is meant by the nongraded elementary school?
7. Why does the slow learner fall more and more below the achievement level for his age group as he advances in school?
8. What are desirable characteristics of a mathematics textbook for slow learners?
9. What is the difference between a low achiever and a slow learner?
10. How often should the slow children in the primary grades be regrouped in mathematics?
11. How can the teacher determine the reading difficulty of a mathematics textbook?
12. Describe the differences you would expect to find in the contents of a three-level set of mathematics textbooks for grade 5.
13. At what Grade level should the grouping of slow learners begin?
14. Do you know of low achievers who are above normal in mental ability?

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The Changing American School, Sixty-fifth Yearbook of the National Society for the Study of Education, Part 2. Chicago: University of Chicago Press. 1966. Chapters 2 and 5.

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ENRICHING AND EXPANDING THE MATHEMATICS PROGRAM FOR SUPERIOR LEARNERS

The urgent needs of the scientific and technical age in which we live demand that each pupil receive training in mathematics that is commensurate with his ability and interests. In the past the mathematics curriculum has usually been a body of experiences geared for average pupils. There was seldom an attempt to differentiate the program for superior pupils and for slow learners. As a consequence, standards of achievement, teaching procedures, and instructional materials were adapted to the learning levels and interests of the average rather than the rapid or slow learner. It has now become evident that we need teachers who are skillful, ingenious, and courageous enough to make adjustments in the subject matter of mathematics and in teaching techniques in accordance with the individual needs, ability, and interests of their pupils. Chapter 23 was concerned with the mathematics program for slow learners.

This chapter will consider methods of adjusting the mathematics program for superior children. The following topics are discussed: identifying superior

learners; organizing the mathematics program for superior learners; adjusting the curriculum for superior learners; adjusting instructional procedures for superior learners; adjusting instructional materials for superior learners; mathematical recreations and other means of enrichment.

IDENTIFYING SUPERIOR LEARNERS

Scholastically gifted pupils constitute between 2 and 3 per cent of the entire school population. At this level the intelligence quotient is 130 or above, although pupils with IQ's of 120 are often called superior. In this discussion we shall use the term "superior" rather than "gifted" to designate pupils of outstanding ability. It must be remembered, of course, that many pupils with IQ's below 120 may do extremely good work in mathematics. Ability in mathematics does not necessarily go hand in hand with measured intelligence, although there is a close connection between the two.

Weaver and Bramley have listed the following as characteristics of the mathematically talented child:

1. Sensitivity to, awareness of, and curiosity regarding quantity and the quantitative aspects of things within the environment
2. Quickness in perceiving, comprehending, understanding, and dealing effectively with quantity and the quantitative aspects of things within the environment
3. Ability to think and work abstractly and symbolically when dealing with quantity and quantitative ideas
4. Ability to communicate quantitative ideas effectively to others, both orally and in writing; and to readily receive and assimilate quantitative ideas in the same way
5. Ability to perceive mathematical patterns, structures, relationships, and inter-relationships
6. Ability to think and perform in quantitative situations in a flexible rather than in a stereotyped manner: with insight, imagination, creativity, originality, self-direction, independence, eagerness, concentration, and persistence
7. Ability to think and reason analytically and deductively; ability to think and reason inductively, and to generalize
8. Ability to transfer learning to new or novel "untaught" quantitative situations
9. Ability to apply mathematical learning to social situations, to other curriculum areas, and the like
10. Ability to remember and retain that which has been learned.[1]

Among the many characteristics of the talented and interested pupil, Hlavaty has selected the following four for special emphasis:

1. He is interested in the relevance of mathematics to life—its applications.
2. He is excited by his ability to raise questions and his ability to answer his own questions or those raised by others.
3. He is delighted with the organization of abstract concepts into patterns and structures and with his perception and understanding of such patterns and structures.
4. Finally, he wants to follow through on any of these to see where they lead.[2]

The information given in the pupil's cumulative record is also needed by the teacher in the study of the individual, including personal and school history, test scores and ratings, special interest data, and supplementary materials such as anecdotal records, case studies, and follow-up data.

[1] J. Fred Weaver and C. F. Bramley, "Enriching the Elementary School Mathematics Program for More Capable Children," *Journal of Education*, October 1959, 142:1–40.

[2] Julius H. Hlavaty, *Enrichment Mathematics for the Grades*, Twenty-seventh Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: The Council, 1963), p. 4.

ORGANIZING THE MATHEMATICS PROGRAM FOR SUPERIOR LEARNERS

For many years various plans have been used to group children. They have included grouping on a chronological age basis, on a mental age or ability basis, according to level of achievement, or on the basis of a combination of various factors, including IQ, achievement, and level of social and physical development. The purpose of such plans has always been to reduce the variability among the pupils and to secure more homogeneous groups.

Various studies that have been made of the practice indicate that homogeneous grouping is not only not feasible but is of no value because of the variability among the various traits of learners, even within one area of achievement such as mathematics. Whatever the form of grouping that is used, the proper adaptation of experiences, methods of teaching, and materials of instruction is essential.

In mathematics grouping is necessary at all levels. We form groups on some basis and then regroup within classes in any number of ways, especially when we use units by which to adapt instruction to pupil needs following testing and diagnosis. Grouping within classes should be flexible and adapted to the local situation. Sometimes the teacher will take the whole class as a group in developing a skill needed by all or in working on some activity; at other times the teacher will form several work groupings according to the achievement level in order to arrange specific activities of different levels of complexity. At some times the teacher will wish to work with an individual pupil who needs special help in a particular area.

Many authorities believe that there is a need for teachers who are specialists in mathematics in order to strengthen the program for superior children. In some schools the more able children are taught as a group by special teachers; often they are sent to a special mathematics room for periods of time where they work with a teacher who has a rich mathematical background on a variety of challenging topics of a high level of difficulty. These plans overcome some of the problems that arise in the self-contained classroom where one teacher must deal with the various areas of the curriculum. This is especially necessary if the mathematics potential of the superior children is to be developed to the fullest possible extent.

In some communities special schools are provided for the superior learners. High standards are set up in these schools and the work in mathematics is difficult. Hunter College Elementary School in New York is an illustration. In some schools special classes are arranged, sometimes full time, sometimes part time, in which work is done under a special teacher in such areas as mathematics and science. Sometimes classes are recruited from several schools to get large enough groups. In some areas classes are formed with superior children of several age levels for reasons of economy. Often there is special grouping for only part of the day to give the superior learners the opportunity to explore areas of special interest. There also is grouping in extracurricular activities such as math clubs. In Los Angeles in a number of centers, children of high ability from a group of schools are brought together twice a week for special training in research techniques and the use of the library. In some places community-sponsored groups of children with special interest

in mathematics meet before or after school and on Saturdays where they come into contact with specialists of various kinds who wish to further the education of superior students.[3]

The effectiveness of such grouping procedures depends on the resourcefulness and ingenuity of the individuals in charge of the group. Often this method requires special administrative adjustments in terms of transportation, curriculum content, use of school facilities, instructional equipment, and contacts with parents. It is important that the community understand the purpose of grouping and its values. Children should be grouped only with the consent of their parents.

In some areas the difficulty of recruiting specially prepared teachers has delayed the development of work with groups of superior learners. Courses in the teaching of these children have been set up. Some states, such as California, provide state aid for special classes of the superior learners to stimulate communities to establish them. The California plan was the outgrowth of a three-year experimental project which evaluated the effectiveness of a variety of procedures of grouping children who are fast learners, ranging from ability grouping to clusters of five or six such pupils in mixed classes. In all of these situations the results for the experimental groups exceeded those of control groups in achievement and social relationships.[4] The teachers seemed to favor the cluster-group approach over the gifted-class approach because they found it easier to make necessary adjustments for a small group than for a whole class of superior children. The essential point is that whatever the plan that was used in working with fast learners, the results were superior to those in which no grouping was used.

ADJUSTING THE CURRICULUM FOR SUPERIOR LEARNERS

The mathematics curriculum of the elementary school has been extended in recent years to include a balanced program of arithmetic, geometry, and algebra. The adjustment of such a broad, comprehensive program to the range of ability levels of the students presents many problems.

Children should take part in a basic mathematics program from the kindergarten through the secondary school and beyond. Children of normal ability in the elementary school should be given the opportunity to study the contents of the whole basic program. For the slow learners the more difficult topics should be eliminated and instead topics should be presented that have social significance and cultural values and that these pupils have the mental capacity to master. For superior children the basic program should be extended in both breadth and depth. Superior learners should learn all that the average child learns in a balanced program and in addition should go far beyond in the direction of abstract mathematics. The work in geometry and algebra is very appropriate for them. The slower learners, on the other hand, are able to grasp only the most elementary aspects of these two fields.

The two major approaches to the selection of mathematics for the superior

[3]See M. M. Provus, "Ability Grouping in Arithmetic," *Elementary School Teacher*, Apr ' 1960, 60:391–398; Elinor C. Flagg, "Mathematics for Gifted Children," *Educational Leadership*, March 1962, 19:379–382; J. Fred Weaver, "Differentiated Instruction and School-Class Organization for Mathematical Learning within the Elementary Grades," *The Arithmetic Teacher*, October 1966, 13:495–506.

[4]Ruth Martinson, "The California Study of Programs for Gifted Children," *Exceptional Children*, March 1960, 26:339–343.

learner are academic acceleration and enrichment. Academic acceleration makes available for a superior grade 3 child the textbook for grade 4 whenever he completes the grade 3 textbook. When he completes the grade 4 textbook he is provided with the grade 5 book and proceeds to work through that even if he is still actually in grade 3 or has been "skipped" to grade 4. When a child is academically accelerated through a series of standard textbooks written for the average learner, he is learning a body of content that the average child learns. The only difference is that he learns the material sooner. This program is wholly insufficient and inadequate for the superior child.

Acceleration

Acceleration in mathematics can take place at any point in the educative process from kindergarten to college. Early admission to kindergarten is a form of acceleration widely used in programs for superior learners. The ungraded primary type of organization allows the bright child to complete the three-year program in a shorter period of time. Skipping a grade is probably one of the oldest but most objectionable acceleration methods. It is based on the assumption that a pupil who is sufficiently bright can forego the learning experiences of a given grade and should be challenged by the work of the next higher grade. Another acceleration method is so organized that a class of superior students can complete the work of two or more grades in a fewer number of years. This plan is similar to the ungraded program. There can be little doubt that acceleration i a useful and valuable method of helping bright children at all levels of the school to progress so that they can begin their secondary and college programs at an earlier date. In this age of rapid technological changes, with its demands for highly skilled mathematicians, steps should be taken to eliminate any lag in the education process. Acceleration hastens the beginning of the individual's most productive period so that his entry into a life career is not delayed.

Enrichment

Enrichment is the most widely accepted method of providing satisfactory experiences for superior learners. This approach is desirable whether the children are placed in a self-contained classroom or are grouped in some way according to ability level. Enrichment for the more able children implies that the material covered will be broader than that which is included in the basic program but will be related to that program and continuous with it.

Enrichment can be of two kinds, *horizontal* or *vertical*. By horizontal enrichment is meant the addition of new learning experiences on the level of the pupil's present achievement status. Vertical enrichment, on the other hand, is the provision of advanced work or further specialization in the same area of learning. Enrichment also refers to the process of increasing the quality of the offering with pertinent illustrative materials or of providing wider and deeper understandings in a given area.[5] Vertical enrichment leads to *power* in the field of mathematics. Greater emphasis should be given to vertical enrichment than horizontal enrichment in planning the mathematics program for superior learners.

The list of chapter titles of a recent yearbook of the National Council of Teachers of Mathematics may be used,

[5] Monte S. Norton, "Enrichment as a Provision for the Gifted in Mathematics," *School Sciences and Mathematics*, May 1957, 57:339–345. see also Weaver and Bramley.

in the opinion of the committee that prepared the book, as a reservoir from which the teacher of superior students may draw classroom lessons.[6] The chapter titles are as follows:

1. Non-Decimal Numeration Systems
2. Arithmetic with Frames
3. Modular Arithmetic
4. Short Cuts and Bypaths
5. A Method of Front-End Arithmetic
6. Concepts of Measurement
7. Probability in the Elementary School
8. Geometry in the Grades
9. Topology
10. Some Simple Laws of Physics
11. Tricks and Why They Work
12. Arithmetic for the Fast Learner in English Schools.

Special units in mathematics for superior learners

A number of cities are working on the development of enriched units for superior learners in which emphasis is placed on applications of mathematics and science and the use of library resources. The schools of San Diego County, California, are an example. In 1961, teachers' committees prepared a series of six "Independent Learning Booklets" intended for independent study by gifted children having IQ's of 140 or more in grades 4–6. These booklets, actually in the form of programed learning materials, contain a carefully graded, step-by-step development of each topic. There are periodic test exercises to measure progress. The exercises are keyed to answers that are placed so as to be immediately available for the individual pupil to check his thinking. This plan enables each child to progress independently at his own rate through each booklet with a minimum of help from the teacher. The purpose of these booklets is to use an approach in dealing with topics that is novel for the pupils, one that will create new interests and the desire to go on to further study.

The titles of the booklets given below indicate the nature of the subject matter that the committees believed would be a challenge to the children of top-level ability in grades 3–6.

1. *We Travel with Money.* Children on a trip use American, Guatemalan, Mexican, and English money and learn to compute with foreign systems of money (grades 3 and 4).

2. *We Learn to Measure—We Measure to Learn.* The booklet discusses relationships among units of measure of various kinds and presents the history of measures of length, including metric units. Other topics covered are the history of the calendar, length of shadows, very high and very low temperatures, military time, and approximate measures (grades 3 and 4).

3. *Mental Arithmetic in Subtraction and Division.* This booklet deals with the equal-addition method of subtraction and applies the procedure to short-cut procedures in division by one- and two-place divisors, and the subtraction of measures and fractions (grades 5 and 6).

4. *Understanding the Slide Rule.* This booklet contains a detailed discussion of the construction and method of using the slide rule in number operations. Scientific notation is briefly presented. Introductory work with logarithms is also included (grades 5 and 6).

5. *Estimation—the Decimal Point in Division.* The booklet is essentially a mathematical explanation of the "correct" procedure commonly used in division of decimals in such examples as $4.2\overline{)1.68}$, $.5\overline{)2}$, and $.62\overline{)12.4}$. Estimation

[6] Twenty-seventh Yearbook.

of answers is stressed by a check on answers (grades 5 and 6).

6. *Probability and Statistics.* This booklet consists of two parts. Part I examines mathematical and statistical probability, using simple illustrations of the points involved. Part II deals with the nature of statistics, the ways of organizing facts in tabular and graphic form, measures of central tendency, the range, and the grouping of statistical data (grades 5 and 6).

It is evident that the concepts involved in these booklets are advanced and quite difficult. They are beyond the grasp of all but a few of the most able children in the typical class. Yet the teacher should not hesitate to make such booklets available for these more able children in the classroom or school library. Some of the pupils with a high potential in mathematics will be stimulated to make the effort needed to study their contents. One reader who examined these booklets commented: "What would happen if we took the lid off for these gifted children?"

A human binary computer

A unit dealing with a study of the binary system of numeration may be challenging and interesting for pupils at about the level of the grade 6. The teacher has the class dramatize the operation of a computer by arranging four or five chairs in a row. A different pupil occupies each chair. A chair is assigned a place value to correspond to the place value in a four-place numeral in the binary scale. The teacher uses a ruler to tap on a desk or the floor to produce a sound to represent an electric impulse. When he makes a sound to simulate an impulse, one or more pupils interpret the number of the impulse by raising or lowering their right arms. If four pupils occupy the chairs, there should be a fifth pupil to record the numerals at the chalkboard.

The number of pupils involved in the dramatization of each impulse depends upon the number to be represented. The occupant of the chair in ones' place puts up his right arm to show the first impulse. The pupil at the chalkboard writes 1 to indicate the number of the impulse. When the teacher gives the signal for the second impulse, the pupil in ones' place puts down his right arm because that place is overloaded. The occupant in twos' place then puts up his right arm. The recorder writes 10. Next, the teacher gives the signal for the third impulse. Now the pupil in ones' place puts up his right arm. The right arms of both the pupils in ones' and twos' places represent the number of the impulse, which is recorded as 11.

When the teacher gives the signal for the fourth impulse, ones' place is overloaded. The pupil in the chair representing that place then puts down his right arm. This action overloads twos' place, so the pupil occupying the chair in that place puts down his right arm. The pupil in fours' place then puts up his right arm and the recorder writes 100 on the chalkboard. In a similar manner each succeeding impulse is dramatized by the occupants of the chairs and then the number is recorded on the chalkboard. If a pupil makes an error by having his arm in a wrong position, the teacher states, "We had a short circuit, so we must start from the beginning." The teacher follows this plan until the pupils are able to represent the largest number possible. That number depends upon the number of participants. In case of four occupants of chairs, the largest number is 1111_2, or 15. The largest number for five places is 11111_2, or 31.

Table 24.1 gives the position of the right arm for each occupant of a chair for the first eight numbers represented. U represents a raised arm and D represents an arm down.

TABLE 24.1
Arm Position in a Computer Dramatization

Number	*Chair 4*	*Chair 3*	*Chair 2*	*Chair 1*
1	D	D	D	U
2	D	D	U	D
3	D	D	U	U
4	D	U	D	D
5	D	U	D	U
6	D	U	U	D
7	D	U	U	U
8	U	D	D	D

An examination of the table indicates that a U occurs in the same relative position with respect to the chairs or pupils that 1 occurs in the binary scale for a corresponding four-place numeral. Similarly, D and 0 occupy corresponding places in the table and in a numeral.

The plan described may be extended to include dramatization of a *ternary computer.* Pupils occupy three or more chairs arranged in a row with values assigned to the chairs to correspond to the ordered positions in a numeral in base three. Each place in a numeral in base three may have a frequency of two, therefore both left and right arms are used to represent the impulses. The pupil always expresses the first frequency of a place with his left arm. When the occupant of a chair has both arms up, that place is filled. One more impulse in that place (frequency) will overload the place. The procedure is then similar to that described earlier regarding dramatizing a binary computer.

Possible limitations of enrichment

Madden has pointed out the possible limitations of enrichment:

> A danger of horizontal enrichment standing alone is that it becomes a disorganized experience with too high a portion of mathematical recreations and other activities that have little value in leading to a more complete understanding of mathematics. The result may be a diminished challenge for the pupil and a related loss of interest in mathematics.[7]

Power in mathematics is developed not by disorganized random experiences, however interesting they may be, but by confronting children with challenging questions and problems that are within their power of comprehension. Children should be given many opportunities to discover informal solutions and generalizations. They should be led to see the necessity of dependable data as a basis of thinking and action, and they should become familiar with the ways in which reliable information can be gathered. They should have experiences that will gradually develop in them the methods of logical, systematic, disciplined thought that facilitate all mathematical work and lead to creative thinking.

The teacher should broaden the context and try to raise the mathematics skills to the conceptual level by teaching in such a way that "children will understand relationships, extend them to solutions of new problems, and have time to think, to question, to wonder."[8]

[7]Richard Madden, "Major Issues in the Teaching of Arithmetic," *National Elementary School Principal*, October 1959, 39:17–21.

[8]Charlotte Junge, "Depth Learning in Arithmetic—What Is It?" *The Arithmetic Teacher*, November 1960, 7:341–346.

ADJUSTING INSTRUCTIONAL PROCEDURES FOR SUPERIOR LEARNERS

Differentiated instruction

By differentiated instruction we mean making adjustments of class organization, curriculum, methods, and materials so as to adapt instruction as far as is practical to the wide range of differences in mental ability, achievement, interests, and needs that exist in almost all classes, even in classes in mathematics for superior learners. Under such circumstances the teacher must make a special effort to meet the needs of individual pupils and to enrich the work for all of the more able children, especially those who seem to have unusual ability in mathematics. Special work may be done with cluster groups of the more able children when suitable situations present themselves. Differentiated goals and instruction are practical procedures that can be used in any classroom, regardless of the ability level of the children.

As has been pointed out, grouping, regrouping, and subgrouping as the need arises should be a fundamental principle of teaching for even the superior learners. No plan of grouping will in itself provide effectively for individual differences in a particular group of children. The teacher is the key to the situation. Emphasis should be placed on activities that develop mathematical power rather than on the routine work with skills as done with heterogeneous groups of children.

Methods of developing the mathematical power of the more able

The development of *mathematical power* is a basic purpose of instruction in mathematics as far as the more able pupil is concerned. By power is meant insight and understanding of mathematical relationships at a higher level of thinking and comprehension than the average pupil will be able to achieve.

Among the experiences that can be used to develop mathematical power are the following:

1. Discovering varied methods of solving examples involving number operations
2. Discovering varied methods of solving verbal problems
3. Identifying mathematical properties applied in operations
4. Developing an understanding of how the basic properties of mathematics are applied in short-cut procedures
5. Studying independently the operational procedures in available reference materials
6. Studying independently the topics related to the applications of mathematics
7. Studying subject matter independently through the use of adequate programed materials and scrambled textbooks
8. Studying the newer kinds of computational devices and machines
9. Verbalizing generalizations, rules, and conclusions in concise language
10. Solving and making puzzles requiring the application of basic mathematical properties

Each of these procedures will now be illustrated.

1. A pupil or class can discover various ways of finding a product, such as 15×34:

a.
$$\begin{array}{rcr} 34 & = & 30 + 4 \\ \underline{\times\ 15} & = & \underline{\times\ 15} \\ & & 450 + 60 = \square \end{array}$$

b.
$$\begin{array}{rcr} 34 & & 34 \\ \times\ 10 & + & \times\ 5 \\ \hline 340 & + & 170 = \square \end{array}$$

c. $15 \times 34 = 15 \times (30 + 4) = 450 + 60 = \square$

d. $15 \times 34 = (10 + 5)(30 + 4)$
$= 10 \times (30 + 4) + 5 \times (30 + 4)$
$= \square$

e. $\begin{array}{r} 34 \\ \times 15 \\ \hline \end{array}$ $\quad 10 \times 34 = 340$, $\quad \frac{1}{2} \times 340 = 170$

$340 + 170 = \square$

f. Can you give other methods?

2. Various ways of solving the following problem can be discovered: At the rate of 6 miles in 10 minutes, how many miles will a car travel in an hour?

a. 1 hour = 6 ten-minute periods. Then $6 \times 6 = \square$
b. Add 6 sixes. 6, 12, 18, 24, 30, 36
c. In 1 minute the car travels $\frac{6}{10}$ mile
In 1 hour it will travel $60 \times \frac{6}{10} = \square$
d. Use an equation of the type $\frac{6}{10} = \frac{n}{60}$, $n = \square$

3. The learner can be asked to identify the properties illustrated by the following mathematical sentences and then to give original illustrations of these properties.

a. $34 + 48 = 48 + 34$
b. $3 \times 42 = 3 \times (40 + 2)$
c. $4 \times 0 \times 1 = 0$
d. $74 + 62 + 56 + 28 = (74 + 56) + (62 + 28)$
e. $72 \times 64 = 64 \times 72$
f. $7 \times (9 \times 8) = (7 \times 9) \times 8$
g. $\frac{2}{3} = \frac{2}{3} \times 1 = \frac{2}{3} \times \frac{4}{4} = \frac{8}{12}$

4. Computational short cuts in mathematics may be classified as representative of mathematical properties that the pupil can discover. Three basic properties that apply to short cuts are:

a. *The associative property.* An illustration of the associative property for addition is as follows:
$24 + (46 + 38) = (24 + 46) + 38$
b. *Identity of 1.* The example 25×48 when expressed as $25 \times 48 = 4 \times 25 \times 48 \times \frac{1}{4}$ illustrates the identity element of 1 $(4 \times \frac{1}{4})$.
c. *The distributive property.* The example $19 \times 43 = (20 - 1) \times 43$ illustrates the distributive property

Other properties can be applied in a variety of ways.

5. The pupil may be asked to look up some topic, such as the complementary method of subtraction, the sieve of Eratosthenes, or some similar operational procedure in available reference books. Pupils should be encouraged to volunteer for such activities and should also be encouraged to carry them on independently.

6. The independent study of the topics related to the history of mathematics offers many challenges to the superior learner, such as:

a. The history of our system of numeration

b. Other systems of numeration

c. The reading of such books as E. T. Bell, *Men of Mathematics;*[9] James Newman, *World of Mathematics;*[10] and Lancelot Hogben, *The Wonderful World of Mathematics.*[11]

7. A number of companies have published learning materials that will enable the pupil to study mathematical topics systematically but independently, for example, the series *Mathematics: Programs A, B, and C.*[12]

8. The study of mathematical procedures basic in automatic computing devices and machines offers a real challenge to the superior student. The numeration systems most commonly used in these machines are the binary and decimal systems. The reason for their use should be determined.

9. The answering of questions based on a group of exercises, such as the following, should lead to the formulation of a concise statement or to a valuable mathematical conclusion or generalization.

[9](New York: Simon and Schuster, Inc., 1937).
[10](New York. Simon and Schuster, Inc., 1956).
[11](New York: Doubleday & Company, Inc., 1955).
[12](Morristown, N.J.: Silver Burdett Company).

a. Find the answers to the following:

$4\frac{1}{2} \div 3\frac{1}{2}$ $\quad$ $\frac{3}{4} \div \frac{1}{2}$

$\frac{1}{2} \div \frac{2}{3}$ $\quad$ $6 \div 7\frac{1}{2}$

$5\frac{1}{2} \div 5\frac{1}{2}$ $\quad$ $8\frac{1}{2} \div 4$

b. In which of the examples is the quotient greater than 1? less than 1? equal to 1?

c. See if you can discover general rules that will tell when the quotient of the example is greater than 1; less than 1; equal to 1. The pupil should be directed to write a concise statement concerning the situations described.

10. The pupil should not only work with puzzles but he should also try to find the mathematical basis of the puzzle. For example, perform the following:

a. Select any number, as 7
b. Now add the next higher number: $7 + 8 = 15$
c To that sum add 9: $15 + 9 = 24$
d Divide the sum in (c) by 2: $24 \div 2 = 12$
e. Subtract the original number: $12 - 7 = 5$

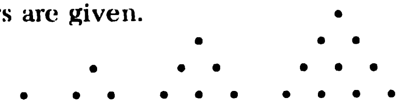
The answer always will be 5. The use of general number will show why the final answer will always be 5.

Let n = the number
Then $n + 1$ = the next number
$2n + 1$ = the sum
$2n + 1 + 9 = 2n + 10$
$(2n + 10) \div 2 = n + 5$
$n + 5 - n = 5$

After the pupil discovers the reason why the puzzle works, have him make a similar puzzle to try on his friends.

Geometric number patterns

One of the best ways to develop power in mathematics is to have the pupil discover number patterns. Identification of geometric number patterns is an illustration of the type of work adapted for this purpose. Geometric numbers are so named because a dot representation of a number forms a geometric figure, such as a *triangular number.* The first four triangular numbers are given.



1 3 6 10

The pupil should be able to discover the pattern that applies to a series of geometric numbers if he makes a diagram to show the first few numbers in the series. He should then find the differences between pairs of consecutive terms. These differences form a series. In the illustration of triangular numbers, the differences 1 (1 — 0), 2 (3 — 1), 3, and 4 are for the first four counting numbers. Therefore the next difference would be 5, hence the next term in the series of triangular numbers would be 10 + 5, or 15, and the following triangular number would be 21. The pupil can always check his work by making a dot diagram to show that a number in a series is correct, as a triangular representation for 15 and 21.

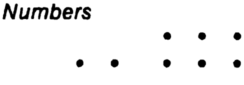
The teacher should write on the chalkboard the numerals for the first few numbers in the series of natural numbers, odd numbers, and even numbers, as follows:

A: {1, 2, 3, 4, 5, 6, 7, 8}
B. {1, 3, 5, 7, 9, 11, 13, 15}
C: {2, 4, 6, 8, 10, 12, 14, 16}

The plan described for triangular numbers should apply to other numbers that can be represented by geometric models. The pupil should make diagrams to represent only regular polygons.

Square Numbers

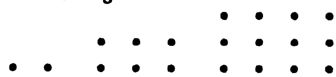
Point representation (array)



Numbers 1 4 9
Set of first five square numbers {1, 4, 9, 16, 25}
Differences 1 (1 — 0), 3, 5, 7, 9

The differences form the series of odd numbers. Therefore, the sum of any number of odd numbers is a square of the number of addends. The sum of the first six odd numbers is 6^2, or 36.

Rectangular Numbers



Arrays

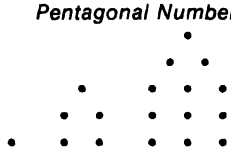
Numbers 2 6 12

Set of first five rectangular numbers $\{2, 6, 12, 20, 30\}$

Differences 2 (2 — 0), 4, 6, 8, 10

The differences form the series of even numbers as given in set C. Therefore the sum of any number of consecutive even numbers is a rectangular number. Thus the sum of the first five even numbers is 30, which is a rectangular number.

Pentagonal Numbers



Arrays

Numbers 1 5 12 22 35

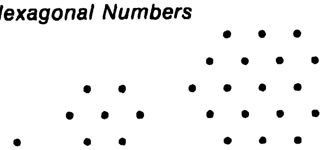
Differences 1, 4, 7, 10, 13

The differences form a series in which each term increases by 3.

The dot diagrams show that a pentagonal number is a combination of models for a triangle and a square. Therefore a number in a series of pentagonal numbers can be formed by adding the correct terms from both series. If the nth term of the square numbers is added to the $(n + 1)$ term of the triangular numbers, the sum will be the nth term of a pentagonal series, as shown by the following:

Triangular		1	3	6	10	15
Square	1	4	9	16	25	36
Pentagonal	1	5	12	22	35	51

Hexagonal Numbers



Point representation

Numbers 1 7 19

Set of first five hexagonal numbers $\{1, 7, 19, 37, 61\}$

Differences 1, 6, 12, 18, 24

The differences form a series in which each term following the first term is a multiple of six. The sequence of differences should enable the pupil to find each succeeding term in the series of hexagonal numbers.

The teacher should have the pupil divide each hexagonal number by 6 and note the quotient. Every hexagonal number is one more than a multiple of 6. The pupil should discover the pattern to enable him to write the next number in the hexagonal series from the division. The pupil renames each term of the series as follows:

Hexagonal numbers	1	7	19	37	61
Renaming numbers	$0 \times 6 + 1$	$1 \times 6 + 1$	$3 \times 6 + 1$	$6 \times 6 + 1$	$10 \times 6 + 1$

In the series of renamed numbers, each hexagonal number is the product of two factors plus a remainder of 1. One of these factors is 6 and the other is a triangular number. Therefore the set of the latter factors forms the set of triangular numbers.

The teacher may have the pupil follow the pattern described to find the *octagonal numbers* or any other set of geometric numbers.

Motivation of superior learners

How can the teacher help the child set up goals that will challenge him?

What can be done to motivate him? To what extent should external inducements and artificial stimuli such as grades, examinations, rewards, and punishments be used to stimulate him to greater efforts?

Ideally, motivation should help the child to develop purposes, interests, and expectations that will direct his efforts and activities toward the fulfillment of long-range ambitions and goals. Some children at an early age reveal a marked aptitude in mathematics, which should be guided and developed by the school. The experiences of young children in the home and elsewhere may have stimulated them to explore a variety of uses of mathematics in daily life. This impetus should be expanded and encouraged by the whole staff of the school. It is an unfortunate fact that lack of motivation accounts for the large number of talented youth who fail to complete their secondary education or to continue the study of mathematics at the college level.

The following are some of the techniques of motivation that are used by the schools:

1. Honor rolls
2. Invitations of membership in mathematics clubs
3. Student interviews with counselors and teachers
4. Letters to parents reporting unusual achievements
5. Scholarship luncheons and banquets
6. Mathematics contests
7. The availability of a wide variety of books and mathematical devices and instruments for examination by those interested. In some elementary schools mathematics workrooms are set up in which children can work when they wish
8. Centers of interest in classrooms.

ADJUSTING INSTRUCTIONAL MATERIALS FOR SUPERIOR LEARNERS

Mathematics textbooks

The mathematics textbook for superior learners should include not only the essentials of the basic program for all learners but also additional material that is more difficult and more abstract in nature. More attention should be given to the study of the numeration system and the properties and principles underlying number operations. Fewer illustrations are needed and the reading difficulty of the discussion of topics can be increased. The problem content can be broadened in scope and complexity, and the work in algebra and geometry can be extended and made more difficult.

The teaching guide should offer suggestions concerning the use of concrete, manipulative materials and visual aids in order to supplement work in the lower grades. Superior learners, however, need much less contact with such learning aids than the children of average and low ability. The rate of presenting topics in textbooks for superior learners can be more rapid than for children of other levels of ability.

Special provisions for enrichment

Mathematics textbooks suggest a wide variety of activities that enrich the learning of mathematics for the more able children in grades 1–6. Some of the more widely used methods are the following:

1. Exploration of the uses of mathematics in all curriculum areas, especially science, health, music, and social studies
2. Starred problems within instructional units that require independent

research, reading, and local inquiry

3. Lists of topics and problems for special investigation and report
4. Starred units of work of above average difficulty
5. Challenges for the more capable learners
6. Solving equations and formulas
7. Mathematical puzzles and recreations
8. Geometrical constructions and proofs
9. Activities leading to the discovery of principles, generalizations, and relationships
10. Field work requiring the application of mathematical procedures, especially geometry
11. Mathematics scrapbooks—individual or class
12. "Brain twisters"
13. The preparation of exhibits, displays, and collections
14. Excursions and field trips
15. Mathematics clubs
16. A section of the textbook is labeled "Challenges" or "Extension," in which the material is keyed to the main body of the text. The extensions are designed to create power in dealing with number.

A number of publishers have made available enrichment materials in mathematics for elementary schools following:

Ginn & Company, Ginn Enrichment Program, grades 2–8
Harcourt, Brace & World, Inc.
Harper & Row, Publishers, Inc., Arithmetic Enrichment Program, grades 3–8
Holt, Rinehart and Winston, Inc., Programed Units in Modern Mathematics
Silver Burdett Company, Mathematics Enrichment Program: A, B, and C
Webster Publishing Division, McGraw-Hill, Inc., Exploring Mathematics on Your Own

The school library

The resources of the library should be used continually to enrich the work in mathematics. The library is the heart of an enrichment program for the superior learners with special interest in mathematics. These students are likely to browse widely among available printed materials of all kinds, seeking information they desire on matters of interest. Independent reading and study is a high type of learning that should be encouraged and facilitated by having available a well-selected variety of printed materials, including general books, reference books, magazines, bulletins, schedules, and the like.

The books listed below are excellent sources. Ideally all of these books should be available in a classroom library.[13]

1. Encyclopedias and reference books:

Britannica Junior
Encyclopedia Britannica
Compton's Pictured Encyclopedia
The World Almanac
The World Book

2. Books with historical materials:

Adler, I., *Time in Your Life*. New York: The John Day Company, 1955
Gardner, M., "The Remarkable Love of Prime Numbers," *Scientific American*, March 1964, pp. 120–128

[13]See Clarence E. Hardgrove, *The Elementary and Junior High School Library* (Washington, D.C.: The National Council of Teachers of Mathematics, 1960).

Hogben, L., *Mathematics for the Millions*. New York: W. W. Norton & Company, Inc., 1937

———, *The Wonderful World of Mathematics*. New York: Doubleday & Company, Inc., 1955

Sanford, Vera, *A Short History of Mathematics*. Boston: Houghton Mifflin Company, 1930

Smith, D. E., *Number Stories of Long Ago*. Boston: Ginn and Company, 1919

Smith, D. E., and J. Ginsburg, *Numbers and Numerals*., New York: Bureau of Publications, Teachers College, Columbia University, 1937

3. Recreations, games, puzzles:

Bakst, A., *Mathematical Puzzles and Pastimes*. Princeton, N.J.: D. Van Nostrand Company, Inc., 1954

Bendick, Jeanne, *How Much and How Many?* New York: McGraw-Hill, Inc., 1947

Collins, F. A., *Fun with Figures*. New York: Appleton-Century-Crofts, Inc., 1948

Enrichment Program for Arithmetic, pamphlets for grades 3–6, eight per grade. New York: Harper & Row, Publishers, Inc., 1956

Friend, N., *Numbers' Fun and Fact*. New York: Charles Scribner's Sons, 1954

Heath, R. V., *Mathemagic*. New York: Dana Publications, 1953

Meyer, J. S., *Fun with Mathematics*. New York: Harcourt, Brace & World, Inc., 1952

Smith, D. E., *Wonderful Wonders of One, Two, Three*. Rochester, N.Y.: Macfarland, 1937

Spitzer, H. F., *Practical Classroom Procedures for Enriching Arithmetic*. St. Louis, Mo.: Webster Publishing Division, McGraw-Hill, Inc., 1956

MATHEMATICAL RECREATIONS AND OTHER MEANS OF ENRICHMENT

Sources of mathematical recreations

The following books contain many illustrations of mathematical recreations.

Adler, Irving, *The Magic House of Numbers*. New York: The John Day Company, 1957

Cullins, F. A., *Fun with Figures*. New York: Appleton-Century-Crofts, Inc., 1948

Freeman, Ira, and Mae Freeman, *Fun with Figures*. New York: Random House, Inc., 1946

Glenn, W. H., and D. A. Johnson, *Fun with Mathematics*. St. Louis, Mo.: Webster Publishing Division, McGraw-Hill, Inc., 1960

Heath, Royal V., *Mathemagic*. New York: Dover Publications, 1953

Schaaf, William A., *Recreational Mathematics*. Washington, D.C.: The National Council of Teachers of Mathematics, 1955

Spitzer, Herbert F., *Practical Classroom Procedures for Enriching Arithmetic*. St. Louis, Mo.: Webster Publishing Division, McGraw-Hill, Inc., 1956

There is a wide range of mathematical recreations. Each of the texts mentioned provides many types. A sample of a mathematical recreation of a geometric nature follows.

Geometric puzzles

1. Lay 6 matches on the table. Place them 1 inch apart. Add 5 matches to them and make 9 (Fig. 24.1).

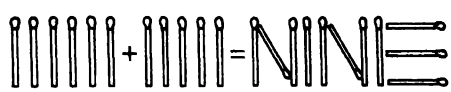


Figure 24.1

2. Draw a 2-inch circle on a piece of paper. Draw a star in the circle, as shown in Fig. 24.2, using one continuous line and without retracing any line.

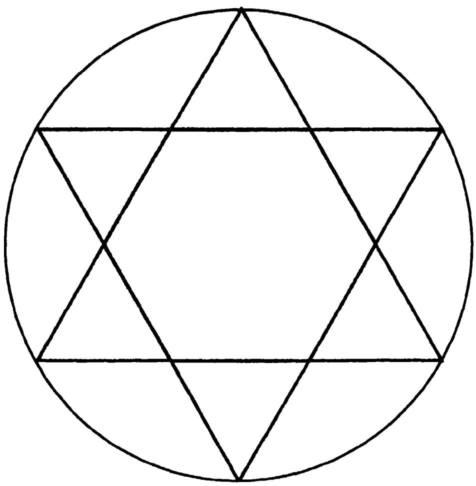


Figure 24.2

3. In the first part of Figure 24.3, six equal areas are formed by 13 matches. How can one get six equal areas with one less match?

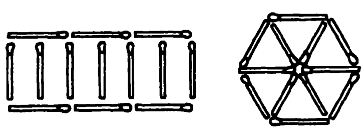


Figure 24.3

Field work in mathematics

There are many experiences in the nature of field work that offer excellent opportunities to enrich the learning of mathematics by applying it in such concrete situations as the following:

1. Measuring changes in shadow lengths at different times of the day
2. Laying out a school garden
3. Finding the widths of lots in the vicinity
4. Finding the length of a city block
5. Laying out a piece of ground equal to an acre in some nearby field
6. Finding the area of the school playground
7. Checking the dimensions of the school baseball diamond
8. Estimating the time required to walk a block, then checking with a watch. Several trials are advisable
9. Laying out a map showing traffic hazards found by a survey of the vicinity of the school.

A mathematics table or corner

On a table in the corner of the classroom, the teacher and children can place interesting materials related to mathematics, such as number games, number puzzles, stunts with numbers, scrapbooks, strange measuring devices, collections of old coins, pictures showing uses of numbers, and similar materials. In some schools it is possible to secure pictures, exhibits, and collections of mathematics materials from libraries, museums, and businesses. To add to the attractiveness of the corner, a small library and a bulletin board can be used. These may be cared for by a small committee of children. All children can bring materials to school to be a part of the mathematics corner. The materials may be evaluated from time to time and steps taken to make the corner more interesting and attractive.

Math clubs

In most junior high schools and in a growing number of elementary schools, there are math clubs that are made up of groups of children especially interested in arithmetic and other branches of mathematics. Usually a teacher sponsors the group. Programs are geared to the interests of the children. The clubs freely draw on the resources of the school and community to secure dis-

cussion leaders who can tell them about important developments in everyday uses of mathematics and the requirements of business, industry, and science. Often such clubs take charge of a number of school assembly programs. They also sometimes sponsor displays of films about mathematics, contests, and even debates or forums. A good math club can do much to enrich the work in mathematics in elementary schools.

Summary

1. Every pupil needs some form of enrichment. The more able pupil should participate in activities that lead to the development of mathematical power.

2. Enrichment should be both vertical and horizontal.

3. There should be some kind of grouping of pupils according to ability or achievement in mathematics.

4. Limited acceleration is a desirable form of procedure for meeting the problem of individual differences provided the pupil does not skip a grade. Acceleration and differentiation of subject matter are inseparable in an effective program in mathematics in the elementary school.

5. The structure of the numeration system should receive great emphasis in the program for the more able pupils.

6. The more able pupils should be encouraged to do independent study of topics in reference books and in supplementary textbooks.

EXERCISES

1. Try to apply the list of principles underlying the planning of the mathematics program given on page 384 to the evaluation of the program of some elementary school. In what respect is the program adequate? What are some of the limitations of the program?
2. Prepare a list of criteria to be used in selecting children talented in mathematics in the primary grades; in the intermediate grades.
3. What methods do you think can be added to the list on page 439 to motivate the more able learners?
4. Why is acceleration by skipping a grade an undesirable method of providing for superior learners?
5. What methods of grouping pupils according to their aptitude in elementary mathematics are used in local schools? What method of grouping can be used by the teacher in a typical classroom?
6. Give other illustrations of methods of developing mathematical power in addition to those given in this chapter.
7. What is the difference between extension and enrichment?
8. What work with operations with number bases other than base ten do you think should be taught in the elementary school?
9. Have a demonstration of a binary computer as described on page 433. Also demonstrate a computer that operates with numbers expressed in base three.
10. Use the method described on page 438 and show the first four octagonal numbers. Use the pattern for finding octagonal numbers and write the next two numbers in the series.

SELECTED READINGS

Barbe, W. B., *Psychology and Education for the Gifted: Selected Readings.* New York: Appleton-Century-Crofts, Inc., 1965.

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Hogben, L., *Mathematics in the Making.* New York: Doubleday & Company, Inc., 1960.

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Individualizing Instruction, Sixty-first Yearbook of the National Society for the Study of Education, Part 1. Chicago: University of Chicago Press, 1962. Chapters 3, 11, 12, and 14.

Newman, J. R., *The World of Mathematics.* New York: Simon and Schuster, Inc., 1956. Volumes I–IV.

Spitzer, H. F., *Practical Classroom Procedures for Enriching Arithmetic.* St. Louis, Mo.: Webster Publishing Division, McGraw-Hill, Inc., 1956.

HOW TO PREPARE ESSENTIAL MATERIALS AND LEARNING AIDS

The teacher may equip the mathematics classroom with either commercial or homemade materials. The following directions will enable her to make the most of the essential materials for classroom use that are described in this text.

Flannel board

A flannel board can be made by covering a piece of masonite or similar material (24 by 30 inches) with flannel having a good nap (Fig. A.1). Make fractional disks lined with flannel about 10 inches in diameter. Cut these disks into halves, thirds, fourths, sixths, and eighths. Keep two disks whole.

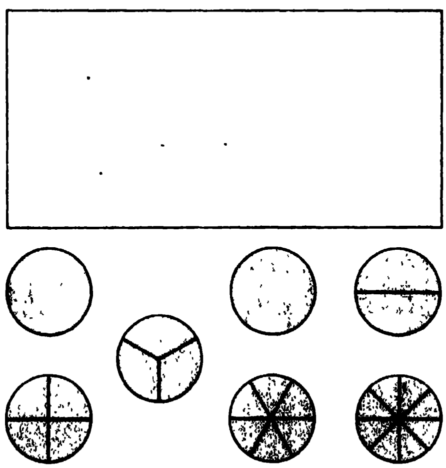


Figure A.1

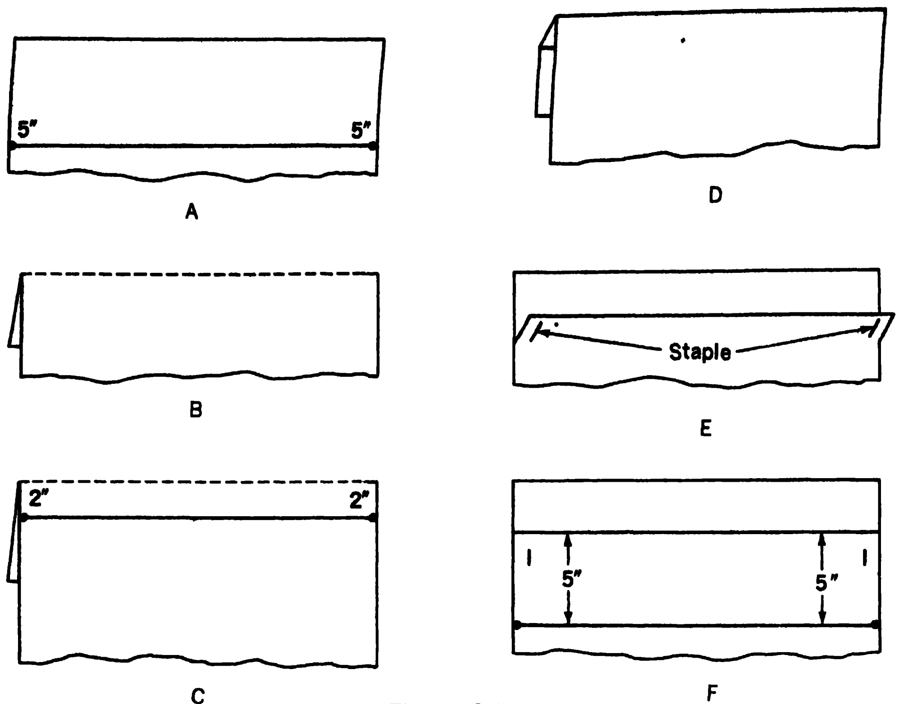


Figure A.2

Place-value charts

A place-value chart may be made of wood or oak tag. For the primary grades the teacher may find it helpful to use a place-value chart consisting of three separate sections, each made of oak tag. Each section is used to represent a place in the number system, as ones, tens, and hundreds. For grades 4–6, one chart can be subdivided into three sections or parts to show the given places. Since the same teacher seldom will be teaching at all six grade levels, directions for making each type of pocket chart follow.

Three pocket charts for primary grades

Use three sheets of oak tag, each 20 by 26 inches, to make the pocket charts.

The steps in making the card holders are as follows (see also Fig. A.2):

1. Place the paper on a table with the short edges at top and bottom. Measure down 5 inches from the top along each side and place dots. Join the dots with a line, as in Figure A.2(A).

2. Fold the top back and under along this line and crease firmly, as in Figure A.2(B).

3. Measure down 2 inches from the crease along each side and join the dots with a line, as in Figure A.2(C).

4. Fold the top back and under along this line and crease firmly, as in Figure A.2(D).

5. Open up flat on the table. Lift the lower creased fold up over the upper crease. The upper crease will slide down under the fold and you will have the first pocket. Crease firmly and staple down each edge of the pocket so that it stays in place, as shown in Figure A.2(E).

6. Now to make the second pocket. Measure down 5 inches from the top

edge of the first pocket on each side of the chart and place dots. Join the dots with a line, as in Figure A.2(F). Form the next pocket by repeating the process described above.

7. Repeat direction (6) until you have three pockets on the card holder.

8. Make two more similar card holders, following directions (1–7) above, so as to have a total of three pocket charts.

9. Each card holder now needs a stiff backing. Get a packing carton for this purpose. You will need three sides of a carton. Each side should be at least as large as a card holder.

Lay the back of the card holder on the cardboard backing and cut the cardboard to the size of the pocket chart. Then staple the cardboard and the chart. You may choose to bind the edges of the chart with masking tape to improve its appearance.

10. Finally, use a white crayon or a small brush with white poster paint to print above the top pocket the name of each card holder, as shown in Figure A.3.

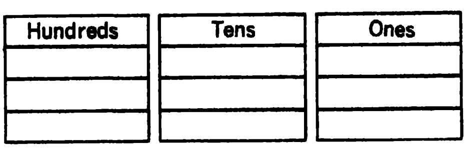


Figure A.3

Cards for card holders

You need 20 sheets of 9- by 12-inch red construction paper to make the cards for the pocket charts. You also need a supply of rubber bands to make bundles of cards. Use miniature rubber bands for the tens' bundles.

1. On your paper cutter you can cut five sheets at a time.

2. Using the 9-inch edge of the paper, measure and cut three 3-inch strips, as in Figure A.4(A).

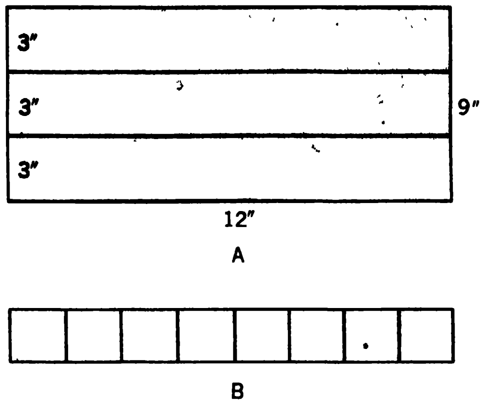


Figure A.4

3. From each 3- by 12-inch strip you can cut eight 3- by $1\frac{1}{2}$-inch cards, as in Figure A.4(B).

4. Since you get 24 cards from each 9- by 12-inch sheet, you will get 480 cards—an adequate supply for the operations needed in the primary grades.

5. To cut these cards, pick up five of the 3- by 12-inch strips at a time. Cut $1\frac{1}{2}$- by 3-inch cards by cutting off $1\frac{1}{2}$-inch strips from the 12-inch edge. As each group of five cards drops, pick up the cards and lay them aside. When the next five cards drop, place them with the previous group and make a bundle of 10 cards, or 1 tens' bundle. Have children help make 45 bundles of 10 cards.

6. To make a hundreds' bundle, pick up 10 tens and wrap them together with a rubber band. Make 2 hundreds' bundles, keeping the bands on the tens.

7. You now have 2 hundreds' bundles, 25 tens' bundles, and 30 single cards. Keep the cards in a box.

One-piece pocket chart

The pocket chart for the intermediate grades should be in one piece, divided into three sections. The teacher in the lower grades may also prefer to use a pocket chart of this kind. The

chart may be made of wood or of oak tag. Oak tag 20 by 26 inches is suitable for making a pocket chart. The steps in making the chart are as follows:

1. Place the paper on a table with one of the 26-inch edges nearest you. Measure down 5 inches from the top edge of the paper and draw a line parallel to this edge. Fold the top along this edge and crease firmly.

2. Repeat steps (2–4) just listed.

3. Divide the sheet longitudinally to form three equal pockets in each row. At the points of intersection with the rows of pockets, insert staples.

4. Beginning on the right, label the sections *ones*, *tens*, *hundreds* to show whole numbers. Beginning on the left, label the pockets *ones*, *tenths*, *hundredths* to show decimals. The same pockets may be used for both whole numbers and decimals provided different labels are given.

5. Fasten the top of the chart to a wire coat hanger.

Use cards approximately $1\frac{1}{2}$ by $2\frac{1}{2}$ inches to insert in the pockets to represent numbers. Tongue depressors or splints may be substituted for cards.

At the level of grade 4 or above, it should not be necessary to use a bundle of 10 cards to represent each digit one place to the left of a given place. Cards of different colors may be used to show the values represented in the respective places. Thus if a red card represents 1 one in ones' place, a blue card may represent 1 ten in tens' place and a green card may represent 1 hundred in hundreds' place. Similarly, different-colored cards may be used to represent the places to the right of ones' place.

At the next higher level of understanding of place value, a card of the same color should be used to represent a digit in a given pocket. If all of the cards are red, three of these cards in ones' pocket represent 3; in tens' place, 30; and in hundreds' place, 300. The pupil who reaches this stage in understanding place value is ready to interpret the value of a digit in a number. He should then know that a 3 two places to the left of ones' place represents 300, as in the number 347.

Fraction kit for pupils

Each pupil in grades 5 and 6 should have a kit of circular cutouts for demonstrating fractional numbers. Have these cutouts made of oak tag or construction paper. Supply each pupil with 12 circular disks approximately 5 inches in diameter. He should use two of these to represent wholes. He should cut two other disks to represent halves and then cut the rest of the disks in pairs to represent thirds, fourths, sixths, and eighths.

In grade 5 it may not be necessary to have cutouts to represent thirds and sixths. In that case 8 circular disks are needed. The teacher should remember that it is necessary to supply a pupil with a model cutout equal to a third. Then he can fold a third so as to form two equal parts, each of which is equal to a sixth.

The radius of a circle may be used in dividing the circle into sixths. Draw a circle and mark a point on the circumference. Open compasses equal to the radius. From the given point mark in succession on the circle five other points. Connect the six points with the center of the circle. Each sector now marked is a sixth of the circle. Cut the sixths by cutting along the radii you drew.

Rectangular squares and strips

Supply each pupil in grades 3–6 with squares and rectangular strips to be used to objectify the work with whole

numbers and decimals. A piece of construction paper 15 by 21 inches can be ruled into $\frac{3}{4}$-inch squares to provide the necessary squares and rectangular strips. A paper of this size will provide 20 rows and 28 columns, each $\frac{3}{4}$-inch wide, making a total of 560 $\frac{3}{4}$-inch squares (20 by 28). Cut these squares to form large squares and rectangular strips as follows:

1. 3 large squares containing 100 $\frac{3}{4}$-inch squares
2. 20 strips of 10 $\frac{3}{4}$-inch squares
3. 60 $\frac{3}{4}$-inch squares cut as follows:
   a. 15 single squares
   b. 5 strips of 2 squares
   c. 5 strips of 3 squares
   d. 5 strips of 4 squares.

The pupil should use the ruled side of the squares and the strips to represent whole numbers. Figure A.5 shows the number 138. The 8 ones may be represented by 8 small squares, two groups of 4 squares, or any other combination of squares having a sum of 8.

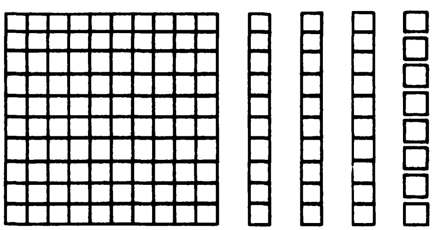


Figure A.5

The pupil should use the unruled side of the squares and the strips to represent decimals. The unruled side of a large square represents a whole, a full single strip represents a tenth, and a small square represents a hundredth.

Figure A.6 shows how to represent the number 1.34.

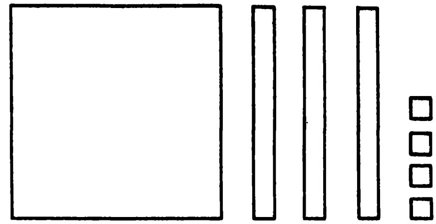


Figure A.6

Have the pupil keep the cutouts, squares, and rectangular strips in two manila envelopes. Be certain that he has a flat-top desk on which to manipulate these exploratory materials when he uses them in class.

"Everybody show" cards

Each child should be supplied with 10 cards 2 by 3 inches. Each card should contain one of the 10 digits (Fig. A.7).

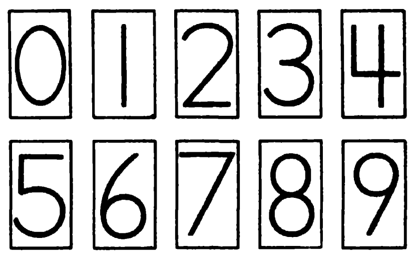


Figure A.7

The digits should be placed to have from $\frac{1}{2}$ inch to 1 inch of space at the bottom of the card. The digits should be printed neatly and legibly and almost as big as the card.

The pupil should always arrange the cards on his desk in the order shown when he participates in the game called Everybody Show.

The 10 cards are adequate for all practice on number facts in addition through the eighteens except for the elevens. When the answer to a fact is 11, two cards containing the digit 1 are needed. These cards can be used in practicing the basic facts in each of the four processes. Thus for the grouping 4×8, the pupil would show 32 in the card holder; for the grouping $9\overline{)54}$, he would show 6.

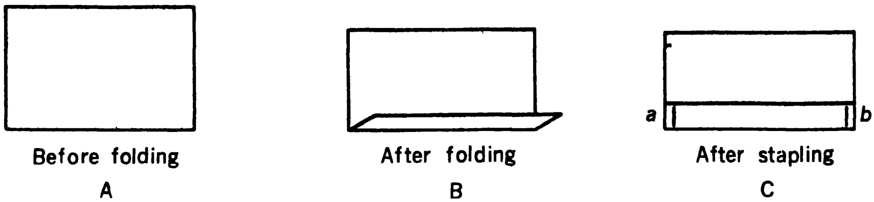


Figure A. 8

Each pupil needs a card holder in order to play "Everybody Show." To make a card holder, use a sheet of red construction paper 4 by 6 inches (Fig. A.8). Lay it with the 6-inch edges at top and bottom, as in (A). Fold up $\frac{1}{2}$ inch on the bottom edge to make a pocket, as in (B). Staple the ends at points *a* and *b* so that the pocket is held firmly in place, as in (C).

GLOSSARY

ABACUS An abacus is an ancient counting frame or device consisting of movable beads on parallel rods. Each rod represents a place in a numeral. Today the abacus is used in schools chiefly to teach the concept of place value. An empty rod performs the same function that zero performs in holding a place in a numeral.

ACUTE ANGLE An acute angle is an angle having a measure between 0° and 90°.

ADDEND An addend is a number to be added to find a sum. The addends in $4 + 7 + 9$ are 4, 7, and 9.

ADDING BY ENDINGS Adding by endings involves the addition of a two-digit number and a one-digit number in one mental operation.

ADDITIVE INVERSE The sum of any number and its additive inverse is zero.

ALGORISM An algorism is a method of writing and performing any of the four basic operations. An algorism for multiplication is shown at the right.

$$\begin{array}{r} 24 \\ \times\ 3 \\ \hline 72 \end{array}$$

ANGLE An angle is a geometric figure formed by two rays, not in the same line, that have the same endpoint.

ARITHMETIC EXPRESSION An arithmetic expression consists of one or more symbols to represent numbers. If two or more numbers are represented, they are connected by conventional signs to indicate the operations to be performed. Examples of arithmetic

expressions are 5, $3 + 7$, 2×6, $18 \div 3$ (see Numeral).

ARRAY An array is an orderly arrangement of the elements of a set in rows, with the same number of elements in each row.

ASSOCIATIVE PROPERTY FOR ADDITION According to this property the way in which three addends are grouped does not affect the sum, as $(a + b) + c = a + (b + c)$.

ASSOCIATIVE PROPERTY FOR MULTIPLICATION According to this property the way in which three factors are grouped does not affect the product, as $(a \times b) \times c = a \times (b \times c)$.

BASE (PERCENTAGE FORMULA) The base in the percentage formula is the number used for applying the rate to find the percentage. In the equation 3% of $80 = n$, the base is 80 ($p = br$).

BASE (SYSTEM OF NUMERATION) The base of a system of numeration is the number of units that must be grouped in a given place in a numeral to equal one in the next place to the left in that numeral. The base of any system of numeration is the same as the number of digits used in that system, provided the system involves the property of place value.

BASIC FACT A basic fact in addition or multiplication is any combination of a pair of one-digit numbers with the sum or product. A basic fact in subtraction or division is the corresponding fact derived from the opposite operation, excluding division by 0. Illustrations of basic facts are $5 + 0 = 5$, $12 - 4 = 8$, $3 \times 7 = 21$, and $14 \div 2 = 7$.

BASIC NUMBER PAIR A basic number pair consists of two one-digit numbers that may be used to form a basic fact in addition and multiplication.

BINARY (NUMBER BASE) The base of a binary numeration system is two. The only digits used in this system are 1 and 0.

BINARY (OPERATION) A binary operation involves only two numbers. All of the basic operations are binary. When more than two numbers are to be added or multiplied, they must be grouped.

BRACES { } Braces are symbols or signs used to designate a set. Braces are also used as symbols of grouping, for example, $3 - \{2[4 - (2 - 1)]\}$.

CARDINAL NUMBER A cardinal number is the number of a set. A cardinal number answers the question, How many?

CARTESIAN PRODUCT The Cartesian product of two sets is the set of ordered pairs obtained by choosing the first element in each pair from the first set and the second element of each pair from the second set. Thus, $\{(1, 2)\} \times \{(a, b)\} = \{(1, a), (1, b), (2, a), (2, b)\}$.

CASTING OUT NINES Casting out nines in a numeral is a procedure used in checking the computation with numbers expressed in base ten.

CLOCK ARITHMETIC Clock arithmetic is another name for modular arithmetic.

CLOSURE A set of numbers is closed with respect to an operation if the answer obtained by applying that operation is an element in that set. Thus, the set of counting numbers is closed with respect to addition and multiplication, as $3 + 5 = 8$ and $12 \times 8 = 96$. The set of counting or natural numbers is not closed with respect to subtraction and division, since the answers to the examples $3 - 4$ and $3 \div 4$ are not in this set.

COMMUTATIVE PROPERTY FOR ADDITION The order of adding two numbers does not affect the sum, as $a + b = b + a$.

COMMUTATIVE PROPERTY FOR MULTIPLICATION The order of multiplying two factors does not affect the product, as $a \times b = b \times a$.

COMPLEX FRACTION A complex fraction contains a fraction in the numerator, in the denominator, or in both terms.

COMPOSITE NUMBER A composite number has other whole-number factors besides itself and 1, for example, 6, which has factors of 2 and 3.

CONGRUENT Two geometric figures are congruent when they have the same size and shape.

DECADE A decade is a set of ten consecutive whole numbers beginning with a multiple of 10.

DECIMAL A decimal is a fractional numeral in which the denominator is not written because it is indicated by the position of the decimal point. Every denominator in a decimal numeral is a power of 10.

DENOMINATOR In the fractional numeral $\frac{a}{b}$, b is the denominator.

DENSENESS There is an infinite set of numbers between any two rational numbers. This property of rational numbers is known as denseness.

DIGIT A digit is any one of the 10 symbols used in the decimal system of numeration.

DIRECTED NUMBERS See Signed numbers.

DISJOINT SETS Two sets are disjoint when they have no common elements.

DISTRIBUTIVE PROPERTY According to this property, if an indicated sum is to be multiplied by a number, each addend must be multiplied by that number and these products added. Thus, $a(b + c) = ab + ac$.

DIVIDEND In a division example, the dividend is the number to be divided, as 34 in the example $2\overline{)34}$. The dividend corresponds to the product in multiplication.

DIVISIBLE One number, n, is divisible by another number, p, if p is a factor of n. Thus 12 is divisible by 2, 3, 4, and 6 because each number is a factor of 12.

DIVISOR In a division example, the divisor is the number by which the dividend is to be divided, as $\frac{2}{3}$ in the example $6 \div \frac{2}{3}$. The product of the divisor and the quotient is equal to the dividend.

DUODECIMAL (NUMBER BASE) A duodecimal system of numeration has a base of twelve. This system has 12 digits: $\{0, 1, 2, \cdots 9, T, E\}$.

ELEMENTS OF A SET Anything that belongs to the set is an element, or member, of the set. The elements of the set {red, blue, green} are red, blue, and green.

EMPTY SET The empty set contains no elements. The designation for the empty set is $\{\ \}$ or ϕ. The set of even numbers in the set $\{1, 3, 5, 7\}$ is designated as the set $\{\ \}$.

ENDPOINT The endpoint of a line segment is the point from which it is possible to move in only one direction and remain in the segment.

EQUAL SETS Equal sets have the same elements, for example, $\{a, b, c\}$ and $\{b, c, a\}$.

EQUALITY See Equation.

EQUATION An equation is a mathematical statement indicating that two expressions name the same number. The equation $3 = 2 + 1$ is true, but the equation $3 = 2$ is false.

EQUILATERAL TRIANGLE A triangle is equilateral if its sides are congruent.

EQUIVALENT SETS Two sets are equivalent if their elements can be brought into one-to-one correspondence.

EULER DIAGRAM An Euler diagram

contains two or more closed curves (usually circles) to show how sets are related. An Euler diagram is often known as a Venn diagram.

EXCESS OF NINES The excess of nines in a number is equal to the remainder obtained when the number is divided by 9. The excess of nines in 437 is 5. The excess of nines is used in checking an example by casting out nines.

EXPANDED FORM OF A NUMERAL The expanded form of a numeral is the notation that indicates the total value of each of its digits. The expanded form of the numeral 725 is $7 \times 10^2 + 2 \times 10^1 + 5 \times 10^0$ or $7 \times 100 + 2 \times 10 + 5$.

EXPONENT The exponent ^{3} in 4^3 shows that 4 is to be used as a factor three times. The expression 4^3 is read "4 to the third power," and it means $4 \times 4 \times 4$. The 4 is called the base and the ^{3} is called the exponent.

FACE VALUE (DIGIT) The face value of a digit in a numeral is the cardinal value of that digit. The face value of 2 in 27 is cardinal value two.

FACTOR If two or more numbers are multiplied, each number is a factor of the product. In the expression $9 \times 24 = 216$, 9 and 24 are factors of 216.

FIELD A field is a structure that is closely associated with the study of arithmetic and algebra. The postulates of a field are frequently known as properties. The properties include closure, associativity, commutativity, distributivity, identity, and inverse.

FINITE SET A finite set has a specific number of elements. The set $\{2, 4, 6, 8\}$ contains four elements, hence it is a finite set.

FRACTIONAL NUMERAL A fractional numeral is in the form $\frac{a}{b}$ in which a and b are whole numbers and $b \neq 0$.

FUNDAMENTAL THEOREM OF ARITHMETIC According to this theorem, every composite number can be factored uniquely into prime factors.

GENERALIZATION A generalization is a statement that is true for every member of a set. An example of a generalization is: A number is divisible by 2 if it ends in 0, 2, 4, 6, or 8.

GEOMETRIC FIGURE A geometric figure is a set of points.

IDENTITY ELEMENT FOR ADDITION The identity element for addition is 0. Zero added to any number is that number.

IDENTITY ELEMENT FOR MULTIPLICATION The identity element for multiplication is 1. Any number multiplied by 1 is that number.

IMPROPER SUBSET Every set is an improper subset of itself. Some mathematicians refer to the empty set as an improper subset.

INEQUALITY An inequality is a number sentence showing that two expressions are names for different numbers, for example, $3 \neq 5$ or $n - 5 < 2$.

INFINITE SET A set with an unlimited (infinite) number of elements is an infinite set, as the set $\{1, 2, 3, 4, \cdots\}$. The leaders indicate that the set is infinite.

INTEGER The set of integers includes the set of positive whole numbers, negative whole numbers, and 0.

INTERSECTION OF TWO SETS The intersection of two sets is the set of elements included in both sets. The intersection of sets A and B is expressed as $A \cap B$.

INVERSE OPERATION Addition and subtraction are inverse operations, as are multiplication and division. An inverse or opposite operation undoes or

neutralizes the operation of which it is the opposite.

IRRATIONAL NUMBER An irrational number cannot be expressed by the quotient of two integers, as $\sqrt{3}$.

ISOSCELES TRIANGLE An isosceles triangle has two congruent sides.

LEAST (LOWEST) COMMON DENOMINATOR (LCD) The LCD is the smallest number that is divisible by all denominators in the set of denominators.

LINE SEGMENT A line segment is the set of points on the shortest path connecting two points in a plane.

LOWEST COMMON MULTIPLE (LCM) The LCM of a set of whole numbers is the smallest number divisible by all the numbers in the set. The LCM of 10, 12, and 15 is 60.

MATCHING SETS Matching sets are equivalent sets.

MATHEMATICAL SENTENCE A mathematical sentence is usually an equation or an inequality. It indicates the relationship between two quantities, as $3 \neq 5$, $4 < 5$, or $n - 2 = 7$.

MEASUREMENT Measurement is the process of finding the number of standard units there are in an object or thing. A board 4 feet long is 4 times the measure of the standard unit of 1 foot.

METRIC SYSTEM The metric system of measures is based upon the decimal system of numeration. The standard units of length, of mass, and of liquids are the meter, the gram, and the liter, respectively.

MIXED FRACTIONAL NUMERAL (MIXED NUMBER) A mixed fractional numeral represents the same number as the sum of a whole number and a fraction, as $3\frac{1}{4}$. The numeral $3\frac{1}{4}$ represents the same number as the numeral $3 + \frac{1}{4}$.

MODULAR ARITHMETIC The set of numbers modulus 5 (mod 5) is $\{0, 1, 2, 3, 4\}$. If a modulus is a prime number, the modulus system forms a number field and is useful in helping pupils learn number properties.

MULTIPLE A multiple of a number is the product of any integer and that number.

MULTIPLICAND AND MULTIPLIER Multiplicand and multiplier are technical names for the factors in an example involving multiplication.

MULTIPLICATIVE INVERSE See Reciprocal.

NATURAL NUMBER A natural number is a member of the set of counting numbers, for example, $\{1, 2, 3, \cdots\}$.

NEGATIVE NUMBER A negative number is a number that is less than 0.

NULL SET See Empty set.

NUMBER LINE A number line is a line having numbers corresponding to points on the line.

NUMBER PERIOD A number period is the system of grouping places in a numeral to facilitate the reading of the number represented. Beginning at the right of a whole number, one finds the first three places to the left are designated as units, the next three places as thousands, the next three as millions, the next three as billions, and so on. An illustration of number periods is 2,345,000.

NUMBER RAY A number ray is a ray in which numbers are assigned to points on it.

NUMBER SYSTEM A number system is a set of numbers and one or more operations.

NUMERAL A numeral is a symbol or expression used to represent a number.

NUMERATOR In the fractional numeral $\frac{a}{b}$, a is the numerator.

OBTUSE ANGLE An obtuse angle is an angle having a measure between 90°

and 180°.

OPEN SENTENCE An open sentence is a number sentence that contains one or more variables. Such a sentence is neither true nor false until each variable is replaced by a numeral.

ORDERED PAIR An ordered pair (a, b) has two properties: (1) $(a, b) \neq (b, a)$ if $a \neq b$; (2) $(a, b) = (c, d)$, if and only if $a = c$ and $b = d$.

ORDINAL NUMBER An ordinal number answers the question, Which one? in a set of numbers. Each ordinal is the set of all preceding ordinals. The "third member" of a set is a particular member of the set that contains at least three members and implies the existence of a first member and a second member.

OVERLAPPING SETS Two sets are overlapping if they have one or more elements in common, but neither set is a subset of the other set.

OVERLOADED PLACE A place in a numeral is overloaded when the number in that place cannot be expressed as a single digit.

PERCENTAGE A percentage is the result that is obtained from finding a percent of a number. In the example 3% of 80 = 2.4, the percentage is 2.4.

PERCENTAGE FORMULA The percentage formula is $p = br$, where p = percentage, r = rate, and b = base.

PLACE VALUE Place value is the property of our numeration system that gives a digit a different value depending upon the position the digit holds in a numeral.

POLYGON A polygon is a plane closed figure bounded by line segments, with no two adjacent segments on the same line.

POLYHEDRON A polyhedron is a closed three-dimensional figure having four or more faces. Each face is a polygon and its interior.

POLYNOMIAL FORM OF A NUMERAL See Expanded form of a numeral.

POSITIVE NUMBER A positive number is any number greater than 0.

PRIME FACTORIZATION Prime factorization, or complete factorization, consists in finding all of the prime factors of a number, for example, $12 = 2 \times 2 \times 3$.

PRIME NUMBER A prime number has no whole-number factor except itself and 1. The first prime number is 2.

PRODUCT The product is the answer obtained when the operation of multiplication is performed on a pair of numbers. In the example $3 \times 7 = 21$, the product is 21 and the factors are 3 and 7.

PROPER SUBSET A proper subset of set S is any set not equal to S.

PURE IMAGINARY NUMBER A pure imaginary number can be written as ai, where a is a real number and i is $\sqrt{-1}$. A complex number may be expressed as $a + bi$, where a and b are real numbers.

QUADRILATERAL A quadrilateral is a four-sided polygon.

QUOTIENT Divisor × quotient = dividend; therefore dividend ÷ divisor = quotient. The quotient is the result in an example in division.

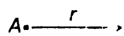
RAGGED DECIMALS Ragged decimals are decimals to be added or subtracted that are expressed as decimals to an unlike number of places, as .3 + .24 + .175.

RATIO The quotient of two numbers is their ratio, excluding a divisor of 0.

RATIONAL NUMBER A rational number is a number that is the quotient of two integers with the divisor not zero. Every rational number may be represented by a fractional numeral.

RAY A ray is a set of points extending indefinitely in one direction from a

point called an endpoint. In the illustration, r is the ray and A is the endpoint. A ray is always a part of a line.



REAL NUMBERS The set of real numbers includes the set of whole numbers, the set of integers, the set of rational numbers, and the set of irrational numbers. The set of real numbers can be paired off with all points on a line. There is a point on a number line for each real number and a real number for each point.

RECIPROCAL The quotient of 1 divided by a number is the reciprocal of that number. There is no reciprocal for 0. The product of any number and its reciprocal is 1. Reciprocal is another name for multiplicative inverse.

REGION (PLANE) A region of a plane is a subset of the plane.

REGULAR POLYGON A regular polygon has congruent sides and congruent angles.

RENAME To rename a number is to express the number with different numerals; for example, 5 may be renamed as $2 + 3$, $6 - 1$, $\frac{1}{2} \times 10$, or $\frac{20}{4}$.

REPEATING DECIMAL A repeating decimal results from expressing a fraction as a decimal. If the denominator of the fraction has factors other than 2 and 5, the decimal will be nonterminating, as $\frac{1}{3} = .333 \cdots$. Generally, repeating decimals refer to nonterminating decimals. However, not all nonterminating decimals are repeating decimals.

REVERSIBILITY Reversibility refers to the inverse relationship between a pair of operations, such as addition and subtraction.

RHOMBUS A rhombus is a parallelogram having four congruent sides.

RIGHT ANGLE A right angle is an angle having a measure of 90°.

SCALENE TRIANGLE A scalene triangle has no two sides that are congruent.

SCIENTIFIC NOTATION Scientific notation is the convention of writing a number as the product of two factors. One factor is less than 10 and equal to or greater than 1. The second factor is a power of 10. In scientific notation, $3{,}750{,}000 = 3.75 \times 10^6$.

SET A set is a collection of objects, things, or numbers. The word "set" is generally accepted as an undefined term in mathematics.

SIEVE OF ERATOSTHENES This sieve is a means of screening the prime numbers from the set of counting numbers.

SIGNED NUMBERS The set of signed numbers is the union of the set of positive numbers, the set of negative numbers, and 0.

SOLUTION SET A number belongs to the solution set of an open sentence if that number makes the sentence true and is a member of the universal set.

SUBSET Each element or combination of elements in a set forms a set that is called a subset of the given set.

SYSTEM OF NUMERATION Any system of symbols representing numbers is called a system of numeration. Our system of numeration is called a decimal system because the base is ten.

TERMS (FRACTION) The numerator and denominator of a fractional numeral are called its terms.

TOTAL VALUE (DIGIT) The total value of a digit in a numeral is the product of the digit's cardinal value and its positional value. The total value of the 7 in 714 is 7×100, or 700.

TRAPEZOID A trapezoid is a quadrilateral with one pair of parallel sides.

TWIN PRIME NUMBERS Two consecutive odd numbers that are prime are called twin prime numbers.

UNION OF TWO SETS The union of two sets is the set of all elements that are in either the first set or the second set or both. The union of set A with set B is represented as $A \cup B$.

UNIT FRACTION A unit fraction is a fraction having a numerator of 1, as in the fraction $\frac{1}{4}$.

UNIVERSAL SET The universal set is the set of all members for any category or discussion.

VARIABLE A variable is a symbol used to represent any element in a set of numbers. Thus, the variable n in the equation $n - 2 = 5$ represents any number in the set of whole numbers. If 7 replaces n in the given equation, the sentence is true. A variable is a symbol that holds a place for a numeral.

VENN DIAGRAM See Euler diagram.

SELECTED ANSWERS

Chapter 4 – page 46

1. C
2a. A and D
2b. E and B
2c. B and C
2d. A and B°
2e. A and B°
3. Subset
4. Disjoint
5. Subset
6. B
7. b, c, and d

°other answers possible

Chapter 4 – page 50

1a. {1, 2, 3, 4, 5}
1b. {1, 2, 3, 4, 5, 6, 7}
1c. {3, 4, 5, 6, 7}
1d. A or {1, 2, 3, 4}
1e. ϕ or {0}
1f. {(3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 7), (5, 5), (5, 6), (5, 7)}
1g. {(3, 9), (4, 0)}
1h. {1, 2}
1i. C or {5, 6, 7}
1j. ϕ or { }
2. {1, 2} and {3, 4}
3. {0, 1, 2, 3} and {4, 5, 6, 7, 8, 9}
4. The set of things which are green or hats.
5. The set of green hats.
6a. {5}
6b. {1, 2, 6, 7}
6c. {1, 2, 3, 4, 5}
6d. {0, 3, 4, 5}

Chapter 4 – page 54

1. A and E, B and D
2a. {x, z, 1, 2, 3}
2b. {0, 1, 2, 3}
2c. A or {x, z}
2d. ϕ or { }
2e. {(1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1)}
3. $N(A) = 2$; $N(B) = 3$, $N(C) = 4$; $N(D) = 3$; $N(E) = 2$
5. b, d, and e
6. b and e
7. a, b, and e
8. The variable holds a place for a numeral.
9. e
10. Rational numbers

Chapter 5 – page 72

5. 7.25×10^{11}
7a. 211_4
7b. 365_8
7c. 4220_5
8a. 121
8b. 2094
8c. 27
8d. 5338
9a. $24_5 = 14$
9b. $130_5 = 40$
9c. $3200_5 = 425$
9d. $4412_5 = 607$
10a. 112_7
10b. 1022_8
10c. 16_8
10d. 224_5
10e. 110_2
10f. 1544_6

Chapter 6 – page 80

1a. Comm. +
1b. Identity +
1c. Comm. ×
1d. Identity ×
1e. Comm. +
1f. Identity ×
1g. Comm. +
1h. Assoc. +
1i. Inverse ×
1j. Dist.
1k. Comm. ×
1l. Dist.
1m. Assoc. +
1n. Assoc. ×
1o. Inverse +
1p. Dist.
1q. Inverse ×
1r. Dist.
2. Whole numbers closed for add. and mult.; even numbers (positive) closed for add. and mult., odd numbers closed for mult.; {0, 1} closed for mult.
3. 1
4. 0
5. d
6. b
7. b

Chapter 6 – page 88

1. Rational numbers
5. Natural numbers; integers
6. A = R, B = Ra, C = ϕ; D = W
7. Rational, real, and complex
8. For every x in S there is a ^-x such that $^-x + x = 0$.
9. The set is not closed for division.
10. Integers
11. Dist
12. a and d

Chapter 9 – page 152

10. sum in a, d, and f, addend in b, c, and e
11a. 43
11b. 18
11c. 32
11d. 63
11e. 59
11f. 101
12a. =
12b. <
12c. =
12d. <
12e. >
12f. >
13a. 21
13b. 20
13c. 40
13d. 8
13e. 10

13f. 11
14. *A:* $32 + 48 = 80$,
$48 + 32 = 80$,
$80 - 32 = 48$;
$80 - 48 = 32$;
B. $18 + 29 = 47$,
$29 + 18 = 47$;
$47 - 29 = 18$,
$47 - 18 = 29$

Chapter 10—page 180

4. $3(30 + 2) =$
$3 \times 30 + 3 \times 2$
6. If $1 \div 0 = n$ then $1 = 0 \times n$ contradicting the fact that $0 \times n = 0$.
9. *A:* $4 \times 4 = 16$, $16 \div 4 = 4$,
B: $7 \times 8 = 56$, and so on
10a. ratio
10b. ratio
10c. 15 mpg is a rate, comparing 15 mi. with 60 mi. is a ratio.
10d rate
10e. rate
13. Basic pattern is
$a(10 - 1) + a = a \cdot 10$.

Chapter 11—page 203

7. The error is a multiple of 9.
8. 50×30, 60×40
11. A. $12 \times 15 = 180$,
$15 \times 12 = 180$,
$180 \div 12 = 15$,
$180 \div 15 = 12$,
B: $16 \times 57 = 912$;
$57 \times 16 = 912$,
$912 \div 16 = 57$,
$912 \div 57 = 16$

Chapter 12—page 220

1. 7, 14, 21, 28, 35, 42, 49
2. $\{60\}$, multiples of 60 less than 61
4. 32, 33, 34, 35, 36, 48, 49, 50, 51, 52, other answers possible
5. 31, 37, 41, 43, 47
6. A. 72; *B.* 90, *C:* 180, *D:* 144
9. $1 - 2 = 1 + 1 = 2 \pmod 3$
10. $1 \div 2 = 1 \times 2 = 2 \pmod 3$

14a. 2 (Mod 5)
14b. 2 (Mod 5)
14c. 4 (Mod 5)
14d. 4 (Mod 5)
14e. 4 (Mod 5)
14f. 4 (Mod 5)
14g. $3 + 2 = 0 \pmod 5$
14h. $2 \times 3 = 1 \pmod 5$
14i. Closed for all but division.

Chapter 14—page 268

5. $(\frac{7}{8} \times 1\frac{1}{7}) \times 3\frac{1}{3} = 3\frac{1}{3}$
7a. $m > n$
7b. $m < n$
7c. n is a multiple of m.
8. $n \times \frac{2}{3} = \frac{1}{4}$
11. $\frac{7}{8} \div \frac{5}{6} > 1$, $\frac{2}{3} \div \frac{1}{4} < 1$;
$\frac{1}{4} \div \frac{5}{9} < 1$; $\frac{11}{12} \div \frac{9}{10} > 1$;
$\frac{5}{6} \div \frac{30}{36} = 1$; $\frac{5}{8} \div \frac{7}{15} > 1$
12. A: $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$, $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$,
$\frac{1}{6} \div \frac{1}{3} = \frac{1}{2}$; $\frac{1}{6} \div \frac{1}{2} = \frac{1}{3}$
B: $\frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$; and so on
C: $12 \times \frac{3}{4} = 9$, and so on
D: $\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$, and so on

Chapter 15—page 289

4a. $7 \times (\frac{1}{10})^2 + 5 \times (\frac{1}{10})^1$
4b. $4 \times 10 + 2 \times 10^0 + 1 \times \frac{1}{10} + 5 \times (\frac{1}{10})^2$
4c. $1 \times (\frac{1}{10})^3 + 5 \times (\frac{1}{10})^1$
4d. $1 \times 10^2 + 5 \times (\frac{1}{10})^2$
5a. .094
5b. .0005
5c. 3005
6a. 30, .06
6b. 6, .03
6c. .3, .0006
6d. 6, .003
7. It indicates how but not why.
8. $\frac{3 \times 10 + 4}{2} = \frac{2 \times 10 + 14}{2}$
$= \frac{2 \times 10}{2} + \frac{14}{2}$
$= 1 \times 10 + 7 = 1.7$
9 $n - 1$
11 *A* $.3 + .5 = .8$;
$.5 + .3 = .8$,
$.8 - .3 = .5$,
$.8 - .5 = .3$
B: $.2 \times 8 = 1.6$ and so on
C: $.5 \times 24 = 12$ and so on
D: $.18 \times 1.2 = .216$ and so on
12a. $\frac{1}{2} = \frac{1}{2} \times \frac{10}{10}$
12b. $\frac{.1}{.4} \times \frac{10}{10} = \frac{1}{4}$
12c. $\frac{1.5}{2.5} \times \frac{10}{10} = \frac{15}{25}$
12d. $\frac{7}{1.8} \times \frac{10}{10} = \frac{70}{18}$

Chapter 17—page 316

4. 3 sets of n members combined with a set of 5 members is a set of 28 members.
7. Combining, separating, and comparing sets.

Chapter 18—page 325

1. Set of all points
7. $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, $\overline{AC}$, $\overline{BD}$, $\overline{AE}$, $\overline{EC}$, $\overline{BE}$, $\overline{ED}$
8. Infinite number
9. One
10. One
11. 6 if no 3 points are on same line
12. Set of lines or set of closed curves
13. Left to right, curved segment; curve; line segment, line, ray.

Chapter 18—page 336

1 c
2a. (A)
2b. (D)
2c. (B)
2d. (E) concave
2e. (F)
2f (C)
3a. (F)
3b. (C)
3c. (B), (D), or (E)

Chapter 18—page 339

1. 0 or 1; 0, 1, or 2; 0, 1, 2, or 3
2. c
4. Cube
5. Zero; one, two; three

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